ON MULTIPLICATIVE GRAPHS AND THE PRODUCT CONJECTURE

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ABSTRACT

We characterize those directed and undirected paths P and cycles C which have the property that the class of graphs not admitting a homomorphism into P, respectively C, is closed under taking the product (conjunction). Whether all undirected complete graphs have the property is a longstanding open question due to S. Hedetniemi.

1. INTRODUCTION

In what follows G, H, etc., could be graphs or digraphs; similarly, the edge gg' could mean the undirected edge $\{g, g'\}$ or the directed arc $\overrightarrow{g\,g'}$. The product $G \times H$ (also known as the categorical product or conjunction [7, 15]) has the vertex set $V(G) \times V(H)$ and the edges $(g, h)(g', h')$ where gg' is an edge of G and hh' an edge of H. A homomorphism $f: G \to H$ is a mappint $f: V(G) \to V(H)$ for which $f(g)f(g')$ is an edge of H whenever gg' is an edge of G. The existence, respectively non-existence, of a homomorphism $f: G \to H$ will be denoted by $G \to H$, respectively $G \not\to H$. Note that G is n-colourable just if $G \to K_n$. Also note that the composition of two homomorphisms is again a homomorphism.

It follows from the definitions that any n-colouring of G gives rise to an n-colouring of $G \times H$; thus $\chi(G{\times}H) \leqq \min(\chi G, \chi H)$. S. Hedetniemi [8] conjectured that $\chi(G{\times}H) = \min(\chi G, \chi H)$. To establish this "product conjecture" it remains to show, for each n, that

$$G \not\to K_n \text{ and } H \not\to K_n \text{ imply } G \times H \not\to K_n$$

for all undirected graphs G and H. While this is easy to verify for $n = 1$ and $n = 2$ (cf. section 2), its proof for $n = 3$ is quite difficult and was accomplished only recently by El-Zahar and Sauer [4]. It seems conceivable that the conjecture may be false for large enough n, cf. [17]. We shall call a (directed respectively undirected) graph W multiplicative if $G \not\to W$ and $H \not\to W$ imply $G \times H \not\to W$ for all (directed respectively undirected) graphs G and H. In other words, W is multiplicative just if $\{G: G \not\to W\}$ is closed under taking products; it is important to note here that the graphs G are taken to be undirected graphs if W is undirected, and directed graphs when W is directed.

(It follows from Remark 3.4 in [17] that K_t, $r \geq 3$, is <u>not</u> multiplicative when viewed as a <u>directed</u> graph, although according to [4] it is multiplicative as an undirected graph; a similar example was constructed by D. Duffus, W. Sands and R. Woodrow.)

Thus the product conjecture asserts that all complete undirected graphs are multiplicative. By investigating the multiplicativity of graphs in general, we hope to gain insights relevant for the eventual proof of multiplicativity - or non-multiplicativity - of complete undirected graphs. We concentrate here on the multiplicativity of simple families of graphs - namely directed and undirected paths, directed and undirected cycles, and transitive tournaments. (Note that the El-Zahar - Sauer theorem asserts the multiplicativity of the undirected 3-cycle.) Not surprizingly, we find some graphs which are not multiplicative; we also demonstrate the multiplicativity of all other graphs in our families.

2. GENERAL REMARKS

Here we outline some standard properties of the product and describe the methods used in demonstrating multiplicativity and non-multiplicativity of graphs.

<u>LEMMA 1</u>. (a) $G \times H \to G$ and $G \times H \to H$

(b) If $X \to G$ and $X \to H$ then $X \to G \times H$

(c) $G \to G \times H$ if and only if $G \to H$

<u>Proof</u>: For (a), verify that the projections $(g, h) \mapsto g$ and $(g, h) \mapsto h$ are homomorphisms; for (b) use any homomorphisms $f: X \to G$ and $f': X \to H$ to define a homomorphism $X \to G \times H$ by $x \mapsto \big(f(x), f'(x)\big)$. Finally (c) follows from (a) and (b).

A subgraph G' of G is a retract of G ([10, 11, 16]) if there is a homomorphism (called a retraction) $r: G \to G'$ such that $r(g) = g$ for each $g \in V(G')$.

LEMMA 2. (a) If $G \not\to H$ and $H \not\to G$, then $W = G \times H$ is not multiplicative. (b) If W' is a retract of W, then W is multiplicative if and only if W' is.

