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ON MULTIPLICATION OF PERRON-INTEGRABLE FUNCTIONS

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Let K be a compact n-dimensional interval, $\zeta : K \to R$, $\varphi : K \to R$; it is proved that the product $\zeta\varphi$ is Perron-integrable, if ζ is Perron-integrable and if φ is of strongly bounded variation (especially if φ is n-times continuously differentiable). This result is a particular case of a Theorem on General Perron Integral. There is obtained a formula for integration by parts. There is an example to show that the assumptions on φ cannot be replaced by certain weaker assumptions.

1. NOTATIONS AND PRELIMINARIES

1.1. Let R be the real line, R^+ – the set of positive reals, $\mathcal{N}$ – a finite nonempty set. Let n be a positive integer. Usually functions from intervals in R^n to R are the object of the theory of the Perron integral. In this paper there will be considered intervals in $R^{\mathcal{N}}$ and functions from these intervals to R; this makes it possible to introduce simple notations for various operations like projections, restrictions, etc.

Let $\emptyset \neq \mathcal{L} \subset \mathcal{M} \subset \mathcal{N}$. $x \in R^{\mathcal{M}}$ means that $x : \mathcal{M} \to R$ and $x_{\mathcal{L}}$ is the restriction of x to $\mathcal{L}$; espeacially $x_{\{m\}}$ is the restriction of x to the one-point set $\{m\}$ for $m \in \mathcal{M}$. $w \in R^{\{m\}}$ may be represented by the couple (m, α), $\alpha = w(m) \in R$. Let $|\mathcal{M}|$ be the number of elements of $\mathcal{M}$. In a natural way, $R^{\mathcal{M}}$ is a linear space. A bijection F of $\{1, 2, \dots |\mathcal{M}|\}$ on $\mathcal{M}$ gives to rise to a linear isomorphism $F' : R^{\mathcal{M}} \to R^{|\mathcal{M}|}$, F' being defined by $F'(x) = (x(F(1)), x(F(2)), \dots, x(F(|\mathcal{M}|)))$.

$\mathcal{M} - \mathcal{L}$ is the complement of $\mathcal{L}$ in $\mathcal{M}$; let $\mathcal{M} - \mathcal{L} \neq \emptyset \neq \mathcal{L}$. If $u \in R^{\mathcal{L}}$, $v \in R^{\mathcal{L} - \mathcal{M}}$, then $u \times v = v \times u$ is such an element z of $R^{\mathcal{M}}$ that $z_{\mathcal{L}} = u$, $z_{\mathcal{M} - \mathcal{L}} = v$; obviously z exists and is unique.

1.2. Let $a, b \in R^{\mathcal{M}}$; define

$$\langle a, b\rangle = \{x \in R^{\mathcal{M}} \mid \min(a(m), b(m)) \leqq x(m) \leqq \max(a(m), b(m)) \quad \text{for} \quad m \in \mathcal{M}\} ;$$

$\langle a, b\rangle$ is a compact interval in $R^{\mathcal{M}}$, which may be degenerate. Put

$$\operatorname{diam}(\langle a, b\rangle) = \max_{m\in\mathcal{M}}(|b(m) - a(m)|) .$$

If $\mathcal{L} \subset \mathcal{M}$, $\mathcal{L} \neq \emptyset \neq \mathcal{M} - \mathcal{L}$, $\mathcal{U} \subset R^{\mathcal{L}}$, $\mathcal{V} \subset R^{\mathcal{M}-\mathcal{L}}$, define

$$\mathcal{U} \times \mathcal{V} = \mathcal{V} \times \mathcal{U} = \{u \times v \mid u \in \mathcal{U},\ v \in \mathcal{V}\} .$$

Especially, if $s, t \in R^{\mathcal{L}}$, $w, z \in R^{\mathcal{M}-\mathcal{L}}$, then

$$\langle s, t\rangle \times \langle w, z\rangle = \langle w, z\rangle \times \langle s, t\rangle = \langle s \times w,\ t \times z\rangle$$

and if $a_m, b_m \in R^{\{m\}}$ for $m \in \mathcal{M}$, then the Carthesian product $\prod_{m\in\mathcal{M}} \langle a_m, b_m\rangle$ may be defined by induction on $|\mathcal{M}|$. Obviously $\prod_{m\in\mathcal{M}} \langle a_m, b_m\rangle = \langle a, b\rangle$, $a, b \in R^{\mathcal{M}}$ being defined by $a_{\{m\}} = a_m$, $b_{\{m\}} = b_m$ for $m \in \mathcal{M}$.

1.3. If $a, b \in R^{\mathcal{M}}$, $\alpha \in R^+$, write

$$a < b\,, \quad \text{if} \quad a(m) < b(m) \quad \text{for} \quad m \in \mathcal{M}\,,$$

$$a \leqq b\,, \quad \text{if} \quad a(m) \leqq b(m) \quad \text{for} \quad m \in \mathcal{M}\,,$$

$$\|a\| = \max_{m\in\mathcal{M}} |a(m)|\,,$$

$$\operatorname{sgn}(b - a) = \prod_{m\in\mathcal{M}} \operatorname{sgn}(b(m) - a(m))\,,$$

$$\mathcal{B}(a, \alpha) = \{x \in R^{\mathcal{M}} \mid \|x - a\| \leqq \alpha\}$$

$\mathcal{B}(a, \alpha)$ is the closed ball with the center a and radius α.

1.4. If $\mathcal{V} \subset R^{\mathcal{M}}$, let $\mathfrak{K}(\mathcal{V})$ be the set of compact intervals, which are contained in $\mathcal{V}$. Let $\mathfrak{K}_0(\mathcal{V})$ be the set of compact nondegenerate intervals, which are contained in $\mathcal{V}$. For $I \in \mathfrak{K}_0(R^{\mathcal{M}})$ denote by $\operatorname{Int} I$ the interior of I. Let $J, I, H \in \mathfrak{K}_0(R^{\mathcal{M}})$. Write $J = I \dotplus H$ if $J = I \cup H$ and $\operatorname{Int} I \cap \operatorname{Int} H = \emptyset$. J is called the direct sum of I and H.

1.5. The following Lemma may be proved by induction on the number of elements of $\mathcal{M}$.

Lemma. *If $J, I, H \in \mathfrak{K}_0(R^{\mathcal{M}})$, $J = I \cup H$, $I \neq J \neq H$, $I = \langle u, v\rangle$, $u < v$, $H = \langle w, z\rangle$, $w < z$, then there exists such a $j \in \mathcal{M}$ that*

(1.1) $$u_{\{m\}} = w_{\{m\}}\,, \quad v_{\{m\}} = z_{\{m\}} \quad \textit{for} \quad m \in \mathcal{M}\,, \quad m \neq j$$

and one of the following relations holds:

$$u_{\{j\}} < w_{\{j\}} \leqq v_{\{j\}} < z_{\{j\}}\,, \tag{1.2}$$

$$w_{\{j\}} < u_{\{j\}} \leqq z_{\{j\}} < v_{\{j\}}\,. \tag{1.3}$$

1.6. Lemma. *If $J, I, H \in \mathfrak{K}_0(R^{\mathcal{M}})$, $J = I \dotplus H$, $I = \langle u, v\rangle$, $u < v$, $H = \langle w, z\rangle$, $w < z$, then there exists such a $j \in \mathcal{M}$ that*

$$u_{\{m\}} = w_{\{m\}}\,,\quad v_{\{m\}} = z_{\{m\}} \quad \textit{for} \quad m \in \mathcal{M}\,,\quad m \neq j \tag{1.4}$$

and one of the following relations holds:

$$u_{\{j\}} < w_{\{j\}} = v_{\{j\}} < z_{\{j\}}\,, \tag{1.5}$$

$$w_{\{j\}} < u_{\{j\}} = z_{\{j\}} < v_{\{j\}}\,. \tag{1.6}$$

This follows by Lemma 1.5.

1.7. Let $M \in \mathfrak{K}_0(R^{\mathcal{M}})$. A map $\Xi : \mathfrak{K}(M) \to R$ is called *additive*, if $\Xi(L) = 0$ for any compact degenerate interval $L \subset M$ and if $\Xi(J) = \Xi(I) + \Xi(H)$, whenever $J, I, H \in \mathfrak{K}_0(M)$, $J = I \dotplus H$.

A map $\Theta : \mathfrak{K}_0(M) \to R$ is called *superadditive*, if the following condition is fulfilled: if $J, I_1, I_2, \ldots, I_k \in \mathfrak{K}_0(M)$, $I_j \subset J$, $\operatorname{Int} I_i \cap \operatorname{Int} I_j = \emptyset$ for $i, j = 1, 2, \ldots, k$, $i \neq j$, then $\Theta(J) \geqq \sum_{i=1}^{k} \Theta(I_i)$.

The set of such superadditive maps $\Theta : \mathfrak{K}_0(M) \to R$ that $\Theta(I) \geqq 0$ for $I \in \mathfrak{K}_0(M)$ will be denoted by $Y(M)$.

1.8. Let $M \in \mathfrak{K}_0(R^{\mathcal{M}})$, $\emptyset \neq \mathcal{L} \subset \mathcal{M}$, $\xi : M \to R$, $u \in M_{\mathcal{L}}$. If $\mathcal{M} - \mathcal{L} \neq \emptyset$, define $\xi^u : M_{\mathcal{M}-\mathcal{L}} \to R$ by $\xi^u(v) = \xi(u \times v)$. If $\mathcal{L} = \mathcal{M}$, let $\xi^u = \xi(u) \in R$; if $v \in M$, then $\xi^u(v_\emptyset)$ may stand for $\xi(u)$.

Let $w \in M$. $\xi^{w_{\mathcal{K}}}$ is defined for all $\mathcal{K}$, $\emptyset \neq \mathcal{K} \subset \mathcal{M}$. Define in addition $\xi^{w_\emptyset} = \xi$ (in spite of that $w_\emptyset$ is not defined).

1.9. Let $M \in \mathfrak{K}_0(R^{\mathcal{M}})$, $\xi : M \to R$, $j \in \mathcal{M}$, $u, v \in M_{\{j\}}$. Define $\Delta(u, v)\,\xi : M_{\mathcal{M}-\{j\}} \to R$ by $\Delta(u, v)\,\xi = \xi^v - \xi^u$. If $\emptyset \neq \mathcal{L} \subset \mathcal{M}$, $f, g \in M_{\mathcal{L}}$, define $\Delta(f, g)\,\xi = \prod_{l \in \mathcal{L}} \Delta(f_{\{l\}}, g_{\{l\}})\,\xi$; of course, the right hand side does not depend on the order in which the operators $\Delta(f_{\{l\}}, g_{\{l\}})$ are applied. Observe that $\Delta(f, g)\,\xi \in R$, if $\mathcal{L} = \mathcal{M}$.

Put $f_\emptyset \times g_{\mathcal{L}} = g$, $f_{\mathcal{L}} \times g_\emptyset = f$; it can be shown that

$$\Delta(f, g)\,\xi = \sum_{\mathcal{J} \subset \mathcal{L}} (-1)^{|\mathcal{J}|}\, \xi^{f_{\mathcal{J}} \times g_{\mathcal{L}-\mathcal{J}}}\,. \tag{1.7}$$

Especially, if $\mathscr{L} = \mathscr{M}$, then

$$(1.8) \qquad \Delta(f, g)\,\xi = \sum_{\mathscr{J} \subset \mathscr{M}} (-1)^{|\mathscr{J}|}\, \xi(f_{\mathscr{J}} \times g_{\mathscr{M}-\mathscr{J}})\,.$$

If $J \in \mathfrak{K}(M)$, $J = \langle f, g\rangle$, $f, g \in M$, $f \leqq g$, define $\Delta(J)\,\xi = \Delta(f, g)\,\xi$. $\Delta(J)\,\xi = 0$, if $J \subset M$ is a degenerate interval (in $R^{\mathscr{M}}$). Moreover,

$$(1.9) \qquad \Delta(\langle t, y\rangle)\,\xi = \Delta(t, y)\,\xi \operatorname{sgn}(y - t) \quad \text{for} \quad y, t \in M\,.$$

1.10. Lemma. *Let $M \in \mathfrak{K}_0(R^{\mathscr{M}})$, $\xi : M \to R$. Define $\Xi : \mathfrak{K}(M) \to R$ by $\Xi(J) = \Delta(J)\,\xi$. Then Ξ is additive.*

This follows from the definition of $\Delta(J)\,\xi$ and from Lemma 1.6.

1.11. Lemma. *Let $M \in \mathfrak{K}_0(R^{\mathscr{M}})$, $\Xi : \mathfrak{K}(M) \to R$, $q \in M$. Define $\xi : M \to R$ by $\xi(x) = \Xi(\langle q, x\rangle) \operatorname{sgn}(x - q)$. If Ξ is additive, then*

$$(1.10) \qquad \Delta(J)\,\xi = \Xi(J) \quad \textit{for} \quad J \in \mathfrak{K}(M)\,.$$

Proof. Lemma 1.11 can be verified easily, if $|\mathscr{M}| = 1$. Let l be a positive integer. Assume that Lemma 1.11 is valid, if $|\mathscr{M}| \leqq l$ and let $|\mathscr{M}| = l + 1$. Choose $m \in \mathscr{M}$ and put $\mathscr{L} = \mathscr{M} - \{m\}$. Fix $u \in M_{\{m\}}$ and define $\Theta : \mathfrak{K}(M_{\mathscr{L}}) \to R$ by $\Theta(I) = \Xi(I \times \langle q_{\{m\}}, u\rangle)$. Θ is additive (cf. Lemma 1.6).