Proof: (a) follows from Lemma 1(c): $G \not\to W$, $H \not\to W$, but $G \times H = W \to W$. (b) follows from the fact that $W \to W'$ and $W' \to W$, hence $\{G: G \not\to W\} = \{G: G \not\to W'\}$.

Lemma 2(a) yields many examples of non-multiplicative graphs, as there are many ways of constructing pairs of graphs G, H with $G \not\to H$ and $H \not\to G$ [12, 13]. To mention just two examples - for directed cycles $\vec{C}_k \not\to \vec{C}_m$ when $k \not\equiv 0 \pmod{m}$, and for undirected graphs, letting G_m^k to be any graph of chromatic number k and smallest odd cycle of length m, $G_k^k \not\to G_{k+2}^{k+2}$ because of the smallest odd cycle in G_k^k, and $G_{k+2}^{k+2} \not\to G_k^k$ because of chromatic number. Lemma 2(b) allows us to restrict our attention to graphs which contain no proper retracts. For instance, since K_3 is multiplicative [4], any 3-chromatic graph with a triangle is likewise multiplicative (cf. also Example 2).

Let W be a fixed graph. A set $\mathcal{O} \subseteq \{G: G \not\to W\}$ is called a complete set of obstructions for W if

(1) For each G with $G \not\to W$ there is an $X \in \mathcal{O}$ such that $X \to G$.

(2) For each $X, X' \in \mathcal{O}$ there is an $X^* \in \mathcal{O}$ such that $X^* \to X$ and $X^* \to X'$.

LEMMA 3. W is multiplicative if and only if there is a complete set of obstructions for W.

Proof: If W is multiplicative then $\mathcal{O} = \{G: G \not\to W\}$ is a complete set of obstructions: (1) is trivial, and (2) follows for $X^* = X \times X'$ from Lemma 1(a) and the multiplicativity of W. On the other hand, if $\mathcal{O}$ is a complete set of obstructions for W and if $G \not\to W$ and $H \not\to W$, then there exist $X, X' \in \mathcal{O}$ such that $X \to G$, $X' \to H$, and hence an $X^* \in \mathcal{O}$ such that $X^* \to X \to G$, $X^* \to X' \to H$. By Lemma 1(b) $X^* \to G \times H$, whence $G \times H \not\to W$.

Naturally, the method of Lemma 3 is only interesting if we can find "small" or "simple" complete sets of obstructions. For instance, $\{K_2\}$ is a complete set of obstructions for $W = K_1$: any graph that is not 1-colourable contains an edge. Hence K_1 is multiplicative. Similarly, $\{C_3, C_5, \ldots\}$ is a complete set of obstructions for $W = K_2$ (any non-bipartite graph contains an odd cycle; (2) follows from the fact that $C_{k+2} \to C_k$). Hence K_2 is also multiplicative. No simple complete set of obstructions is known for K_n with $n > 2$ and the multiplicativity of these graphs is much harder to establish. Until recently, the product conjecture ($G \not\to K_n$, $H \not\to K_n$ imply $G \times H \not\to K_n$) for $n > 2$ was only established for particular classes of connected graphs G and H - e.g. both having a K_{n-1} [3, 18], or one having each vertex in a K_{n-1} [1]. Then El-Zahar and Sauer introduced an elegant new method for proving the multiplicativity of K_3.

Let W be a fixed graph and G an arbitrary graph. The map-graph $\mathcal{M}(G, W)$ is defined as follows: the vertices of $\mathcal{M}(G, W)$ are the mappings $\phi: V(G) \to V(W)$ and the edges of $\mathcal{M}(G, W)$ are just those $\phi\phi'$ for which $\phi(g)\phi'(g')$ is an edge of W whenever gg' is an edge of G. Note that $\mathcal{M}(G, W)$ is directed or undirected depending on G and W being directed or undirected. Although normally our graphs have no loops, the map-graph may have loops - cf. Lemma 4(a). El-Zahar and Sauer [4] introduced and used $\mathcal{M}(G, C_3)$ to demonstrate the multiplicativity of C_3; the connection of $\mathcal{M}(G, W)$ to the multiplicativity of W is explained in Lemma 4(f). Following their usage we shall write W(G) for $\mathcal{M}(G, W)$. We may assume that each G and W below has at least one edge.