Define $\vartheta : M_{\mathscr{L}} \to R$ by $\vartheta(v) = \Theta(\langle q_{\mathscr{L}}, v\rangle) \operatorname{sgn}(v - q_{\mathscr{L}}) = \Xi(\langle q, v \times u\rangle) \operatorname{sgn} . (v - q_{\mathscr{L}}) = \xi^u(v) \operatorname{sgn}(u - q_{\{m\}})$. As $|\mathscr{L}| = l$, by assumption $\Delta(I)\,\vartheta = \Theta(I) = \Xi(I \times \langle q_{\{m\}}, u\rangle)$ for $I \in \mathfrak{K}(K_{\mathscr{L}})$. $\Delta(I \times \langle q_{\{m\}}, u\rangle)\,\xi = \operatorname{sgn}(u - q_{\{m\}}) [\Delta(I)\,\xi^u - \Delta(I)\,\xi^{q_{\{m\}}}]$. $\xi^{q_{\{m\}}}(w) = 0$ for $w \in K_{\mathscr{L}}$ so that $\Delta(I)\,\xi^{q_{\{m\}}} = 0$ and

$$(1.11) \quad \Delta(I \times \langle q_{\{m\}}, u\rangle)\,\xi = \operatorname{sgn}(u - q_{\{m\}})\,\Delta(I)\,\xi^u = \Delta(I)\,\vartheta = \Xi(I \times \langle q_{\{m\}}, u\rangle)\,.$$

(1.11) holds for $I \in \mathfrak{K}(K_{\mathscr{L}})$, $u \in K_{\{m\}}$ and that (1.10) holds also.

1.12. Lemma. *Let $M \in \mathfrak{K}_0(R^{\mathscr{M}})$, $\xi : M \to R$, $q, x, y \in M$. Then*

$$(1.12) \qquad \Delta(q, y)\,\xi = \sum_{\mathscr{L} \subset \mathscr{M}} \Delta(q_{\mathscr{M}-\mathscr{L}} \times x_{\mathscr{L}},\, x_{\mathscr{M}-\mathscr{L}} \times y_{\mathscr{L}})\,\xi\,.$$

Proof. By (1.8) the right hand side of (1.12) is equal to

$$\sum_{\mathscr{L} \subset \mathscr{M}} \sum_{\mathscr{H} \subset \mathscr{M}} (-1)^{|\mathscr{H}|}\, \xi(q_{(\mathscr{M}-\mathscr{L}) \cap \mathscr{H}} \times x_{\mathscr{L} \cap \mathscr{H} \cup (\mathscr{M}-\mathscr{L}) \cap (\mathscr{M}-\mathscr{H})} \times y_{\mathscr{L} \cup (\mathscr{M}-\mathscr{H})})\,.$$

Let $\mathscr{J}, \mathscr{K} \subset \mathscr{M}$, $\mathscr{J} \cap \mathscr{K} = \emptyset$. The relations $(\mathscr{M} - \mathscr{L}) \cap \mathscr{H} = \mathscr{J}$, $\mathscr{L} \cap (\mathscr{M} - \mathscr{H}) = \mathscr{K}$ are fulfilled iff $\mathscr{H} = \mathscr{J} \cup \mathscr{I}$, $\mathscr{L} = \mathscr{K} \cup \mathscr{I}$ for some $\mathscr{I} \subset \mathscr{M} - \mathscr{J} -$

$-\mathscr{K}$. Thus the right hand side of (1.12) is equal to

$$\sum_{\mathscr{J},\mathscr{K}\subset\mathscr{M},\mathscr{J}\cap\mathscr{K}=\emptyset}\ \sum_{\mathscr{I}\subset\mathscr{M}-\mathscr{J}-\mathscr{K}} (-1)^{|\mathscr{J}|+|\mathscr{I}|}\,\xi(q_{\mathscr{J}}\times x_{\mathscr{M}-\mathscr{J}-\mathscr{K}}\times y_{\mathscr{K}})\,.$$

As $\sum_{\mathscr{I}\subset\mathscr{M}-\mathscr{J}-\mathscr{K}} (-1)^{|\mathscr{I}|} = 0$ if $\mathscr{M}-\mathscr{J}-\mathscr{K}\neq\emptyset$, the above expression is equal to $\sum_{\mathscr{J}\subset\mathscr{M}} (-1)^{|\mathscr{J}|}\,\xi(q_{\mathscr{J}}\times y_{\mathscr{M}-\mathscr{J}})$. (1.12) is fulfilled by (1.8).

1.13. Lemma. *Let $M\in\mathfrak{R}_0(R^{\mathscr{M}})$, $q\in M$ and let $\Xi:\mathfrak{R}(M)\to R$ be additive. Define $\xi: M\to R$ by $\xi(w)=\Xi(\langle q,w\rangle)\operatorname{sgn}(w-q)$. Then*

$$(1.13)\qquad \xi(y)-\xi(x)=\sum_{\emptyset\neq\mathscr{L}\subset\mathscr{M}}\Xi(\langle q_{\mathscr{M}-\mathscr{L}},x_{\mathscr{M}-\mathscr{L}}\rangle\times$$
$$\times\langle x_{\mathscr{L}},y_{\mathscr{L}}\rangle)\operatorname{sgn}(x_{\mathscr{M}-\mathscr{L}}-q_{\mathscr{M}-\mathscr{L}})\operatorname{sgn}(y_{\mathscr{L}}-x_{\mathscr{L}})\,.$$

Proof. By the definition of ξ it follows that $\xi(q_{\mathscr{J}}\times w_{\mathscr{M}-\mathscr{J}})=0$ if $\emptyset\neq\mathscr{J}\subset\mathscr{M}$. By (1.8) $\xi(y)=\Delta(q,y)\,\xi$, $\xi(x)=\Delta(q,x)\,\xi$, so that by (1.12) $\xi(y)-\xi(x)=\sum_{\emptyset\neq\mathscr{L}\subset\mathscr{M}}\Delta(q_{\mathscr{M}-\mathscr{L}}\times x_{\mathscr{L}},x_{\mathscr{M}-\mathscr{L}}\times y_{\mathscr{L}})$. By the definition of $\langle a,b\rangle$ in 1.2, by the definition of $\Delta(f,g)\,\xi$ and $\Delta(\langle f,g\rangle)\,\xi$ in 1.9 and by (1.10) it follows that

$$\Delta(q_{\mathscr{M}-\mathscr{L}}\times x_{\mathscr{L}},x_{\mathscr{M}-\mathscr{L}}\times y_{\mathscr{L}})\,\xi=$$
$$=(\langle q_{\mathscr{M}-\mathscr{L}}\times x_{\mathscr{L}},x_{\mathscr{M}-\mathscr{L}}\times y_{\mathscr{L}}\rangle)\,\xi\operatorname{sgn}(x_{\mathscr{M}-\mathscr{L}}-q_{\mathscr{M}-\mathscr{L}})\operatorname{sgn}(y_{\mathscr{L}}-x_{\mathscr{L}})=$$
$$=\Xi(\langle q_{\mathscr{M}-\mathscr{L}}\times x_{\mathscr{L}},x_{\mathscr{M}-\mathscr{L}}\times y_{\mathscr{L}}\rangle)\operatorname{sgn}(x_{\mathscr{M}-\mathscr{L}}-q_{\mathscr{M}-\mathscr{L}})\operatorname{sgn}(y_{\mathscr{L}}-x_{\mathscr{L}})\,.$$

Finally $\langle q_{\mathscr{M}-\mathscr{L}}\times x_{\mathscr{L}},x_{\mathscr{M}-\mathscr{L}}\times y_{\mathscr{L}}\rangle=\langle q_{\mathscr{M}-\mathscr{L}},x_{\mathscr{M}-\mathscr{L}}\rangle\times\langle x_{\mathscr{L}},y_{\mathscr{L}}\rangle$ (cf. 1.2), which makes the proof of (1.13) complete.

1.14. Definition. Let $M\in\mathfrak{R}_0(R^{\mathscr{M}})$, $\varphi: M\to R$. For $L\in\mathfrak{R}_0(M)$ put $\operatorname{Var}(\varphi,L)=\sup\sum_{i=1}^{k}|\Delta(J_i)\,\varphi|$, sup being taken over such finite systems $\{J_i\in\mathfrak{R}_0(L)\mid i=1,2,\ldots,k\}$ that $\operatorname{Int}J_i\cap\operatorname{Int}J_j=\emptyset$ for $i\neq j$. φ is said to be of *bounded variation*, if $\operatorname{Var}(\varphi,M)<\infty$.

The set of functions $\varphi: M\to R$ of bounded variation is denoted by $\mathrm{BV}(M)$.

$\varphi: M\to R$ is said to be of *strongly bounded variation*, if $\varphi\in BV(M)$ and $\varphi^u\in\mathrm{BV}(M_{\mathscr{L}})$ for every $\mathscr{L}\subset\mathscr{M}$, $\emptyset\neq\mathscr{L}\neq\mathscr{M}$ and every $u\in M_{\mathscr{M}-\mathscr{L}}$. The set of functions $\varphi: M\to R$ of strongly bounded variation will be denoted by $\mathrm{SBV}(M)$.

1.15. Let $M\in\mathfrak{R}_0(R^{\mathscr{M}})$, $\varphi\in\mathrm{SBV}(M)$ and let $\xi: M\to R$ be a bounded Baire function. For $\emptyset\neq\mathscr{L}\subset\mathscr{M}$, $v\in M_{\mathscr{M}-\mathscr{L}}$, $J\in\mathfrak{R}_0(M)$ write $\int_{J_{\mathscr{L}}}\xi^v\,\mathrm{d}\varphi^v$ for (V) – $\int_{J_{\mathscr{L}}}\xi^v\,\mathrm{d}\varphi^v$ (see the definition in 2.3) or, equivalently, for the Ward integral. It is well known, that the dominated convergence theorem holds for this type of integral (cf. [1]) and that $\int_J \xi^v \, d\varphi^v$ exists, if ξ is continuous. Thus $\int_{J_{\mathscr{L}}} \xi^v \, d\varphi^v$ exists, if ξ is a bounded Baire function.

It is easy to deduce from Definition 2.1 that

$$(1.14) \qquad \left| \int_{J_{\mathscr{L}}} \xi^v \, d\varphi^v \right| \leqq \sup_{u \in J_{\mathscr{L}}} |\xi^v(u)| \,. \quad \mathrm{Var}\,(\varphi^v, J_{\mathscr{L}}) \,.$$

Moreover, if $H, I, L \in \mathfrak{K}_0(M_{\mathscr{L}})$, $H \dotplus I = L$, then

$$(1.15) \qquad \int_H \xi^v \, d\varphi^v + \int_I \xi^v \, d\varphi^v = \int_L \xi^v \, d\varphi^v \,.$$

This property is called the additivity of the integral; it need not be fulfilled for the Lebesgue-Stieltjes integral, if the latter is defined by means of the measure, which is obtained in the usual manner from the function $\Phi : \mathfrak{K}_0(M_{\mathscr{L}}) \to R$ defined by $\Phi(L) = \Delta(L)\, \varphi^v$.

Also, Fubini Theorem may be applied to $\int_{J_{\mathscr{L}}} \xi^v \, d\varphi^v$ (see [3] or [1]). $\int_{J_{\mathscr{L}}} \xi \, d\varphi$ is defined to be the map from $M_{\mathscr{M}-\mathscr{L}}$ to R which maps v on $\int_{J_{\mathscr{L}}} (\xi^v) \, d(\varphi^v)$.

Let $L \in \mathfrak{K}_0(M)$. Define for $\mathscr{L} \subset \mathscr{M}$, $\emptyset \neq \mathscr{L} \neq \mathscr{M}$

$$(1.16) \qquad S(\xi, \varphi, L, \mathscr{L}) = (-1)^{|\mathscr{L}|} \, \Delta(L_{\mathscr{M}-\mathscr{L}}) \int_{L_{\mathscr{L}}} \xi \, d\varphi$$

and define

$$(1.17) \qquad S(\xi, \varphi, L, \emptyset) \;\; = \Delta(L) \, (\xi\varphi) \,,$$

$$(1.18) \qquad S(\xi, \varphi, L, \mathscr{M}) = (-1)^{|\mathscr{M}|} \int_L \xi \, d\varphi \,.$$

The right hand side of (1.16) will be interpreted in such a way that (1.16) holds for all $\mathscr{L} \subset \mathscr{M}$. Finally define

$$(1.19) \qquad T(\xi, \varphi, L) = \sum_{\mathscr{L} \subset \mathscr{M}} S(\xi, \varphi, L, \mathscr{L}) \,.$$

1.16. A remark on the symbols. Reals are denoted by small Greek letters from the beginning of the alphabet, $\alpha, \beta, \gamma, \delta, \varepsilon, \varkappa$. Finite sets are denoted by $\mathscr{H}, \mathscr{I}, \mathscr{J}, \mathscr{K}, \mathscr{L}, \mathscr{M}, \mathscr{N}$, intervals by H, I, J, K, L, M. Elements of $R^{\mathscr{N}}, R^{\mathscr{M}}, \ldots$ are denoted by small Roman letters. Functions from intervals to R are denoted by small Greek letters $\lambda, \mu, \nu, \vartheta, \varphi, \psi, \xi, \zeta, \eta, \omega$. Interval functions are denoted by capital Greek letters $\Gamma, \Theta, \Lambda, \Xi, \Phi, \Psi, \Omega$. By Q, S, T, U, V, W, Z there are denoted functions with various domains of definition. $\mathscr{U}, \mathscr{V}, \mathscr{W}$ are subsets of $R^{\mathscr{N}}, R^{\mathscr{M}}, \ldots$ There are some especial symbols like Δ (the difference operator, cf. 1.9), $Y(M)$ (cf. 1.7) $\mathfrak{K}(M)$, $\mathfrak{K}_0(M)$ (cf. 1.4), $\mathscr{B}(a, \alpha)$ (cf. 1.3)).

2. RESULTS

2.1. Definition. Let $\mathscr{M}$ be a nonempty finite set, $M \in \mathfrak{K}_0(R^{\mathscr{M}})$, $U : \mathfrak{K}_0(M) \times M \to R$. U is called *variationally integrable*, if there exists such an additive function $\Xi : \mathfrak{K}(M) \to R$, that to every $\varepsilon \in R^+$ there exist such $\omega : M \to R^+$ and $\Omega \in Y(M)$ that $\Omega(M) \leqq \varepsilon$ and

$$|U(J, t) - \Xi(J)| \leqq \Omega(J) \quad \text{for} \quad t \in J \in \mathfrak{K}_0(M)\,, \quad J \subset \mathscr{B}(t, \omega(t))\,.$$

Ξ is called *the variational integral* of U and $\Xi(J)$ is sometimes denoted by $(\mathrm{V}) - \int_J U$.

2.2. It is easy to show (by means of Lemma 1.1 from [3]) that there exists at most one Ξ fulfilling the conditions of Definition 2.1. There are two definitions of the integral, which are equivalent to Definition 2.1. One of them is based on major and minor functions and is a generalization of a well known definition by Perron, in the other one there is made use of the integral sums and this definition is a direct generalization of the definition of the integral by Riemann. For the above definitions of the integral and their equivalence see [1] and also [3], Note 1.3. The generalization of the Riemann integral was introduced in [4], the variational integral was introduced in [2].

2.3. Let $\lambda : R^{\mathscr{M}} \to R$ be defined by $\lambda(y) = \prod_{m \in \mathscr{M}} y(m)$. Let $M \in \mathfrak{K}_0(R^{\mathscr{M}})$. $\zeta : M \to R$ is called *variationally integrable*, if $U : \mathfrak{K}_0(M) \times M \to R$ defined by $U(J, t) = \zeta(t)\, \Delta(J)\, \lambda$ is variationally integrable (observe that $\Delta(J)\, \lambda$ is the Euclidean volume of J). Write $(\mathrm{V}) - \int_J \zeta \, \mathrm{d}\lambda$ instead of $(\mathrm{V}) - \int_J U$ in such a case. If ζ is Perron-integrable, let $(\mathrm{P}) - \int_J \zeta \, \mathrm{d}\lambda$ be the Perron integral of ζ over J. It follows from the equivalence of the definitions of the integral mentioned in 2.2 that ζ is variationally integrable iff it is Perron-integrable and that $(\mathrm{V}) - \int_J \zeta \, \mathrm{d}\lambda = (\mathrm{P}) - \int_J \zeta \, \mathrm{d}\lambda$ in such a case.

More generally, let $\varrho : M \to R$. $\zeta : M \to R$ called variationally ϱ-integrable, if $U : \mathfrak{K}_0(M) \times M \to R$ defined by $U(J, t) = \zeta(t)\, \Delta(J)\, \varrho$ is variationally integrable. Write $(\mathrm{V}) - \int_J \zeta \, \mathrm{d}\varrho$ instead of $(\mathrm{V}) - \int_J U$ in such a case. Again, there is an equivalent definition of the integral by means of major and minor functions and the integral is usually called Ward integral.

2.4. The main result of this paper are Theorems 2.6 and 2.7. The following notations and assumptions are common for both of them.

(2.1) $\mathscr{N}$ is a nonempty finite set, $c, d \in R^{\mathscr{N}}$, $c < d$, $K = \langle c, d \rangle$, $q \in K$, $U : \mathfrak{K}_0(K) \times K \to R$ is variationally integrable, $\Xi : \mathfrak{K}(K) \to R$ being its variational integral, $\xi : K \to R$ is defined by $\xi(x) = \Xi(\langle q, x \rangle) \operatorname{sgn}(x - q)$.

(2.2) there exist such $\eta : K \to R^+$ and $\Theta \in Y(K)$ that $|U(J, t)| \leqq \eta(t)\, \Theta(J)$ for $t \in J \in \mathfrak{K}_0(K)$.

(2.3) $\varphi \in \mathrm{SBV}(K)$, $W : \mathfrak{K}(K) \times K \to R$ is defined by $W(J, t) = U(J, t)\, \varphi(t)$.

2.5. It will be proved later (cf. Lemmas 3.1 and 3.7) that ξ is a bounded function of the first Baire class, if (2.1) and (2.2) hold. Thus $T(\xi, \varphi, J)$ (cf. (1.19)) is well defined for $J \in \mathfrak{K}_0(K)$ provided that (2.1)–(2.3) hold.

Define $\Gamma_T : \mathfrak{K}(K) \to R$ by

$$\Gamma_T(J) = T(\xi, \varphi, J)\,, \quad \text{if} \quad J \in \mathfrak{K}_0(K)\,, \tag{2.4}$$

$\Gamma_T(J) = 0$ if J is a compact degenerate interval.

2.6. Theorem. *Let* (2.1)–(2.3) *be fulfilled. Let* φ *be continuous. Then* W *is variationally integrable.*

Moreover Γ_T, *is the variational integral of* W.

2.7. Theorem. *Let* (2.1)–(2.3) *be fulfilled. Let there exist continuous increasing functions* $\vartheta_i : K_{\{i\}} \to R$ *for* $i \in \mathcal{N}$ *that*

$$\Theta(J) = \prod_{i \in \mathcal{N}} \Delta(J_{\{i\}})\, \vartheta_i \quad \text{for} \quad J \in \mathfrak{K}_0(K)\,. \tag{2.5}$$

Then W *is variationally integrable. Moreover,* Γ_T *is the variational integral of* W.

2.8. Note. In conditions of Theorem 2.7 ξ is continuous (cf. 3.5).

2.9. To prove Theorems 2.6 and 2.7 it is to be verified that Γ_T is the variational integral of W. The continuity properties of ξ (which guarantee that Γ_T is well defined) are examined in § 3. In § 4 there are established some properties of functions from $\mathrm{SBV}(K)$, which are needed later. The proofs of Theorems 2.6 and 2.7 have a large part in common; this common part is contained in § 5. The proof of Theorem 2.6 is finished in § 6, the proof of Theorem 2.7 is finished in § 7. In § 8 there is an example to demonstrate, that the assumptions on φ cannot be replaced by certain weaker assumptions.

2.10. Theorem. *Let* $\zeta : K \to R$ *be Perron integrable,* $\varphi \in \mathrm{SBV}(K)$. *Then the product* $\zeta\varphi : K \to R$ *is Perron integrable.*

Moreover, define $\Xi : \mathfrak{K}(K) \to R$ *by* $\Xi(J) = (\mathrm{P}) - \int_J \zeta \,\mathrm{d}\lambda$ *if* $J \in \mathfrak{K}_0(K)$ (*cf.* 2.3), $\Xi(J) = 0$ *if* $J \in \mathfrak{K}(K)$ *is degenerate and define* ξ *by* (2.1). *Then* ξ *is continuous and*

$$(\mathrm{P}) - \int_J \zeta\varphi \,\mathrm{d}\lambda = T(\xi, \varphi, J) \quad \text{for} \quad J \in \mathfrak{K}_0(K)\,. \tag{2.6}$$

For the proof put $U(J, t) = \zeta(t)\,\Delta(J)\,\lambda$, $\eta(t) = |\zeta(t)|$, $\vartheta_i(u) = u(i)$ for $u \in K_{\{i\}} \subset R^{\{i\}}$, $i \in \mathcal{N}$. U is variationally integrable by 2.3 and Ξ is the variational integral of U by 2.1. Define W by (2.3). (2.1)–(2.3) and (2.5) are fulfilled, so that Theorem 2.7 may be applied. It follows that W is variationally integrable and, by 2.3, $\zeta\varphi$ is Perron integrable. Moreover, Γ_T is the variational integral of W and by (2.4) and 2.3

$$T(\xi, \varphi, J) = \Gamma_T(J) = (\mathrm{V}) - \int_J W = (\mathrm{V}) - \int_J \zeta\varphi\,\mathrm{d}\lambda = (\mathrm{P}) - \int_J \zeta\varphi\,\mathrm{d}\lambda\,.$$

ξ is continuous by Note 2.8.

The proof of Theorem 2.10 is complete.

2.11. For illustration, let (2.6) be written in detail for $\mathcal{N} = \{1, 2\}$ with somewhat changed notations. Let $J = \langle a, b\rangle$, $a, b \in R^{\{1,2\}}$, $a < b$ and write $a_1, a_2, \ldots$ instead of $a_{\{1\}}, a_{\{2\}}, \ldots$, $\langle a_1, b_1\rangle$ instead of $J_{\{1\}}$, $\xi(b_1, \cdot)$ instead of $\xi^{b_{\{1\}}}$, etc. (2.6) reads in this case

$$(\mathrm{P})\int_J \zeta\varphi\,\mathrm{d}\lambda = \int_J \xi\,\mathrm{d}\varphi - \int_{\langle a_1,b_1\rangle} \xi(\cdot, b_2)\,\mathrm{d}\varphi(\cdot, b_2) +$$
$$+ \int_{\langle a_1,b_1\rangle} \xi(\cdot, a_2)\,\mathrm{d}\varphi(\cdot, a_2) - \int_{\langle a_2,b_2\rangle} \xi(b_1, \cdot)\,\mathrm{d}\varphi(b_1, \cdot) +$$
$$+ \int_{\langle a_2,b_2\rangle} \xi(a_1, \cdot)\,\mathrm{d}\varphi(a_1, \cdot) + \xi(b_1, b_2)\,\varphi(b_1, b_2) -$$
$$- \xi(b_1, a_2)\,\varphi(b_1, a_2) - \xi(a_1, b_2)\,\varphi(a_1, b_2) + \xi(a_1, a_2)\,\varphi(a_1, a_2)\,.$$

(2.6) is a formula for integration by parts.

2.12. Let $\zeta : R^{\mathcal{N}} \to R$ be locally Perron integrable, let $\varphi : R^{\mathcal{N}} \to R$ be $|\mathcal{N}|$-times continuously differentiable and have a compact support. Let ψ be the mixed derivative of φ of order $|\mathcal{N}|$, i.e. if $\mathcal{N} = \{1, 2, \ldots, n\}$, then $\psi = (\partial^n/\partial x_{\{1\}} \ldots \partial x_{\{n\}})\,\varphi$. Choose $q \in R^{\mathcal{N}}$ and define $\xi : R^{\mathcal{N}} \to R$ by $\xi(x) = (\mathrm{P}) - \int_{\langle q,x\rangle} \zeta\,\mathrm{d}\lambda\,.\,\mathrm{sgn}\,(x - q)$. Let $K \in \mathfrak{K}_0(R^{\mathcal{N}})$. Then, by Theorem 2.10, $(\mathrm{P}) - \int_K \zeta\varphi\,\mathrm{d}\lambda$ exists. If $q \in K \in \mathfrak{K}_0(R^{\mathcal{N}})$ and if K contains the support of φ, then by (2.6)

$$(\mathrm{P}) - \int_K \zeta\varphi\,\mathrm{d}\lambda = (-1)^n \int_K \xi\,\mathrm{d}\varphi = (-1)^n \int_K \xi\psi\,\mathrm{d}\lambda\,.$$

Thus ζ may be identified with a distribution.

2.13. Let $\varrho \in \mathrm{BV}(K)$ and let $\zeta : K \to R$ be variationally ϱ-integrable. It can be deduced from Theorem 2.6 in an analogous manner as in 2.10 that $\zeta\varphi$ is variationally ϱ-integrable, whenever $\varphi \in \mathrm{SBV}(K)$ is continuous.

Let there exist continuous functions $\varrho_i \in \mathrm{BV}(K_{\{i\}})$ for $i \in \mathcal{N}$ and let $\varrho(x) = \prod_{i \in \mathcal{N}} \varrho_i(x_{\{i\}})$. It can be deduced from Theorem 2.7 that $\zeta\varphi$ is variationally ϱ-integrable for $\varphi \in \mathrm{SBV}(K)$.

In both cases $(\mathrm{V}) - \int_J \zeta\varphi \,\mathrm{d}\varrho = T(\zeta, \varphi, J)$.

3. CONTINUITY PROPERTIES OF ξ

In this section it is assumed that (2.1) and (2.2) hold.

3.1. Lemma. *ξ is bounded.*

Proof. Choose $\varepsilon \in R^+$ and find $\omega : K \to R^+$ and $\Omega \in Y(K)$ by Definition 2.1. By (1.13)

$$(3.1) \qquad |\zeta(y) - \xi(x)| \leqq \sum_{\emptyset \neq \mathcal{L} \subset \mathcal{N}} |\Xi(\langle q_{\mathcal{N}-\mathcal{L}}, x_{\mathcal{N}-\mathcal{L}}\rangle \times \langle x_{\mathcal{L}}, y_{\mathcal{L}}\rangle)| \quad \text{for} \quad x, y \in K\,.$$

Keep x fixed. In the sum on the right hand side there can be taken into account only such $\mathcal{L}$'s that either $\mathcal{L} = \mathcal{N}$ or $\langle q_{\mathcal{N}-\mathcal{L}}, x_{\mathcal{N}-\mathcal{L}}\rangle$ is nondegenerate (in $R^{\mathcal{N}-\mathcal{L}}$). By Lemma 1,1 from [3] there exists such a finite set $\{(J_i, t_i) \mid i = 1, 2, \ldots, k\}$ that $J_i \in \mathfrak{K}_0(\langle q_{\mathcal{N}-\mathcal{L}}, x_{\mathcal{N}-\mathcal{L}}\rangle)$, $t_i \in J_i \subset \mathcal{B}(t_i, \omega(x_{\mathcal{L}} \times t_i))$, $\mathrm{Int}\, J_i \cap \mathrm{Int}\, J_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{k} J_i = \langle q_{\mathcal{N}-\mathcal{L}}, x_{\mathcal{N}-\mathcal{L}}\rangle$. If $\|x_{\mathcal{L}} - y_{\mathcal{L}}\| \leqq \min_{i=1,2,\ldots,k} \omega(x_{\mathcal{L}} \times t_i)$, then by Definition 2.1 (cf. 1.3), then

$$|U(J_i \times \langle x_{\mathcal{L}}, y_{\mathcal{L}}\rangle, t_i \times x_{\mathcal{L}}) - \Xi(J_i \times \langle x_{\mathcal{L}}, y_{\mathcal{L}}\rangle)| \leqq \Omega(J_i \times \langle x_{\mathcal{L}}, y_{\mathcal{L}}\rangle)|\,,$$
$$i = 1, 2, \ldots, k\,.$$

By the additivity of Ξ, the superadditivity of Ω, by $\Omega(K) \leqq \varepsilon$ and (2.2)

$$(3.2) \qquad |\Xi(\langle q_{\mathcal{N}-\mathcal{L}}, x_{\mathcal{N}-\mathcal{L}}\rangle \times \langle x_{\mathcal{L}}, y_{\mathcal{L}}\rangle)| \leqq \sum_{i=1}^{k} |U(J_i \times \langle x_{\mathcal{L}}, y_{\mathcal{L}}\rangle, t_i \times x_{\mathcal{L}})| + $$
$$+ \Omega(J_i \times \langle x_{\mathcal{L}}, y_{\mathcal{L}}\rangle) \leqq \sum_{i=1}^{k} \eta(t_i \times x_{\mathcal{L}})\, \Theta(J_i \times \langle x_{\mathcal{L}}, y_{\mathcal{L}}\rangle) + \varepsilon\,.$$

For $\mathcal{L} = \mathcal{N}$, $\|y - x\| \leqq \omega(x)$

$$|U(\langle x, y\rangle, x) - \Xi(\langle x, y\rangle)| \leqq \Omega(\langle x, y\rangle)$$

and

$$(3.3) \qquad |\Xi(\langle x, y\rangle)| \leqq \eta(x)\, \Theta(\langle x, y\rangle) + \varepsilon\,.$$

It follows by (3.1), (3.2), (3.3) that to every $x \in K$ there exists such a neighbourhood $\mathcal{U}(x)$ in K that ξ is bounded on $\mathcal{U}(x)$. As K is compact, ξ is bounded.