LEMMA 4. (a) W(G) has loops if and only if $G \to W$

(b) $G \times W(G) \to W$

(c) $G \to W(W(G))$ by a one-to-one homomorphism

(d) $W \to W(G)$ by an isomorphism onto an induced subgraph of W(G)

(e) $G \times H \to W$ if and only if $H \to W(G)$

(f) W is multiplicative if and only if $W(G) \to W$ whenever $G \not\to W$

(g) $G \to G'$ implies $W(G') \to W(G)$ for each W

(h) $W \to W'$ implies $W(G) \to W'(G)$ for each G.

Proof: Since each loop of W(G) is a homomorphism $G \to W$, (a) follows. For the homomorphism in (b) take $(g, \phi) \mapsto \phi(g)$. Similarly, in (c) assign to each $g \in V(G)$ the map $\Phi_g \in W(W(G))$ defined by $\Phi_g(\phi) = \phi(g)$ for all $\phi \in W(G)$. An isomorphism for (d) is obtained by assigning to each vertex v of W the constant map $\phi^v \in W(G)$ which maps all vertices of G to v. To prove (e), let $f\colon G \times H \to W$ be a homomorphism and let, for each $h \in V(H)$, the mapping $f_h \in W(G)$ be defined by $f_h(g) = f(g, h)$ for all $g \in V(G)$; f_h will be called the map induced from f by h. A homomorphism $H \to W(G)$ is obtained by mapping each h to its f_h. Conversely, $H \to W(G)$ implies $G \times H \to G \times W(G) \to W$ by Lemma 1(b) and (b) above. Proving (f) note that if W is multiplicative and if $G \not\to W$ as well as $W(G) \not\to W$, then $G \times W(G) \not\to W$ contrary to (b); on the other hand if $G \nrightarrow W$ implies $W(G) \to W$, and if $G \nrightarrow W$, $H \nrightarrow W$, but $G \times H \to W$, then according to (e) $H \to W(G)$, which (taken together with $W(G) \to W$) contradicts $H \nrightarrow W$. To prove (g), let $f: G \to G'$ be a homomorphism; associate with each $\phi \in W(G')$ the mapping $\phi \circ f \in W(G)$.

3. DIGRAPHS

The directed path $\vec{P}_n$ has vertices $0, 1, \ldots, n$ and arcs $\overrightarrow{01}, \overrightarrow{12}, \ldots, \overrightarrow{(n-1)n}$. The directed cycle $\vec{C}_n$ has vertices $0, 1, \ldots, n-1$ and arcs $\overrightarrow{01}, \overrightarrow{12}, \ldots, \overrightarrow{(n-2)(n-1)}, \overrightarrow{(n-1)0}$. The transitive tournament TT_n has vertices $0, 1, \ldots, n-1$ and all arcs $\overrightarrow{ij}$ with $i < j$.

We first deal with the case of transitive tournaments which are simple. This case is illustrative of the method used throughout this section.

EXAMPLE 1. Each transitive tournament TT_n is multiplicative. We shall in fact observe that $\mathcal{O} = \{\vec{P}_k : k \geqq n\}$ is a complete set of obstructions for TT_n. First of all, $\vec{P}_k \nrightarrow TT_n$ for $k \geqq n$ because the longest directed walk in TT_n has length $n - 1$; thus $\mathcal{O} \subseteq \{D: D \nrightarrow TT_n\}$. It also follows that $\vec{P}_k \to D$ for some $k \geqq n$ implies $D \nrightarrow TT_n$. The converse of this statement, which asserts the validity of (1) in the definition of a complete set of obstructions for TT_n, also holds: Assume that no $\vec{P}_k$ with $k \geqq n$ satisfies $\vec{P}_k \to D$. Then let $\alpha(v)$ denote the maximum length of a directed walk in D ending at the vertex v, for all $v \in V(D)$; by the assumption, $\alpha(v)$ is well defined and smaller than n. Then α is a homomorphism $D \to TT_n$. Thus $D \nrightarrow TT_n$ implies $\vec{P}_k \to D$ for some $k \geqq n$. To complete the proof that $\mathcal{O}$ is a complete set of obstructions for TT_n, note that if $k \geqq m$ then $\vec{P}_m \to \vec{P}_k$ as well as $\vec{P}_m \to \vec{P}_m$, verifying the condition (2). The rest follows from Lemma 3.