3.2. Let $\mathscr{M}$ be a nonempty finite set, $M \in \mathfrak{K}_0(R^{\mathscr{M}})$, $\Theta \in Y(M)$, $\varepsilon \in R^+$. Denote by $\mathscr{S}(\Theta, \mathscr{M}, \varepsilon)$ the set of such $x \in M$ that for every $\delta \in R^+$ there exists such a $J \in \mathfrak{K}_0(M)$ that $x \in J \subset \mathscr{B}(x, \delta)$ and $\Theta(J) \geqq \varepsilon$. Obviously, $\mathscr{S}(\Theta, \mathscr{M}, \varepsilon)$ is finite (possibly empty). Define $\mathscr{S}(\Theta, \mathscr{M}) = \bigcup_{k=1}^{\infty} \mathscr{S}(\Theta, \mathscr{M}, k^{-1})$. $\mathscr{S}(\Theta, \mathscr{M})$ is countable (i.e. empty, finite or infinite countable).

$M - \mathscr{S}(\Theta, \mathscr{M})$ is the set of such $x \in M$ that if $x \in J_k \in \mathfrak{K}_0(M)$ for $k = 1, 2, \ldots$ and $\operatorname{diam} J_k \to 0$ for $k \to \infty$, then $\Theta(J_k) \to 0$ for $k \to \infty$.

For $\mathscr{J} \subset \mathscr{M}, \emptyset \neq \mathscr{J} \neq \mathscr{M}$ define $\Theta[\mathscr{J}] : \mathfrak{K}(M_{\mathscr{J}}) \to R$ by $\Theta[\mathscr{J}](I) = \Theta(M_{\mathscr{M}-\mathscr{J}} \times I)$. Obviously $\Theta[\mathscr{J}] \in Y(M_{\mathscr{J}})$.

Define $\mathscr{S}(\Theta, \mathscr{J}) = \mathscr{S}(\Theta[\mathscr{J}], \mathscr{J})$. $\mathscr{S}(\Theta, \mathscr{J}) \subset M_{\mathscr{J}}$ is countable. $M_{\mathscr{J}} - \mathscr{S}(\Theta, \mathscr{J})$ is the set of such $u \in M_{\mathscr{J}}$ that if $u \in L_k \in \mathfrak{K}_0(M_{\mathscr{J}})$ for $k = 1, 2, \ldots$, $\operatorname{diam} L_k \to 0$ for $k \to \infty$ and $I \in \mathfrak{K}_0(M_{\mathscr{M}-\mathscr{J}})$, then $\Theta(I \times L_k) \to 0$ for $k \to \infty$.

3.3. Lemma. *If $\Theta \in Y(M)$, $m \in \mathscr{J} \subset \mathscr{M}$, $u \in \mathscr{S}(\Theta, \mathscr{J})$, then $u_{\{m\}} \in \mathscr{S}(\Theta, \{m\})$.*

Proof. Let $u \in M_{\mathscr{J}}$, $u_{\{m\}} \notin \mathscr{S}(\Theta, \{m\})$. Then to every $\varepsilon \in R^+$ there exists such $\delta \in R^+$ that if $u_{\{m\}} \in H \in \mathfrak{K}_0(M_{\{m\}})$, $H \subset \mathscr{B}(u_{\{m\}}, \delta)$, then $\Theta(H \times M_{\mathscr{M}-\{m\}}) \leqq \varepsilon$. It follows that $u \in M_{\mathscr{J}} - \mathscr{S}(\Theta, \mathscr{J})$ and the proof of Lemma 3.3 is complete.

3.4. Lemma. (i) *Let $\mathscr{K} \subset \mathscr{N}$, $\emptyset \neq \mathscr{K} \neq \mathscr{N}$, $u \in K_{\mathscr{N}-\mathscr{K}}$. Let $v \in K_{\mathscr{K}}$, $v_{\{k\}} \notin \mathscr{S}(\Theta, \{k\})$ for $k \in \mathscr{K}$. Then ξ^u is continuous at v.*

(ii) *ξ is continuous at $x \in K$, if $x_{\{j\}} \notin \mathscr{S}(\Theta, \{j\})$ for $j \in \mathscr{N}$.*

Let there be proved (i). Put $x = u \times v$ and for $w \in K_{\mathscr{K}}$ put $y = u \times w$. If $\mathscr{L} \subset \mathscr{N}$, $\mathscr{L} \cap (\mathscr{N} - \mathscr{K}) \neq \emptyset$, then $\langle x_{\mathscr{L}}, y_{\mathscr{L}} \rangle$ is a degenerate interval; if $\mathscr{L} \subset \mathscr{K}$, then $\langle q_{\mathscr{N}-\mathscr{L}}, x_{\mathscr{N}-\mathscr{L}} \rangle \times \langle x_{\mathscr{L}}, y_{\mathscr{L}} \rangle = \langle q_{\mathscr{N}-\mathscr{L}}, u \times v_{\mathscr{K}-\mathscr{L}} \rangle \times \langle v_{\mathscr{L}}, w_{\mathscr{L}} \rangle$. By (3.1)

$$|\xi^u(w) - \xi^u(v)| \leqq \sum_{\emptyset \neq \mathscr{L} \subset \mathscr{K}} |\Xi(\langle q_{\mathscr{N}-\mathscr{L}}, u \times v_{\mathscr{K}-\mathscr{L}} \rangle \times \langle v_{\mathscr{L}}, w_{\mathscr{L}} \rangle)| . \tag{3.4}$$

Choose $\varepsilon \in R^+$ and find $\omega : K \to R^+$ and $\Omega \in Y(K)$ by Definition 2.1 (putting $\mathscr{M} = \mathscr{N}$ and $M = K$). Let $\emptyset \neq \mathscr{L} \subset \mathscr{K}$ and let the interval $\langle q_{\mathscr{N}-\mathscr{L}}, u \times v_{\mathscr{K}-\mathscr{L}} \rangle$ be nondegenerate. By Lemma 1.1 from [3] there exists such a finite set

$$\{(I_i, t_i) \in \mathfrak{K}_0(\langle q_{\mathscr{N}-\mathscr{L}}, u \times v_{\mathscr{K}-\mathscr{L}} \rangle) \times \langle q_{\mathscr{N}-\mathscr{L}}, u \times v_{\mathscr{K}-\mathscr{L}} \rangle \mid i = 1, 2, \ldots, l\}$$

that $t_i \in I_i \subset \mathscr{B}(t_i, \omega(t_i \times v_{\mathscr{L}}))$, $\operatorname{Int} I_i \cap \operatorname{Int} I_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, l$ and $\bigcup_{i=1}^{l} I_i = \langle q_{\mathscr{N}-\mathscr{L}}, u \times v_{\mathscr{K}-\mathscr{L}} \rangle$. Put $\delta(\mathscr{L}) = \min_{i=1,2,\ldots,l} \omega(t_i \times v_{\mathscr{L}})$. Let $\|w - v\| \leqq \delta(\mathscr{L})$ and let $\langle v_{\mathscr{L}}, w \rangle$ be a nondegenerate interval. Then

$$t_i \times v_{\mathscr{L}} \in \langle q_{\mathscr{L}}, u \times v_{\mathscr{K}-\mathscr{L}} \rangle \times \langle v_{\mathscr{L}}, w_{\mathscr{L}} \rangle \subset \mathscr{B}(t_i \times v_{\mathscr{L}}, \delta(\mathscr{L}))$$

and by Definition 2.1

$$\left|U(I_i \times \langle v_{\mathscr{L}}, w_{\mathscr{L}}\rangle, t_i \times v_{\mathscr{L}}) - \Xi(I_i \times \langle v_{\mathscr{L}}, w_{\mathscr{L}}\rangle)\right| \leqq \Omega(I_i \times \langle v_{\mathscr{L}}, w_{\mathscr{L}}\rangle)$$
$$\text{for} \quad i = 1, 2, \ldots, l\,.$$

It follows by the additivity of Ξ and the superadditivity of Ω and by (2.2) by $\Omega(K) \leqq \varepsilon$ that

$$(3.5) \qquad \left|\Xi(\langle q_{\mathscr{N}-\mathscr{L}}, u \times v_{\mathscr{K}-\mathscr{L}}\rangle \times \langle v_{\mathscr{L}}, w_{\mathscr{L}}\rangle\right| \leqq$$
$$\leqq \sum_{i=1}^{l} \eta(t_i \times v_{\mathscr{L}})\, \Theta(I_i \times \langle v_{\mathscr{L}}, w_{\mathscr{L}}\rangle) + \varepsilon\,.$$

By the assumption that $v_{\{k\}} \notin \mathscr{S}(\Theta, \{k\})$ for $k \in \mathscr{K}$ and by Lemma 3.3 it follows that $v_{\mathscr{L}} \notin \mathscr{S}(\Theta, \mathscr{L})$ for $\emptyset \neq \mathscr{L} \subset \mathscr{K}$ so that $\Theta(I_i \times \langle v_{\mathscr{L}}, w_{\mathscr{L}}\rangle)$ can be made arbitrarily small by making $\|w - v\|$ small. Thus the continuity of ξ^u follows from (3.4) and (3.5), as $\varepsilon \in R^+$ is arbitrary. The proof of (i) is complete. The proof of (ii) is analogous.

3.5. Let (2.5) hold. It is easy to prove that $\mathscr{S}(\Theta, \mathscr{J}) = \emptyset$ for $\emptyset \neq \mathscr{J} \subset \mathscr{N}$ (cf. Lemma 3.3). By Lemma 3.4 (ii) ξ is continuous.

3.6. Lemma. *Let $\chi : K \to R$ and let for $m \in \mathscr{N}$ there exist a countable set $\mathscr{T}(m) \subset \subset K_{\{m\}}$ that*

(i) *if $\mathscr{K} \subset \mathscr{N}$, $\emptyset \neq \mathscr{K} \neq \mathscr{N}$, $u \in K_{\mathscr{N}-\mathscr{K}}$, $v \in K_{\mathscr{K}}$ $v_{\{m\}} \notin \mathscr{T}(m)$ for $m \in \mathscr{K}$, then χ^u is continuous at v,*

(ii) *if $x \in K$, $x_{\{m\}} \notin \mathscr{T}(m)$ for $m \in \mathscr{N}$, then χ is continuous at x.*

Then χ is a function of the Baire class.

Proof. By (2.1) $K = \langle c, d\rangle$, $c, d \in R^{\mathscr{N}}$, $c < d$. The sets $\mathscr{T}(m)$ can be enlarged, so that without loss on generality it may be assumed in addition that $\mathscr{T}(m)$ is dense in $K_{\{m\}}$ and $c_{\{m\}}, d_{\{m\}} \in K_{\{m\}}$ for $m \in \mathscr{N}$. Find such sets $\mathscr{C}_{i,m} \subset \mathscr{T}(m)$ for $i = 1, 2, \ldots$, $m \in \mathscr{N}$ that the following conditions are fulfilled:

$$\mathscr{C}_{i,m} = \{w_{i,m,j} \mid j = 0, 1, 2, \ldots, i\}\,,$$
$$w_{i,m,0} = c_{\{m\}}\,, \quad w_{i,m,i} = d_{\{m\}}\,,$$
$$w_{i,m,j} < w_{i,m,j+1} \quad \text{for} \quad j = 0, 1, 2, \ldots, i-1\,,$$
$$\mathscr{C}_{i,m} \subset \mathscr{C}_{i+1,m}\,, \quad \bigcup_{i=1}^{\infty} \mathscr{C}_{i,m} = \mathscr{T}(m)\,.$$

Observe that these conditions imply that

$$(3.6) \qquad \text{if} \quad v \notin \mathscr{T}(m)\,, \quad w_{i,m,j(i)} < v < w_{i,m,j(i)+1}$$

for $i = 1, 2, \ldots$ then $\|w_{i,m,j(i)+1} - w_{i,m,j(i)}\| \to 0$ for $i \to \infty$.

Let $\mathscr{C}_i = \prod_{m \in \mathscr{N}} \mathscr{C}_{i,m}$ (cf. 1.2).

For $x \in K - \mathscr{C}_i$ define

$$\mathscr{J}(x, i) = \{m \in \mathscr{N} \mid x_{\{m\}} \in \mathscr{C}_{i,m}\} .$$

Obviously $\mathscr{N} - \mathscr{J}(x, i) \neq \emptyset$ and for $m \in \mathscr{N} - \mathscr{J}(x, i)$ there exists a unique $j \in \in \{0, 1, \ldots, i - 1\}$ that $w_{i,m,j} < x_{\{m\}} < w_{i,m,j+1}$. Put

$$\mathscr{V}(x, i, m) = \{w_{i,m,j}, w_{i,m,j+1}\} ,$$

$$\mathscr{V}(x, i) = \prod_{m \in \mathscr{N} - \mathscr{J}(x,i)} \mathscr{V}(x, i, m) .$$

Let χ_i be the restriction of χ to the finite set $\mathscr{C}_i$ and let $\mu_i : K \to R$ be a continuous extension of χ_i and let the following condition be fulfilled:

$$\min_{y \in \mathscr{V}(x,i)} \chi_i(x_{\mathscr{J}(x,i)} \times y) \leqq \mu_i(x) \leqq \max_{y \in \mathscr{V}(x,i)} \chi_i(x_{\mathscr{J}(x,i)} \times y) . \tag{37}$$

Such an extension can be constructed step by step. At first χ_i is extended to the set of such $x \in K$, that $|\mathscr{J}(x, i)| \geqq |\mathscr{N}| - 1$, then to the set of such $x \in K$ that $|\mathscr{J}(x, i)| \geqq \geqq |\mathscr{N}| - 2$ etc.

Proposition. $\mu_i(x) \to \chi(x)$ *for* $i \to \infty$, $x \in K$.

Let Proposition be proved in case that $|\mathscr{N}| = 1$ i.e. $\mathscr{N} = \{m\}$. If $x \in \mathscr{T}(m)$, then $\mu_i(x) = \chi(x)$ for all sufficiently large x: If $x \in K - \mathscr{T}(i)$, then χ is continuous at x. There exist integers $j(i)$ that

$$w_{i,m,j(i)} < x < w_{i,m,j(i)+1} \quad \text{for} \quad i = 1, 2, \ldots$$

By (3.6) $\|w_{i,m,j(i)+1} - w_{i,m,j(i)}\| \to 0$ and by the continuity of χ at x and by (3.7) $\mu_i(x) \to \chi(x)$ ($\mathscr{J}(x, i) = \emptyset$ as $|\mathscr{N}| = 1$). Thus Proposition holds, if $|\mathscr{N}| = 1$.

In the general case Proposition will be proved by induction. Assume that there is such a positive integer n that Proposition holds, whenever $|\mathscr{N}| \leqq n$. Let $|\mathscr{N}| = = n + 1$, $x \in K$. If possible, find such $m \in \mathscr{N}$ that $x_{\{m\}} \in \mathscr{T}(m)$ and put $u = x_{\{m\}}$. By the induction assumption $\mu_i^u(y) \to \chi^u(y)$ for $i \to \infty$, $y \in K_{\mathscr{N}-\{m\}}$, so that $\mu_i(x) \to \to \chi(x)$ for $i \to \infty$. If $x_{\{m\}} \notin \mathscr{T}(m)$ for $m \in \mathscr{N}$, then there are such integers $j(m, i)$ for $m \in \mathscr{N}$, $i = 1, 2, \ldots$ that

$$w_{i,m,j(m,i)} < x_{\{m\}} < w_{i,m,j(m,i)+1} \quad \text{for} \quad i = 1, 2, \ldots$$

By (3.6) $\|w_{i,m,j(m,i)+1} - w_{i,m,j(m,i)}\| \to 0$ for $i \to \infty$. Moreover, χ is continuous at x by assumption (ii) of Lemma 3.6. Thus by (3.7) $\mu_i(x) \to \chi(x)$ for $i \to \infty$. It follows that Proposition is valid, whenever $|\mathscr{N}| = n + 1$ and, by induction, Proposition is valid, if $\mathscr{N}$ is any finite set. Proposition is proved and this makes the proof of Lemma 3.6 complete.

3.7. Lemma. *ξ is of the first Baire class.*

Proof. Put $\mathscr{T}(m) = \mathscr{S}(\Theta, \{m\})$ for $m \in \mathscr{N}$, $\chi = \xi$. By Lemma 3.4 the assumptions of Lemma 3.6 are fulfilled and by Lemma 3.6 ξ is of the first Baire class.

4. FUNCTIONS OF BOUNDED VARIATION AND OF STRONGLY BOUNDED VARIATION

Let $\mathscr{M}$ be a finite set, $M \in \mathfrak{K}_0(R^{\mathscr{M}})$, $q \in M$.

4.1. Definition. For $\varphi \in \mathrm{BV}(M)$ define $V[\varphi] : M \to R$ by $V[\varphi](y) = \mathrm{Var}(\varphi, \langle q, y\rangle)$ . . $\mathrm{sgn}(y - q)$ if $\langle q, y\rangle$ is a nondegenerate interval, $V[\varphi](y) = 0$ otherwise.

4.2. Lemma. *Let $\varphi \in \mathrm{BV}(M)$. Define $\Psi : \mathfrak{K}(M) \to R$ by $\Psi(J) = \mathrm{Var}(\varphi, J)$ if $J \in \mathfrak{K}_0(M)$, $\Psi(J) = 0$ otherwise. Then Ψ is additive.*

The proof is omitted. It follows from Lemmas 4.2 and 1.11 that

$$\Delta(J)\, V[\varphi] = \mathrm{Var}(\varphi, J) \quad \text{for} \quad J \in \mathfrak{K}_0(M)\,. \tag{4.1}$$

Moreover,

$$\Delta(J)\, V[\varphi] \geqq \Delta(I)\, V[\varphi] \quad \text{for} \quad I, J \in \mathfrak{K}(M)\,, \quad I \subset J\,. \tag{4.2}$$

4.3. Lemma. *Let $\psi : M \to R$ and let $\Delta(J)\,\psi \geqq 0$ for $J \in \mathfrak{K}(M)$. Then $\psi \in \mathrm{BV}(M)$ and $\mathrm{Var}(\psi, J) = \Delta(J)\,\psi$ for $J \in \mathfrak{K}_0(M)$.*

The proof is omitted.

4.4. Lemma. *Let $\mathscr{J}$ be a nonempty finite set, $J \in \mathfrak{K}_0(R^{\mathscr{J}})$, $\zeta : J \to R, f, g \in J$. Then*

$$\zeta(g) - \zeta(f) = \sum_{\emptyset \neq \mathscr{I} \subset \mathscr{J}} \Delta(f_{\mathscr{I}}, g_{\mathscr{I}})\, \zeta^{f_{\mathscr{J}-\mathscr{I}}}\,.$$

Proof. By (1.8)

$$\Delta(f_{\mathscr{I}}, g_{\mathscr{I}})\, \zeta^{f_{\mathscr{J}-\mathscr{I}}} = \sum_{\mathscr{H} \subset \mathscr{I}} (-1)^{|\mathscr{I}-\mathscr{H}|}\, \zeta^{f_{\mathscr{J}-\mathscr{H}}}(g_{\mathscr{H}})\,.$$

Hence

$$\sum_{\emptyset \neq \mathscr{I} \subset \mathscr{J}} \Delta(f_{\mathscr{I}}, g_{\mathscr{I}})\, \zeta^{f_{\mathscr{J}-\mathscr{I}}} = \sum_{\mathscr{H} \subset \mathscr{J}} (-1)^{|\mathscr{H}|}\, \zeta^{f_{\mathscr{J}-\mathscr{H}}}(g_{\mathscr{H}})\, Q(\mathscr{H})\,,$$

$$Q(\mathscr{H}) = \sum_{\mathscr{H} \subset \mathscr{I} \subset \mathscr{J}, \mathscr{I} \neq \emptyset} (-1)^{|\mathscr{I}|}\,.$$

It is easy to show that $Q(\emptyset) = -1$, $Q(\mathscr{H}) = 0$ for $\emptyset \neq \mathscr{H} \neq \mathscr{J}$, $\mathscr{H} \subset \mathscr{J}$ and $Q(\mathscr{J}) = (-1)^{|\mathscr{J}|}$. It follows that Lemma 4.4 holds.

4.5. Lemma. *Let* $\varphi^{q_{\mathcal{M}-\mathcal{L}}} \in \mathrm{BV}(M_{\mathcal{L}})$ *for every* $\mathcal{L}$, $\emptyset \neq \mathcal{L} \subset \mathcal{M}$ *(read* $\varphi^{q_\emptyset} = \varphi$ *(cf.* 1.8)). *Let* $\emptyset \neq \mathcal{K} \subset \mathcal{M}$. *For* $\mathcal{K} \subset \mathcal{L} \subset \mathcal{M}$ *define* $\chi[\mathcal{K}, \mathcal{L}] : M_{\mathcal{K}} \to R$ *by* $\chi[\mathcal{K}, \mathcal{K}] = V[\varphi^{q_{\mathcal{M}-\mathcal{K}}}]$, $\chi[\mathcal{K}, \mathcal{L}] = \Delta(M_{\mathcal{L}-\mathcal{K}})\, V[\varphi^{q_{\mathcal{M}-\mathcal{L}}}]$ *if* $\mathcal{K} \neq \mathcal{L}$. *Put* $\psi[\mathcal{K}] = \sum_{\mathcal{K} \subset \mathcal{L} \subset \mathcal{M}} \chi[\mathcal{K}, \mathcal{L}]$ *for* $\emptyset \neq \mathcal{K} \subset \mathcal{M}$. *Then*

$$|\Delta(H)\, \varphi^y| \leqq \Delta(H)\, \psi[\mathcal{K}] \quad \text{if} \quad \mathcal{K} \subset \mathcal{M}, \quad \emptyset \neq \mathcal{K} \neq \mathcal{M}, \quad y \in M_{\mathcal{M}-\mathcal{K}}, \quad H \in \mathfrak{K}(M_{\mathcal{K}}), \tag{4.3}$$

$$|\Delta(H)\, \varphi| \leqq \Delta(H)\, \psi[\mathcal{M}] \quad \text{for} \quad H \in \mathfrak{K}(M). \tag{4.4}$$

Proof. (4.4) holds by (4.1). To prove (4.3), apply Lemma 4.4 with $\mathcal{I} = \mathcal{M} - \mathcal{K}$, $J = M_{\mathcal{M}-\mathcal{K}}$, $\zeta = \Delta(H)\, \varphi$, $f = q_{\mathcal{M}-\mathcal{K}}$, $g = y$. Then

$$\Delta(H)\, \varphi^y - \Delta(H)\, \varphi^{q_{\mathcal{M}-\mathcal{K}}} = \sum_{\emptyset \neq \mathcal{I} \subset \mathcal{M}-\mathcal{K}} \Delta(q_{\mathcal{I}}, y_{\mathcal{I}})\, \Delta(H)\, \varphi^{q_{\mathcal{M}-\mathcal{K}-\mathcal{I}}},$$

$$|\Delta(H)\, \varphi^y| \leqq |\Delta(H)\, \varphi^{q_{\mathcal{M}-\mathcal{K}}}| + \sum_{\emptyset \neq \mathcal{I} \subset \mathcal{M}-\mathcal{K}} |\Delta(H \times \langle q_{\mathcal{I}}, y_{\mathcal{I}} \rangle)\, \varphi^{q_{\mathcal{M}-\mathcal{K}-\mathcal{I}}}|.$$

By (4.1)

$$|\Delta(H)\, \varphi^{q_{\mathcal{M}-\mathcal{K}}}| \leqq \mathrm{Var}\,(\varphi^{q_{\mathcal{M}-\mathcal{K}}}, H) = \Delta(H)\, V[\varphi^{q_{\mathcal{M}-\mathcal{K}}}] = \Delta(H)\, \chi[\mathcal{K}, \mathcal{K}],$$

and for $\emptyset \neq \mathcal{I} \subset \mathcal{M} - \mathcal{K}$ by (4.1) and (4.2)

$$|\Delta(H \times \langle q_{\mathcal{I}}, y_{\mathcal{I}} \rangle)\, \varphi^{q_{\mathcal{M}-\mathcal{K}-\mathcal{I}}}| \leqq \Delta(H \times \langle q_{\mathcal{I}}, y_{\mathcal{I}} \rangle)\, V[\varphi^{q_{\mathcal{M}-\mathcal{K}-\mathcal{I}}}] \leqq$$
$$\leqq \Delta(H \times M_{\mathcal{I}})\, V[\varphi^{q_{\mathcal{M}-\mathcal{K}-\mathcal{I}}}] = \Delta(H)\, \chi[\mathcal{K}, \mathcal{K} \cup \mathcal{I}].$$

It follows that (4.3) holds.

4.6. *If* $\varphi^{q_{\mathcal{M}-\mathcal{L}}} \in \mathrm{BV}(M_{\mathcal{L}})$ *for every* $\mathcal{L}$, $\emptyset \neq \mathcal{L} \subset \mathcal{M}$, *then, by Lemma* 4.5, $\varphi \in \mathrm{SBV}(M)$.

4.7. *If* $\varphi \in \mathrm{SBV}(M)$ *and if* $\psi[\mathcal{K}]$ *are defined as in Lemma* 4.5, $\emptyset \neq \mathcal{K} \subset \mathcal{M}$, *then*

$$\mathrm{Var}\,(\varphi^y, J) \leqq \Delta(J)\, \psi[\mathcal{K}] \quad \text{for} \quad y \in M_{\mathcal{M}-\mathcal{K}}, \quad J \in \mathfrak{K}_0(M_{\mathcal{K}}).$$

4.8. *If* $\varphi \in \mathrm{SBV}(M)$, *then, by Lemma* 4.5,

$$|\varphi(v) - \varphi(u)| \leqq \sum_{m \in \mathcal{M}} |\psi_{\{m\}}(v_{\{m\}}) - \psi_{\{m\}}(u_{\{m\}})|$$

for $u, v \in M$. *Especially,* φ *is bounded.*

4.9. Lemma. *Let* $\varphi \in \mathrm{SBV}(M)$ *and let* $\psi[\mathcal{K}]$ *be defined as in Lemma* 4.5. *For* $m \in \mathcal{M}$ *let* $\mathcal{T}(m) \in M_{\{m\}}$ *be the set of points of discontinuity of* $\psi[\{m\}]$. *Let* $\mathcal{W}$ *be* *the set of points* $y \in M$ *that* $y_{\{m\}} \in \mathcal{T}(m)$ *for some* $m \in \mathcal{M}$. *Let* $t \in M - \mathcal{W}$. *Then to every* $\varepsilon^* \in R^+$ *there exists such a* $\delta^* \in R^+$ *that*

$$\Delta(J_{\mathcal{K}})\,\psi[\mathcal{K}] \leqq \varepsilon^*\,, \quad \text{if} \quad t \in J \in \mathfrak{K}_0(M)\,, \quad J \subset \mathcal{B}(t, \delta^*)\,, \quad \emptyset \neq \mathcal{K} \subset \mathcal{M}\,.$$

Proof. It is easy to see that it is sufficient to prove the following

Proposition. *Let* $t \in M - \mathcal{W}$, $\emptyset \neq \mathcal{K} \subset \mathcal{L} \subset \mathcal{M}$, $\varepsilon_1 \in R^+$. *Then there exists such a* $\delta_1 \in R^+$ *that*

$$\Delta(J_{\mathcal{K}})\,\chi[\mathcal{K}, \mathcal{L}] \leqq \varepsilon_1\,, \quad \text{if} \quad t \in J \in \mathfrak{K}_0(M)\,, \quad J \subset \mathcal{B}(t, \delta_1)$$

($\chi[\mathcal{K}, \mathcal{L}]$ *being defined in Lemma* 4.5).