An <u>oriented</u> path (cycle, walk) is a digraph obtained from an undirected path (cycle, walk respectively) by choosing one direction for each edge. The <u>net length</u> of an oriented walk (e.g. path or cycle) is the absolute value of the difference between the number of edges directed forward, and the number of edges directed backward, with respect to any particular traversal of the walk, cf. Figure 1.

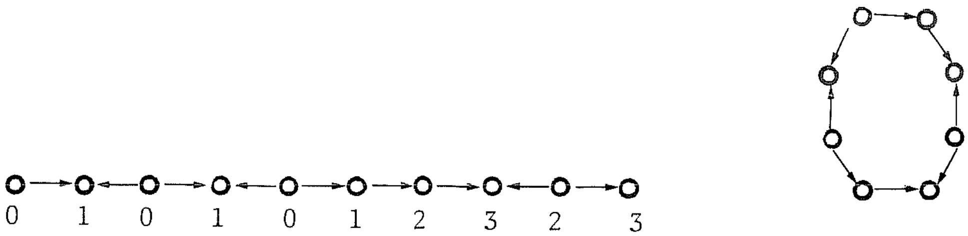

FIGURE 1. A (minimal oriented path of net length 3 and an oriented cycle of net length 0.

An oriented path P of net length k is <u>minimal</u> if no subpath (i.e., subgraph which is a path) of P has net length strictly greater than k; note that there could be subpaths of net length k, cf. Figure 1 again. The <u>level</u> of a vertex v in an oriented path P is defined as follows: Of the two traversals of P choose one in which the number of forward arcs is not smaller than the number of backward arcs; the level of v is then the net length of the initial segment of P, up to v. We have marked the level of all vertices on the path in Figure 1.

<u>LEMMA 5</u>. (a) $P \not\to \vec{P}_n$ for any oriented path P of net length k, $k \geqq n + 1$.

(b) An oriented path P of net length k is minimal if and only if the level of any vertex v of P is at least 0 and at most k.

(c) An oriented path P of net length k is minimal if and only if $P \to \vec{P}_k$.

(d) $D \not\to \vec{P}_n$ if and only if $P \to D$ for some oriented path P of net length $k = n + 1$.

(e) $C \not\to \vec{C}_n$ for any oriented cycle C of net length k, $k \not\equiv 0 \pmod{n}$.

(f) $D \not\to \vec{C}_n$ if and only if $C \to D$ for some oriented cycle C of net length $k \not\equiv 0 \pmod{n}$.

Proof: (a) and (b) are easy to see, and they imply (c) (a homomorphism $P \to \vec{P}_k$ is obtained by mapping v to its level). Moreover (a) implies the "if" part of (d); to see the "only if" part assume that D has no oriented walk of net length $k = n + 1$ (this being equivalent to the assumption that $P \not\to D$ for any oriented path P of net length $k = n + 1$). Now we can define a homomorphism $f: D \to \vec{P}_n$ by letting $f(v)$ be the maximum net length of an oriented path ending at v. The proofs of (e) and (f) are similar and shall be omitted.

THEOREM 1. The set $\mathcal{O}$ of all oriented paths of net length $n + 1$ is a complete set of obstructions for $\vec{P}_n$.

COROLLARY 1. Each directed path $\vec{P}_n$ is multiplicative.

Proof: It remains to show that for any two oriented paths P, P' of net length k there is an oriented path P^* of net length k such that $P^* \to P$ and $P^* \to P'$; taking $k = n + 1$ and Lemma 5(d) proves the Theorem, and the Corollary follows by Lemma 3. Moreover, it will be sufficient to prove this for minimal paths P, P' of net length k, as any oriented path of net length k contains a minimal one. Assume that $P = p_0 \, p_1 \, \dots \, p_a$ and $P' = p'_0 \, p'_1 \, \dots \, p'_b$ and that P and P' traversed in the listed order have at least as many forward arcs as backward arcs. Note that by minimality $\overrightarrow{p_0p_1}$ and $\overrightarrow{p'_0p'_1}$ are both forward arcs. Let $\ell(v)$ denote the level of the vertex v of P or P'.