Let $j \in \mathcal{K}$. By definition of $\chi[\mathcal{K}, \mathcal{L}]$ and by Lemmas 4.2 and 1.10

$$\Delta(J_{\mathcal{K}})\,\chi[\mathcal{K}, \mathcal{L}] = \Delta(M_{\mathcal{L}-\mathcal{K}} \times J_{\mathcal{K}})\,V[\varphi^{q_{\mathcal{M}-\mathcal{L}}}] =$$
$$= \operatorname{Var}(\varphi^{q_{\mathcal{M}-\mathcal{L}}}, M_{\mathcal{L}-\mathcal{K}} \times J_{\mathcal{K}}) \leqq \operatorname{Var}(\varphi^{q_{\mathcal{M}-\mathcal{L}}}, M_{\mathcal{L}-\{j\}} \times J_{\{j\}}) =$$
$$= \Delta(M_{\mathcal{L}-\{j\}} \times J_{\{j\}})\,V[\varphi^{q_{\mathcal{M}-\mathcal{L}}}] = \Delta(J_{\{j\}})\,\chi(\{j\}, \mathcal{L}) \leqq \Delta(J_{\{j\}})\,\psi[\{j\}]\,.$$

Proposition holds, as $\psi[\{j\}]$ is continuous at $t_{\{j\}}$.

4.10. Lemma. *Let* $\varphi \in \mathrm{BV}(M)$ *be continuous. Then* $V[\varphi]$ *is continuous.*

Proof. By Lemmas 1.13 and 4.2

$$|V[\varphi](y) - V[\varphi](x)| \leqq \sum_{\emptyset \neq \mathcal{L} \subset \mathcal{M}} \operatorname{Var}(\varphi, \langle q_{\mathcal{M}-\mathcal{L}}, x_{\mathcal{M}-\mathcal{L}} \rangle \times \langle x_{\mathcal{L}}, y_{\mathcal{L}} \rangle) \tag{4.5}$$

for $x, y \in M$.

If

$$\operatorname{Var}(\varphi, \langle q_{\mathcal{M}-\mathcal{L}}, x_{\mathcal{M}-\mathcal{L}} \rangle \times \langle x_{\mathcal{L}}, u \rangle) \geqq \varepsilon > 0 \tag{4.6}$$

for some $u \in M_{\mathcal{L}}$, then $u \neq x_{\mathcal{L}}$ and – by the continuity of φ – there exists such α, $0 < \alpha < 1$, that

$$\operatorname{Var}(\varphi, \langle q_{\mathcal{M}-\mathcal{L}}, x_{\mathcal{M}-\mathcal{L}} \rangle \times \langle (1 - \alpha)\,x_{\mathcal{L}} + \alpha u, u \rangle) \geqq \tfrac{1}{2}\varepsilon\,. \tag{4.7}$$

Let $x \in M$ and $\mathcal{L}$ be fixed. Then to every $\varepsilon \in R^+$ there exists such a $\delta \in R^+$ that

$$\operatorname{Var}(\varphi, \langle q_{\mathcal{M}-\mathcal{L}}, x_{\mathcal{M}-\mathcal{L}} \rangle \times \langle x_{\mathcal{L}}, y_{\mathcal{L}} \rangle) \leqq \varepsilon \quad \text{if} \quad y \in M\,, \quad \|y - x\| \leqq \delta\,.$$

Otherwise (4.6) holds for some $\varepsilon \in R^+$ and for $u = v_l$, $l = 1, 2, \ldots$, $v_l \to x$ for $l \to \infty$. It follows that there exist such α_l, $0 < \alpha_l < 1$ for $l = 1, 2, 3, \ldots$ and such a subsequence i_l of positive integers that (4.7) holds for $u = v_{i_l}$ and $\alpha = \alpha_l$ and that the intervals $\langle (1 - \alpha_l)\,x_{\mathcal{L}} + \alpha_l v_{i_l}, v_{i_l} \rangle$, $l = 1, 2, 3, \ldots$ are mutually disjoint. This contradicts the assumption that $\varphi \in \mathrm{BV}(M)$. Lemma 4.10 is proved.

4.11. Lemma. *Let $\varphi \in \mathrm{SBV}(M)$ be continuous. Then to every $\varepsilon \in R^+$ there exists such a $\delta \in R^+$ that*

$$\text{(4.8)} \qquad \operatorname{Var}(\varphi^y, J) \leqq \varepsilon\,, \quad \text{if} \quad \mathscr{K} \subset \mathscr{M}\,, \quad \emptyset \neq \mathscr{K} \neq \mathscr{M}\,,$$

$$y \in M_{\mathscr{M}-\mathscr{K}}\,, \quad J \in \mathfrak{K}_0(M_{\mathscr{K}})\,, \quad \operatorname{diam} J \leqq \delta\,,$$

$$\text{(4.9)} \qquad \operatorname{Var}(\varphi, J) \leqq \varepsilon \quad \text{if} \quad J \in \mathfrak{K}_0(M)\,, \quad \operatorname{diam} J \leqq \delta\,.$$

Proof. Let there be proved (4.8). By (4.3) $\operatorname{Var}(\varphi^y, J) \leqq \Delta(J)\,\psi[\mathscr{K}]$, by Lemma 4.10 $\psi[\mathscr{K}]$ is continuouns and (4.8) follows by the compactness of M. The proof of (4.9) is analogous.

4.12. Lemma. *Let $a, b \in R$, $a < b$. Let $\sigma : \langle a, b\rangle \to R$ be nondecreasing and continuous, $\sigma(a) < \sigma(b)$, let $\mathscr{Z} \subset \langle a, b\rangle$ be countable, $\varkappa \in R^+$, $\varkappa \geqq 1$. Then there exist such a function $\mu_\varkappa : \langle a, b\rangle \to R$ and such a set $\mathscr{U}_\varkappa \subset \langle a, b\rangle$ that $\mathscr{U}_\varkappa$ is open in $\langle a, b\rangle$, $\mathscr{Z} \subset \mathscr{U}_\varkappa$, $\mu_\varkappa$ is nondecreasing and*

$$\text{(4.10)} \qquad \mu(b) - \mu(a) \leqq \varkappa^{-1}(\sigma(b) - \sigma(a))\,,$$

$$\text{(4.11)} \qquad \sigma(v) - \sigma(u) \leqq \varkappa^{-1}(\mu_\varkappa(v) - \mu_\varkappa(u))\,, \quad \text{if} \quad u, v \in \langle a, b\rangle\,, \quad \langle u, v\rangle \subset \mathscr{U}_\varkappa\,.$$

Proof. Let $\mathscr{Z} = \{z_i \mid i = 1, 2, \ldots\}$. For any i such that $a < z_i < b$ find $a_i, b_i \in \langle a, b\rangle$, $a_i < z_i < b_i$

$$\text{(4.12)} \qquad \sigma(b_i) - \sigma(a_i) \leqq \varkappa^{-2} \,.\, 2^{-i-1}(\sigma(b) - \sigma(a))$$

and put $\mathscr{V}_i = (a_i, b_i)$. If $z_i = a$, put $a_i = a$ and find such $b_i \in \langle a, b\rangle$, $b_i > a$ that (4.12) holds and put $\mathscr{V}_i = \langle a_i, b_i)$ (a halfopen interval). Proceed analogously, if $z_i = b$. Define $\mathscr{U}_\varkappa = \bigcup_{i=1}^{\infty} \mathscr{V}_i$. It may be written $\mathscr{U}_\varkappa = \bigcup_{j=1}^{k} \mathscr{W}_j$, $\mathscr{W}_j$ being open in $\langle a, b\rangle$ and mutually disjoint, k finite or infinite. Let $\mathscr{W}_j = (u_j, v_j)$ ($\mathscr{W}_j = \langle u_j, v_j)$ if $a \in \mathscr{U}_\varkappa$ and $u_j = a$, $\mathscr{W}_j = (u_j, v_j\rangle$ if $b \in \mathscr{U}_\varkappa$ and $v_j = b$). Define $\mu_\varkappa$ as folllws: if $t \in \langle a, b\rangle - \mathscr{U}_\varkappa$, put $\mu_\varkappa(t) = \varkappa \sum_{W_j \subset \langle a, t\rangle} (\sigma(v_j) - \sigma(u_j))$ ($\mu_\varkappa(t) = 0$ if $t \in \langle a, b\rangle - \mathscr{U}_\varkappa$ and there is no $\mathscr{W}_j \subset \langle a, t\rangle$); if $t \in \mathscr{U}_\varkappa$, find l that $t \in \mathscr{W}_l$ (l is unique) and put $\mu_\varkappa(t) = \varkappa \sum_{W_j \subset \langle a, u_l\rangle} (\sigma(v_j) - \sigma(u_j)) + \varkappa(\sigma(t) - \sigma(u_l))$ ($\mu_\varkappa(t) = \varkappa(\sigma(t) - \sigma(u_l))$, if there is no $\mathscr{W}_j \subset \langle a, u_l\rangle$). $\mu_\varkappa$ is well defined, as $\sum_{j=1}^{k} (\sigma(v_j) - \sigma(u_j)) < \infty$; it is nondecreasing and continuous. Let $u, v \in \langle a, b\rangle$, $u < v$, $\langle u, v\rangle \subset \mathscr{U}_\varkappa$. There exists such an l that $\langle u, v\rangle \subset \mathscr{W}_l$. By the definition of $\mu_\varkappa$

$$\mu_\varkappa(v) - \mu_\varkappa(u) = \varkappa(\sigma(v) - \sigma(u))$$

and (4.11) holds.

Put $\mathscr{E}_j = \{i \mid \mathscr{V}_i \subset \mathscr{W}_j\}$. Obviously $\mathscr{E}_i \cap \mathscr{E}_h = \emptyset$ for $i \neq h$. It may be shown that there exist such a positive integer m, such $q(h) \in \langle u, v \rangle$ for $h = 0, 1, 2, \ldots, m$ and such integers $p(h) \in \mathscr{E}_j$ for $h = 0, 1, 2, \ldots, m - 1$ that $u = q(0)$, $q(h) < q(h + 1)$, $\langle q(h), q(h + 1) \rangle \subset \mathscr{V}_{p(h)}$ for $h = 0, 1, 2, \ldots, m - 1$, $q(m) = v$, $p(h_1) \neq p(h_2)$ for $h_1 \neq h_2$. It follows that

$$\mu_\varkappa(v) - \mu_\varkappa(u) = \sum_{h=0}^{m-1} (\mu_\varkappa(q(h + 1)) - \mu_\varkappa(q(h))) \leqq$$

$$\leqq \sum_{h=0}^{m-1} \varkappa^{-1} . 2^{-p(h)-1}(\sigma(b) - \sigma(a)) \leqq \varkappa^{-1}(\sigma(b) - \sigma(a)) \sum_{i \in \mathscr{E}_j} 2^{-i-1} .$$

By the continuity of $\mu_\varkappa$ it follows that

$$\mu_\varkappa(v_j) - \mu_\varkappa(u_j) \leqq \varkappa^{-1}(\sigma(b) - \sigma(a)) \sum_{i \in \mathscr{E}_j} 2^{-i-1}$$

and finally

$$\mu_\varkappa(b) - \mu_\varkappa(a) \leqq \sum_{j=1}^{k} (\mu_\varkappa(v_j) - \mu_\varkappa(u_j)) \leqq$$

$$\leqq \varkappa^{-1}(\sigma(b) - \sigma(a)) \sum_{j=1}^{k} \sum_{i \in \mathscr{E}_j} 2^{-i-1} = \varkappa^{-1}(\sigma(b) - \sigma(a)) .$$

(4.10) holds.

4.13. Lemma. *Let $a, b \in R$, $a < b$. Let $\sigma : \langle a, b \rangle \to R$ be nondecreasing and continuous, $\sigma(a) < \sigma(b)$, let $\mathscr{Z} \subset \langle a, b \rangle$ be countable, $\gamma \in R^+$, $\gamma \geqq 1$. Then there exists such a $\nu : \langle a, b \rangle \to R$ that ν is continuous and nondecreasing and that*

$$\nu(b) - \nu(a) \leqq \gamma^{-1}(\sigma(b) - \sigma(a)) , \tag{4.13}$$

(4.14) *if $z \in \mathscr{Z}$, $\alpha \in R^+$, $\alpha \geqq 1$, then there exists such a $\beta \in R^+$ that $\alpha(\sigma(v) - \sigma(u)) \leqq \nu(v) - \nu(u)$ whenever $u, v \in \langle a, b \rangle$, $u < v$, $\langle u, v \rangle \subset \mathscr{B}(z, \beta)$.*

Proof. Define $\nu = \sum_{j=1}^{\infty} \mu_{2^{j+1}\gamma}$, $\mu_\varkappa$ having the same meaning as in Lemma 4.10. Obviously ν is continuous and nondecreasing. (4.13) follows from (4.10), (4.14) follows from (4.11).

5. COMMON PART OF PROOFS OF THEOREMS 2.6 AND 2.7

In this section (2.1)–(2.3) are assumed to be valid, but there is made no use of the assumption that φ is continuous or of (2.5). Lemma 2.5 was already proved in section 3 (cf. Lemmas 3.1 and 3.7), so that Γ_T is well defined by (2.4).

5.1. Lemma. *Γ_T is additive.*

Proof. For $\mathscr{L} \subset \mathscr{N}$ define $\Gamma_S[\mathscr{L}] : \mathfrak{K}(K) \to R$ by

$$\text{(5.1)} \qquad \Gamma_S[\mathscr{L}](J) = S(\xi, \varphi, J, \mathscr{L}), \quad \text{if} \quad J \in \mathfrak{K}_0(K),$$
$$\Gamma_S[\mathscr{L}](J) = 0, \quad \text{if } J \text{ is a compact degenerate interval}.$$

It is sufficient to prove that $\Gamma_S[\mathscr{L}]$ is additive for $\mathscr{L} \subset \mathscr{N}$ (cf. (1.19) with $\mathscr{M} = \mathscr{N}$, $M = K$). Let $H, I, J \in \mathfrak{K}_0(K)$, $H \dotplus I = J$, $I = \langle u, v \rangle$, $u < v$, $H = \langle w, z \rangle$, $w < z$. Find $j \in \mathscr{N}$ by Lemma 1.6 (with $\mathscr{M} = \mathscr{N}$). If $j \in \mathscr{L}$, then by Lemma 1.6 $H_{\mathscr{L}} \dotplus I_{\mathscr{L}} = J_{\mathscr{L}}$ and $\Gamma_S[\mathscr{L}]$ is additive by the additivity of the integral (cf. (1.15)). If $j \notin \mathscr{L}$, then $H_{\mathscr{N}-\mathscr{L}} \dotplus I_{\mathscr{N}-\mathscr{L}} = J_{\mathscr{N}-\mathscr{L}}$ and the additivity of $\Gamma_S[\mathscr{L}]$ follows from $\Delta(H_{\mathscr{N}-\mathscr{L}})\,\chi + \Delta(I_{\mathscr{N}-\mathscr{L}})\,\chi = \Delta(J_{\mathscr{N}-\mathscr{L}})\,\chi$, which is valid for any $\chi : K_{\mathscr{N}-\mathscr{L}} \to R$. The proof of Lemma 5.1 is complete.

5.2. To complete the proof of Theorem 2.6 or 2.7 it remains to verify (cf. Definition 2.1) that to any $\varepsilon_2 \in R^+$ there exist such $\omega_2 : K \to R^+$ and $\Omega_2 \in Y(K)$ that $\Omega_2(K) \leqq \varepsilon_2$ and that

$$\text{(5.2)} \qquad |W(J, t) - \Gamma_T(J)| \leqq \Omega_2(J) \quad \text{if} \quad t \in J \in \mathfrak{K}_0(K), \quad J \subset \mathscr{B}(t, \omega_2(t)).$$

This will be done in sections 6 and 7. The aim of section 5 is to establish an estimate for $|W(J, t) - \Gamma_T(J)|$. In accordance with Definition 2.1 and with (2.1) it will be assumed that there are such $\omega : K \to R^+$ and $\Omega \in Y(K)$ that

$$\text{(5.3)} \qquad |U(J, t) - \Xi(J)| \leqq \Omega(J), \quad \text{if} \quad t \in J \in \mathfrak{K}_0(K), \quad J \subset \mathscr{B}(t, \omega(t)).$$

The choice of ω and Ω will be made in sections 6 and 7.

5.3. Let $t, y \in K$. It follows from (1.10), (1.9) and (1.8) that

$$\xi(y) = \sum_{\emptyset \neq \mathscr{M} \subset \mathscr{N}} (-1)^{|\mathscr{M}|+1}\, \xi(t_{\mathscr{M}} \times y_{\mathscr{N}-\mathscr{M}}) + \operatorname{sgn}(y - t)\, \Xi(\langle t, y \rangle).$$

Define $\tau[\mathscr{M}] : K \to R$ for $\emptyset \neq \mathscr{M} \subset \mathscr{N}$ and $\tau_0 : K \to R$ by

$$\text{(5.4)} \qquad \tau[\mathscr{M}](y) = (-1)^{|\mathscr{M}|+1}\, \xi(t_{\mathscr{M}} \times y_{\mathscr{N}-\mathscr{M}}),$$
$$\tau_0(y) = \operatorname{sgn}(y - t)\, \Xi(\langle t, y \rangle).$$

Thus

$$\text{(5.5)} \qquad \xi = \sum_{\emptyset \neq \mathscr{M} \subset \mathscr{X}} \tau[\mathscr{M}] + \tau[\Xi].$$

Observe, that $\tau[\mathscr{M}]$ for $\emptyset \neq \mathscr{M} \subset \mathscr{N}$ and τ_0 are bounded functions of the first

Baire class. Let $\emptyset \neq \mathscr{M} \subset \mathscr{N}$, $\mathscr{L} \subset \mathscr{N}$, $J \in \mathfrak{K}_0(K)$. By (1.16)

$$S(\tau[\mathscr{M}], \varphi, J, \mathscr{L}) = (-1)^{|\mathscr{L}|} \Delta(J_{\mathscr{N}-\mathscr{L}}) \int_{J_{\mathscr{L}}} \tau[\mathscr{M}] \, d\varphi \, .$$

Put $\mathscr{K} = (\mathscr{N} - \mathscr{M}) \cap \mathscr{L}$. If $\mathscr{K} \neq \emptyset \neq \mathscr{L} \cap \mathscr{M}$, then $J_{\mathscr{L}} = J_{\mathscr{L} \cap \mathscr{M}} \times J_{\mathscr{K}}$ and by integration over $J_{\mathscr{L} \cap \mathscr{M}}$ (cf. (5.4) and 1.15 – the Fubini Theorem)

$$\int_{J_{\mathscr{L}}} \tau[\mathscr{M}] \, d\varphi = \Delta(J_{\mathscr{L}-\mathscr{K}}) \int_{J_{\mathscr{K}}} \tau[\mathscr{M}] \, d\varphi \, ,$$

so that

$$S(\tau[\mathscr{M}], \varphi, J, \mathscr{L}) = (-1)^{|\mathscr{L}|} \Delta(J_{\mathscr{N}-\mathscr{K}}) \int_{J_{\mathscr{K}}} \tau[\mathscr{M}] \, d\varphi \, . \tag{5.6}$$

If $\mathscr{K} \neq \emptyset$, $\mathscr{L} \cap \mathscr{M} = \emptyset$, then $\mathscr{K} = \mathscr{L}$ and (5.6) holds by (1.16). If $\mathscr{K} = \emptyset \neq \mathscr{L}$, then $\mathscr{L} \subset \mathscr{M}$, $\int_{J_{\mathscr{L}}} \tau[\mathscr{M}] \, d\varphi = \Delta(J_{\mathscr{L}}) (\varphi \, . \, \tau[\mathscr{M}])$, so that

$$S(\tau[\mathscr{M}], \varphi, J, \mathscr{L}) = \Delta(J) (\varphi \, . \, \tau[\mathscr{M}]) \, . \tag{5.7}$$

If $\mathscr{K} = \emptyset = \mathscr{L}$ then (5.7) holds by (1.17). Thus (5.6) holds for $\mathscr{K} \neq \emptyset$ and (5.7) holds for $\mathscr{K} = \emptyset$. (Observe that always $\mathscr{N} - \mathscr{K} \neq \emptyset$.) Like in 1.15 the right hand side of (5.6) is given such an interpretation that (5.6) holds for $\mathscr{K} = \emptyset$ also.

Put $\mathscr{J} = \mathscr{L} \cap \mathscr{M}$; obviously $\mathscr{L} = \mathscr{J} \cup \mathscr{K}$ and $\mathscr{J} \cap \mathscr{K} = \emptyset$. By (1.19) and (5.6)

$$T(\tau[\mathscr{M}], \varphi, J) = \sum_{\mathscr{L} \subset \mathscr{N}} S(\tau[\mathscr{M}], \varphi, J, \mathscr{L}) = \tag{5.8}$$

$$= \sum_{\mathscr{K} \subset \mathscr{N} - \mathscr{M}} (-1)^{|\mathscr{K}|} \Delta(J_{\mathscr{N}-\mathscr{K}}) \int_{J_{\mathscr{K}}} \tau[\mathscr{M}] \, d\varphi \sum_{\mathscr{J} \subset \mathscr{M}} (-1)^{|\mathscr{J}|} = 0$$

as $\mathscr{M} \neq \emptyset$.

Let $y \in \mathscr{B}(t, \omega(t)) \cap K$. By (5.3)

$$|\Xi(\langle t, y \rangle)| \leqq |U(\langle t, y \rangle, t)| + \Omega(\langle t, y \rangle) \, ,$$

if $\langle t, y \rangle$ is a nondegenerate interval. By (2.2)

$$|\Xi(\langle t, y \rangle)| \leqq \eta(t) \, \Theta(\langle t, y \rangle) + \Omega(\langle t, y \rangle)$$

and, by $\Theta, \Omega \in Y(K)$

$$|\Xi(\langle t, y \rangle)| \leqq \eta(t) \, \Theta(J) + \Omega(J) \tag{5.9}$$

provided that $t \in J \in \mathfrak{K}_0(K)$, $J \subset \mathscr{B}(t, \omega(t))$, $y \in J$ and $\langle t, y \rangle$ is nondegenerate. Of course, $\Xi(\langle t, y \rangle) = 0$ if $\langle t, y \rangle$ is a degenerate interval. Assume that $J = \langle a, b \rangle$, $a, b \in K$, $a < b$. For $\mathscr{L} \subset \mathscr{N}$, $\mathscr{L} \neq \mathscr{N}$, $\mathscr{J} \subset \mathscr{N} - \mathscr{L}$ put $w(J, \mathscr{L}, \mathscr{J}) = a_{\mathscr{J}} \times \times \, b_{\mathscr{N}-\mathscr{L}-\mathscr{J}} \in K_{\mathscr{N}-\mathscr{L}}$.

Let $\emptyset \neq \mathscr{L} \subset \mathscr{N}$. It follows from (1.16), (1.18), (1.14), (1.7) (with $\mathscr{M} = \mathscr{N}$) and (5.9) that

$$(5.10) \qquad |S(\tau_0, \varphi, J, \mathscr{L})| \leqq (\eta(t)\, \Theta(J) + \Omega(J)) \sum_{\mathscr{I} \subset \mathscr{N} - \mathscr{L}} \operatorname{Var}(\varphi^{w(J,\mathscr{L},\mathscr{I})}, J_{\mathscr{L}})$$

– if $\mathscr{L} = \mathscr{N}$ it is meant that $\varphi^{w(J,\mathscr{N},\emptyset)} = \varphi$.

It remains to discuss $S(\tau_0, \varphi, J, \emptyset)$. Define $\varphi[t] : K \to R$ by $\varphi[t](y) = \varphi(t)$. By (1.17)

$$S(\tau_0, \varphi, J, \emptyset) = \Delta(J)(\tau_0 \varphi) = \varphi(t)\, \Delta(J)\, \tau[\Xi] + \Delta(J)(\tau_0(\varphi - \varphi[t]))\,.$$

By Lemma 1.11 $\Delta(J)\, \tau_0 = \Xi(J)$; thus

$$|S(\tau_0, \varphi, J, \emptyset) - \varphi(t)\, \Xi(J)| \leqq \Delta(J)(\tau_0(\varphi - \varphi[t]))\,.$$

By (5.3), (5.9) and (1.8)

$$(5.11) \qquad |S(\tau_0, \varphi, J, \emptyset) - W(J, t)| \leqq |\varphi(t)|\, \Omega(J) + \\ + (\eta(t)\, \Theta(J) + \Omega(J)) \sum_{\mathscr{I} \subset \mathscr{N}} |\varphi(w(J, \emptyset, \mathscr{I}) - \varphi(t)|\,,$$

provided that $t \in J \in \mathfrak{K}_0(K)$, $J \subset \mathscr{B}(t, \omega(t))$. By (5.5) and (5.8)

$$(5.12) \qquad T(\xi, \varphi, J) = T(\tau[\Xi], \varphi, J)\,,$$

by (2.4), (1.19), (5.12), (5.10) and (5.11)

$$(5.13) \qquad |W(J, t) - \Gamma_T(J)| \leqq |\varphi(t)|\, \Omega(J) + (\eta(t)\, \Theta(J) + \Omega(J))\, Q(J, t)\,,$$

$$Q(J, t) = \sum_{\emptyset \neq \mathscr{L} \subset \mathscr{N}} \sum_{\mathscr{I} \subset \mathscr{N} - \mathscr{L}} \operatorname{Var}(\varphi^{w(J,\mathscr{L},\mathscr{I})}, J_{\mathscr{L}}) + \sum_{\mathscr{I} \subset \mathscr{N}} |\varphi(w(J, \emptyset, \mathscr{I}) - \varphi(t)|\,,$$

if $t \in J \in \mathfrak{K}_0(K)$, $J \subset \mathscr{B}(t, \omega(t))$.

6. PROOF OF THEOREM 2.6, CONTINUED

Let $\varepsilon_2 \in R^+$ (cf. 5.2). By 4.8 there exists such a $\varkappa \in R^+$ that $|\varphi(t)| \leqq \varkappa$ for $t \in K$. To $\varepsilon = \varepsilon_2(2\varkappa)^{-1}$ find ω and Ω by Definition 2.1 and define $\Omega_2 \in Y(K)$ by

$$\Omega_2 = \varkappa\Omega + \tfrac{1}{2}\varepsilon_2(\varkappa\, \Theta(K) + \Omega(K))^{-1} (\varkappa\Theta + \Omega)\,.$$

Obviously $\Omega_2(K) \leqq \varepsilon_2$. As φ is continuous, there exists such a $\omega_3 : K \to R^+$ that

$$(6.1) \qquad |\sum_{\mathscr{I} \subset \mathscr{N}} \varphi(w(J, \emptyset, \mathscr{I})) - \varphi(t)| \leqq \tfrac{1}{4}\varepsilon_2(\varkappa\, \Theta(K) + \Omega(K))^{-1}\,, \\ \text{if} \quad J \in \mathfrak{K}_0(K)\,, \quad J \subset \mathscr{B}(t, \omega_3(t))\,.$$

By Lemma 4.11 there exists such a $\omega_4 : K \to R^+$ that

(6.2) $$\sum_{0 \neq \mathscr{L} \subset \mathscr{N}} \sum_{\mathscr{I} \subset \mathscr{N} - \mathscr{L}} \operatorname{Var}\left(\varphi^{w(J,\mathscr{L},\mathscr{I})}, J_{\mathscr{L}}\right) \leqq \tfrac{1}{4}\varepsilon_2(\varkappa\, \Theta(K) + \Omega(K))^{-1}, \quad \text{if} \quad J \in \mathfrak{K}_0(K), \quad J \subset \mathscr{B}(t, \omega_4(t)).$$

Define $\omega_2(t) = \min(\omega(t), \omega_3(t), \omega_4(t))$. It follows from (5.13), (6.1) and (6.2) that $|W(J, t) - \Gamma_T(J)| \leqq \Omega_2(J)$, if $t \in J \in \mathfrak{K}_0(K)$, $J \subset \mathscr{B}(t, \omega_2(t))$, which makes the proof of Theorem 2.6 complete.