CLAIM 1. There exists an oriented path $P^* = p_0^* p_1^* \dots p_c^*$ of net length k such that $P^* \to P$ and $P^* \to P'$; moreover the homomorphisms $P^* \to P$ and $P^* \to P'$ could be chosen in such a way that p_0^* is mapped to p_0 and p_0', and p_c^* is mapped to p_a and p_b'.

Claim 1 will be proved by induction on k. When $k = 0$, P and P' are single points (by minimality) and everything is trivial. Assume then that the claim holds for $k - 1$ and proceed to prove it for k; here again we use induction, this time on q, the total number of vertices v in P and P' with $\ell(v) = 0$. Clearly $q \leqq 2$, and if $q = 2$ then all vertices of P and P' except for p_0 and p_0' have level at least 1. The levels in the oriented paths $P - p_0$ and $P' - p_0'$ are one less than the corresponding levels in P and P' respectively, and hence $P - p_0$ and $P' - p_0'$ are both minimal paths of net length $k - 1$ (cf. Lemma 5(b)). By induction on k, there is a path $P_1^* = p_1^* \dots p_c^*$ of net length $k - 1$ with $P_1^* \to P$ and $P_1^* \to P'$, the homomorphisms taking p_1^* to p_1 and to p_1', and p_c^* to p_a and p_b'. Then $\overrightarrow{p_0^* p_1^*}$ concatenated with P_1^* defines a P^* satisfying the claim. Now suppose that the claim holds for $k - 1$ and all smaller values of q. Without loss of generality, let P contain a vertex p_r with $r \geqq 1$ and $\ell(p_r) = 0$, and among all such subscripts let r be the greatest. Let $\ell = \max \ell(p_i)$ over all i, $0 \leqq i \leqq r$, and let s be the greatest subscript such that $1 \leqq s \leqq r$ and $\ell(p_s) = \ell$. Finally, let t be the smallest subscript such that $\ell(p_t') = \ell$. The paths $P_1 = p_0\, p_1 \dots p_s$ and $P_1' = p_0'\, p_1' \dots p_t'$ are minimal of net length $\ell \leqq k$ and have a total number of level - 0 - vertices strictly smaller than q (p_r is missing). Therefore the induction hypothesis applies and yields a path $P_1^* = p_0^* \dots p_u^*$ of net length ℓ and homomorphisms $P_1^* \to P_1$, $P_1^* \to P_1'$ mapping p_0^* to p_0 and p_0', and p_u^* to p_s and p_t'. The paths $P_2 = p_r\, p_{r-1} \ldots p_s$ and $P'_2 = p'_0\, p'_1 \ldots p'_t$ are also minimal of net length ℓ, and the induction hypothesis again applies, yielding a path $P^*_2 = p^*_w \ldots p^*_u$ of net length ℓ and homomorphisms $P^*_2 \to P_2$, $P^*_2 \to P'_2$ taking p^*_w to p_r and p'_0, and p^*_u to p_s and p'_t. Finally, the paths $P_3 = p_r\, p_{r+1} \ldots p_a$ and $P'_3 = p'_0\, p'_1 \ldots p'_b$ are minimal paths of net length k with the number of level-0 - vertices strictly smaller than q (p_0 is missing). Thus by induction on q there is a path $P^*_3 = p^*_w \ldots p^*_c$ of net length k and two homomorphisms $P^*_3 \to P_3$, $P^*_3 \to P'_3$ taking p^*_w to p_r and p'_0, and p^*_c to p_a and p'_b. Then $P^* = p^*_0 \ldots p^*_u \ldots p^*_w \ldots p^*_c$ ($= P^*_1 + (-P^*_2) + P^*_3$) satisfies the claim.

Note that in fact the set $\mathcal{O}'$ of all minimal oriented paths of net length $n + 1$ is a complete set of obstructions for $\vec{P}_n$.

Before attempting to apply a similar reasoning to directed cycles, we must observe that not all directed cycles are multiplicative: if n is not a prime power then $\vec{C}_n$ is not multiplicative. Indeed, if n is not a prime power then it can be written as $n = mk$ with coprime m and k satisfying $m > 1$, $k > 1$. Then $\vec{C}_n$ is isomorphic to $\vec{C}_m \times \vec{C}_k$, one such isomorphism being given by $i \mapsto (i, i)$ with the first component reduced mod m and the second component reduced mod k. (Note that $(i, i) = (0, 0)$ if and only if i is a common multiple of m and k, i.e., a multiple of n.) Since neither of m, k divides the other, we have $\vec{C}_m \not\to \vec{C}_k$ and $\vec{C}_k \not\to \vec{C}_m$, as observed earlier; hence Lemma 2(a) implies that $\vec{C}_n \cong \vec{C}_m \times \vec{C}_k$ is not multiplicative.