7. PROOF OF THEOREM 2.7, CONTINUED

By 4.8, 4.7, and Lemma 4.3 there exists such a $\varkappa \in R^+$ that $|\varphi(t)| + Q(J, t) \leqq \varkappa$ for $t \in K$, $J \in \mathfrak{K}_0(K)$ (cf. (5.13)). By (5.13)

(7.1) $$|W(J, t) - \Gamma_T(J)| \leqq \varkappa\, \Omega(J) + \eta(t)\, Q(J, t)\, \Theta(J) \quad \text{if} \quad t \in J \in \mathfrak{K}_0(K), \quad J \subset \mathscr{B}(t, \omega(t)).$$

By (4.3) (or (4.4)) $\psi_{\{m\}} : K_{\{m\}} \to R$ is nondecreasing for $m \in \mathscr{N}$ ($\psi_{\{m\}}$ is defined in Lemma 4.5 with $M = K$, $\mathscr{M} = \mathscr{N}$). Let $\mathscr{T}(m)$ be the set of points of discontinuity of $\psi_{\{m\}}$ for $m \in \mathscr{N}$.

Let $\varepsilon_2 \in R^+$. For $m \in \mathscr{N}$ apply Lemma 4.13 replacing $\langle a, b\rangle$ by $K_{\{m\}}$, σ by ϑ_m, $\mathscr{X}$ by $\mathscr{T}(m)$ and γ by $\varepsilon_2^{-1} . 3|\mathscr{N}| \prod_{i \in \mathscr{N} - \{m\}} \Delta(K_{\{i\}})\, \vartheta_i$; it may be assumed without loss on generality that $\varepsilon_2^{-1} . 3|\mathscr{N}| \prod_{i \in \mathscr{N} - \{m\}} \Delta(K_{\{i\}})\, \vartheta_i \geqq 1$ (of course, γ is replaced by $3\varepsilon_2^{-1}$, if $\mathscr{N} = \{m\}$). Find v by Lemma 4.13 and write v_m instead of v. Define $\Omega_3 : \mathfrak{K}_0(K) \to R$ by

$$\Omega_3(J) = \sum_{m \in \mathscr{N}} \Delta(J_{\{m\}})\, v_m \prod_{i \in \mathscr{N} - \{m\}} \Delta(J_{\{i\}})\, \vartheta_i .$$

It is easy to verify that $\Omega_3 \in Y(K)$ (cf. Lemma 1.10) and that $\Omega_3(K) \leqq \frac{1}{3}\varepsilon_2$. To $\varepsilon = \varepsilon_2(3\varkappa)^{-1}$ find ω and Ω by Definition 2.1. Define

$$\Omega_2 = \varkappa\Omega + \varepsilon_2(3\Theta(K))^{-1}\, \Theta + \Omega_3 .$$

Obviously $\Omega_2 \in Y(K)$ and $\Omega_2(K) \leqq \varepsilon_2$. It remains to find such $\omega_2 : K \to R^+$ that (5.2) holds.

Put $\mathscr{W} = \{x \in K, \mid x_{\{m\}} \in \mathscr{T}(m)$ for some $m \in \mathscr{N}\}$. If $t \in K - \mathscr{W}$, then by 4.8 φ is continuous at t. There will be distinguished two cases:

(i) $t \in K - \mathscr{W}$. By 4.8

$$|\varphi(w(J, \emptyset, \mathscr{I})) - \varphi(t)| \leqq \sum_{i \in \mathscr{N}} |\psi_{\{i\}}(w(J, \emptyset, \mathscr{I})_{\{m\}}) - \psi_{\{i\}}(t_{\{m\}})| ,$$

by 4.7

$$\operatorname{Var}(\varphi^{w(J,\mathscr{L},\mathscr{I})}, J_{\mathscr{L}}) \leqq \Delta(J_{\mathscr{L}})\, \psi[\mathscr{L}].$$

Therefore (cf. Lemma 4.9) there exists such a $\omega_5(t) \in R^+$ that

$$\eta(t)\, Q(J, t) \leqq \varepsilon_2(3\Theta(K))^{-1}, \quad \text{if} \quad J \in \mathfrak{K}_0(K), \quad J \subset \mathscr{B}(t, \omega_5(t)).$$

Put $\omega_2(t) = \min(\omega(t), \omega_5(t))$. (5.2) holds by (7.1).

(ii) $t \in \mathscr{W}$. Then $t_{\{m\}} \in \mathscr{T}(m)$ for some $m \in \mathscr{N}$. Put $\alpha = \max(\varkappa\, \eta(t), 1)$, $z = t_{\{m\}}$. By the choice of v_m and by (4.14) there exists such a $\omega_n(t) \in R^+$ that $\varkappa\, \eta(t)\,(\vartheta_m(v) - \vartheta_m(u)) \leqq v_m(v) - v_m(u)$ whenever $u, v \in K_{\{m\}}$, $u < v$, $\langle u, v\rangle \subset \mathscr{B}(t_{\{m\}}, \omega_5(t))$. It follows that

$$\eta(t)\, Q(J, t)\, \Theta(J) \leqq \eta(t)\, \varkappa \prod_{i \in \mathscr{N}} \Delta(J_{\{i\}})\, \vartheta_i \leqq$$

$$\leqq \Delta(J_{\{m\}})\, v_m \prod_{i \in \mathscr{N} - \{m\}} \Delta(J_{\{i\}})\, \vartheta \leqq \Omega_3(J) \quad \text{if} \quad J \in \mathfrak{K}_0(K), \quad J \subset \mathscr{B}(t, \omega_6(t)).$$

Put $\omega_2(t) = \min(\omega(t), \omega_6(t))$. Again, (5.2) holds by (7.1). The proof of Theorem 2.7 is complete.

8. AN EXAMPLE

In this section there will be used common notations, i.e. the basic space will be R^2 and not $R^{\{1,2\}}$, the points in R^2 being denoted by $x = (x_1, x_2)$ with $x_1, x_2 \in R$.

8.1. For $\gamma, \delta \in R^+$ let $J[\gamma, \delta] = \langle 0, \gamma\rangle \times \langle 0, \delta\rangle \subset R^2$, $D[\gamma] = J[\gamma, \gamma] - J[\frac{1}{2}\gamma, \frac{1}{2}\gamma]$ and let $\sigma_\gamma : J[1, 1] \to \{0, 1\}$ be the characteristic function of $D[\gamma]$ for $0 < \gamma \leqq 1$. For $k = 0, 1, 2, \ldots$ let $m(k)$ be an integer, $m(k) \geqq k + 2$, $m(k)/k \to \infty$ for $k \to \infty$. Define $\zeta : J[1, 1] \to R$ by

$$(8.1) \qquad \zeta(x_1, x_2) = \sum_{k=0}^{\infty} (k + 1)^{-1}\, 2^{2m(k)} \sigma_{2^{-k}}(x_1, x_2) \sin(2^{m(k)}\pi x_1) \sin(2^{m(k)}\pi x_2).$$

Let $\lambda : J[1, 1] \to R$ be defined by $\lambda(x_1, x_2) = x_1 x_2$ and let $(\mathrm{P}) - \int_I \psi\, d\lambda$ denote the Perron integral of ψ over I, if I is a nondegenerate subinterval of $J[1, 1]$ and $\psi : J[1, 1] \to R$ is Perron-integrable (cf. 2.3).

8.2. Lemma. *ζ is Perron integrable and* $(\mathrm{P}) - \int_{J[1,1]} \zeta\, d\lambda = 0$.

Proof. For any $\alpha, \beta \in (0, 1)$ ζ is continuous on $J[1, 1] - J[\alpha, \beta]$ and it is well known that it is sufficient to prove that

$$(8.2) \qquad \lim_{\alpha \to 0+, \beta \to 0+} \int_{J[1,1] - J[\alpha,\beta]} \zeta\, d\lambda = 0.$$

It is easy to see that

$$\int_{J[1,1]-J[2^{-k},2^{-k}]} \zeta \, d\lambda = 0 \quad \text{for} \quad k = 1, 2, \ldots . \tag{8.3}$$

Let $\alpha, \beta \in (0, \frac{1}{2})$, $\beta \leqq \alpha$ and let j be the greatest integer that $\alpha < 2^{-j}$.

$$\int_{\langle 0,2^{-j}\rangle \times \langle \beta, 2^{-j}\rangle} \zeta \, d\lambda = 0 ,$$

so that

$$\int_{\mathscr{U}} \zeta \, d\lambda = \int_{\mathscr{V}} \zeta \, d\lambda , \tag{8.4}$$

with $\mathscr{U} = J[2^{-j}, 2^{-j}] - J[\alpha, \beta]$, $\mathscr{V} = \langle \alpha, 2^{-j} \rangle \times \langle 0, \beta \rangle$. $\mathscr{V}$ is contained in the closure of $D(2^{-j})$, hence

$$|\zeta(x_1, x_2)| \leqq (j + 1)^{-1} \, 2^{2m(j)} \quad \text{for} \quad (x_1, x_2) \in \mathscr{V} . \tag{8.5}$$

Let p be the greatest integer that $2^{-j} - p \, . \, 2^{-m(j)+1} \geqq \alpha$, let q be the greatest integer that $q \, . \, 2^{-m(j)+1} \leqq \beta$. Put

$$\mathscr{V}_1 = \langle 2^{-j} - p \, . \, 2^{-m(j)+1}, 2^{-j} \rangle \times \langle 0, \beta \rangle ,$$

$$\mathscr{V}_2 = \langle \alpha, 2^{-j} - p \, . \, 2^{-m(j)+1} \rangle \times \langle 0, q \, . \, 2^{-m(j)+1} \rangle ,$$

$$\mathscr{V}_3 = \langle \alpha, 2^{-j} - p \, . \, 2^{-m(j)+1} \rangle \times \langle q \, . \, 2^{-m(j)+1}, \beta \rangle .$$

Read $\int_{\mathscr{V}_i} \zeta \, d\lambda = 0$ if $\mathscr{V}_i$ is a degenerate interval. Then

$$\int_{\mathscr{V}} \zeta \, d\lambda = \int_{\mathscr{V}_1} \zeta \, d\lambda + \int_{\mathscr{V}_2} \zeta \, d\lambda + \int_{\mathscr{V}_3} \zeta \, d\lambda .$$

It follows from the definition of ζ that $\int_{\mathscr{V}_1} \zeta \, d\lambda = 0 = \int_{\mathscr{V}_2} \zeta \, d\lambda$ so that by (8.4)

$$\int_{\mathscr{U}} \zeta \, d\lambda = \int_{\mathscr{V}_3} \zeta \, d\lambda .$$

$$2^{-j} - p \, . \, 2^{-m(j)+1} - \alpha \leqq 2^{-m(j)+1} ,$$

$$\beta - q \, . \, 2^{-m(j)+1} \leqq 2^{-m(j)+1}$$

and by (8.5) $|\int_{\mathscr{V}_3} \zeta \, d\lambda| \leqq 4(j + 1)^{-1}$ it follows (cf. (8.3)) that $|\int_{J[1,1]-J[\alpha,\beta]} \zeta \, d\lambda| \leqq \leqq 4(j + 1)^{-1}$. 8.2 holds and the proof of Lemma 8.2 is complete.

8.3. Define $\varphi : [1, 1] \to R$ by

$$\varphi(x_1, x_2) = \sum_{k=0}^{\infty} 2^{-2(m(k)-k)} \sigma_{2^{-k}}(x_1, x_2) \sin^3 (2^{m(k)} \pi x_1) \sin^3 (2^{m(k)} \pi x_2) .$$

If $(x_1, x_2) \in D(2^{-k})$, then

$$(\zeta\varphi)(x_1, x_2) = (k + 1)^{-1} 2^{2k} \sin^4(2^{m(k)}\pi x_1) \sin^4(2^{m(k)}\pi x_2)$$

and $\int_{D[2^{-k}]} \zeta\varphi \, d\lambda = \varkappa(k + 1)^{-1}$, $\varkappa \in R^+$ being independent of k. Hence the product $\zeta\varphi$ is not Perron integrable.

If $(x_1, x_2) \in D[2^{-k}]$, then

$$\frac{\partial\varphi}{\partial x_1}(x_1, x_2) = 2^{-m(k)+2k} \, 3 \sin^2(2^{m(k)}\pi x_1) \cos(2^{m(k)}\pi x_1) \sin^3(2^{m(k)}\pi x_2) .$$

It may be shown that $\partial\varphi/\partial x_1$ fulfils the Hölder condition with any exponent ε, $0 < \varepsilon < 1$. Moreover, if $\omega : \langle 0, 1 \rangle \to \langle 0, \infty)$ is continuous, $\omega(0) = 0$, $\omega(\alpha) > 0$ for $\alpha > 0$, $\omega(\alpha)/\alpha \to \infty$ for $\alpha \to 0+$, then there exists such a function $m : \{0, 1, 2, \ldots\} \to \{2, 3, 4, \ldots\}$ that $m(k) \geqq k + 2$, $m(k)/k \to \infty$ for $k \to \infty$ and that

$$\left| \frac{\partial\varphi}{\partial x_1}(v_1, v_2) - \frac{\partial\varphi}{\partial x_2}(u_1, u_2) \right| \leqq \beta\omega(\max(|v_1 - u_1|, |v_2 - u_2|))$$

for $(u_1, u_2), (v_1, v_2) \in J[1, 1]$. Analogous conclusions are valid for $\partial\varphi/\partial x_2$ as $\varphi(x_1, x_2) = \varphi(x_2, x_1)$ for $(x_1, x_2) \in J[1, 1]$.

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