THEOREM 2. Let n be a prime power. Then the set $\mathcal{O}$ of all oriented cycles of net length k, $k \not\equiv 0 \pmod n$, is a complete set of obstructions for $\vec{C}_n$.

COROLLARY 2. $\vec{C}_n$ is multiplicative if and only if n is a prime power.

Proof: In view of Lemma 5(e, f) and Lemma 3, it only remains to prove that for any oriented cycles C and C' of net lengths $k \not\equiv 0 \pmod n$ and $k' \not\equiv 0 \pmod n$ respectively, there exists an oriented cycle C^* of net length $k^* \not\equiv 0 \pmod n$ such that $C^* \to C$ and $C^* \to C'$. Let k^* be the least common multiple of k and k'. Then $k^* \not\equiv 0 \pmod n$: Otherwise $n = p^a$ divides $k^* = kk'/\gcd(k, k')$. Let p^b be the highest power of p which divides both k and k'; then $b < a$, p^b divides $\gcd(k, k')$, and, without loss of generality, k/p^b is not divisible by p. Therefore $n = p^a$ divides k', a contradiction. Since k^* is some xk, there exists an oriented cycle $\tilde{C}$ of net length k^* such that $\tilde{C} \to C$; one such $\tilde{C}$ may be obtained by going x times around C. Similarly, there exists an oriented cycle $\tilde{C}'$ of net length k^* such that $\tilde{C}' \to C'$.

CLAIM 2. There exists an oriented cycle $C^* \subseteq \tilde{C} \times \tilde{C}'$ of net length k^*.

(It may appear at first that Claim 1 should imply Claim 2. However, the paths P, P' in Claim 1 are assumed to be minimal; this can always be assumed for paths, but not for cycles - hence no obvious application of Claim 1 proves Claim 2, and we in fact appeal to a result from homology theory to prove it - cf. below.) Once Claim 2 has been verified, $C^* \to \tilde{C} \times \tilde{C}' \to \tilde{C} \to C$ and $C^* \to \tilde{C} \times \tilde{C}' \to \tilde{C}' \to C'$ as required. To prove the claim, we first note that there exist in $\tilde{C} \times \tilde{C}'$ closed walks (and hence also cycles) of non-zero net length: Since the net length k^* of $\tilde{C}$ and $\tilde{C}'$ is not zero, there exist oriented paths $\tilde{P}$, $\tilde{P}'$ of arbitrarily high net length ℓ such that $\tilde{P} \to \tilde{C}$ and $\tilde{P}' \to \tilde{C}'$ (obtained by going around $\tilde{C}$ and $\tilde{C}'$ arbitrarily many times). By Claim 1, there exists a path $\tilde{P}^*$ of the same net length ℓ such that $\tilde{P} \to \tilde{C}$ and $\tilde{P}' \to \tilde{C}'$ (obtained by going around $\tilde{C}$ and $\tilde{C}'$ arbitrarily many times). By Claim 1, there exists a path $\tilde{P}^*$ of the same net length ℓ with $\tilde{P}^* \to \tilde{P}$ and $\tilde{P}^* \to P'$. Then $\tilde{P}^* \to \tilde{C}$ and $\tilde{P}^* \to \tilde{C}'$, and hence $\tilde{P}^* \to \tilde{C} \times \tilde{C}'$. Thus $\tilde{C} \times \tilde{C}'$ has walks of arbitrarily high net length, which would be impossible if all closed walks had net length 0. Next we notice that the graph $\tilde{C} \times \tilde{C}'$ is embedded on the torus T whose generating cycles are $\tilde{C}$ and $\tilde{C}'$ viewed as topological spaces. Any cycle C^* of $\tilde{C} \times \tilde{C}'$ is a simple closed polygon in T. The projection of C^* to $\tilde{C}$ is a closed walk in $\tilde{C}$ and hence it winds around $\tilde{C}$ a certain number, say x, of times; note that the net length of C^* is xk^*. Similarly, the projection of C^* to $\tilde{C}'$ winds y times around $\tilde{C}'$ and the net length of C^* is yk^*; hence $x = y$. A consequence of the Lefschetz duality theorem of homology theory (e.g. Proposition 9.22 in [6]) asserts that a simple closed polygon in T winds either 0 times around each generating cycle, or p times around one generating cycle and q times around the other, with p and q coprime. Therefore we have $x = 0$ or $x = 1$. Since we have established above that not all cycles can have net length 0, there exist a cycle C^* with $x = 1$, i.e., net length k^*.

4. GRAPHS

All graphs in this section are understood to be undirected graphs, and all edges uv undirected udges $\{u, v\}$. The <u>path</u> P_n has vertices $0, 1, \ldots, n$ and edges $01, 12, \ldots, (n-1)n$; the <u>cycle</u> C_n has vertices $0, 1, \ldots, n-1$ and edges $01, 12, \ldots, (n-2)(n-1), (n-1)0$. We shall show that all paths and cycles are multiplicative; the undirected analogues of transitive tournaments, the complete graphs, are conjectured to be multiplicative in [8]. First we dispose of the trivial case of paths and even cycles.

EXAMPLE 2. Each path P_n and each even cycle C_{2m} is multiplicative. We observed in Section 2 that $P_1 \cong K_2$ is multiplicative; moreover P_1 is a retract of each P_n and each C_{2m} via the retraction that maps all even vertices to 0 and all odd vertices to 1. Thus each P_n and each C_{2m} is multiplicative by Lemma 2(b).

The situation is much less transparent in the case of odd cycles. Our proof of their multiplicativity (Theorem 3) closely parallels the proof of El-Zahar and Sauer [4] showing the multiplicativity of $K_3 \cong C_3$. Where the extensions are evident we abbreviate and appeal to their paper for greater detail.

In what follows we shall be considering the graph $C_n(C_k)$ with an odd n, $n \geqq 5$; recall that its vertices are mappings (not necessarily homomorphisms) of $V(C_k)$ to $V(C_n)$. Let ϕ be such a mapping: We shall say that a vertex v of C_k is a j-point of ϕ if j is the unique integer $0 \leqq j < n/2$ such that $|\phi(v') - \phi(v'')| \equiv j \pmod{n}$, where v' and v'' are the two neighbours of v in C_k. We shall say that an edge uv of C_k has length j with respect to ϕ if j is the unique integer $0 \leqq j < n/2$ such that $|\phi(u) - \phi(v)| \equiv j \pmod{n}$. Note that if ϕ from $C_n(C_k)$ has a j-point v on C_k with $j \neq 0, 2$ then ϕ is an isolated vertex of $C_n(C_k)$, because any ϕ' adjacent to ϕ must have the value $\phi'(v)$ adjacent in C_n to $\phi(v')$ and to $\phi(v'')$. We shall say that a 2-point v of ϕ is filled if $\phi(v)$ is the unique value between $\phi(v')$ and $\phi(v'')$; otherwise it is unfilled.

LEMMA 6. Let n be odd and $n \geqq 5$, and suppose that ϕ from $C_n(C_k)$ has no j-points with $j \neq 0, 2$. Then each unfilled 2-point v of ϕ satisfies:

(a) If $n \equiv 1 \pmod{4}$ then v is incident with exactly one edge of C_k of length congruent to 2 or 3 (mod 4).

(b) If $n \equiv 3 \pmod 4$ then v is incident with exactly one edge of C_k of length congruent to 1 or 2 (mod 4).

Proof: The lengths (understood to be with respect to ϕ) j and j' of the two edges incident on an unfilled 2-point satisfy $j \pm j' \equiv \pm 2 \pmod n$ and $0 \leqq j, j' < \frac{n}{2}$. Of the four possibilities inherent in the $\pm$ notation, $j + j' \equiv 2 \pmod n$ implies $j = j' = 1$ - contrary to the definition of an unfilled 2-point; $j - j' \equiv 2 \pmod n$ (or equivalently $j' - j \equiv -2 \pmod n$) implies $j - j' = 2$ - which is only possible if exactly one of j, j' is congruent to 2 or 3 modulo 4, and exactly one congruent to 1 or 2 modulo 4; and finally $j + j' \equiv -2 \pmod n$ implies $j + j' = n - 2$: if $n - 2 \equiv 3 \pmod 4$, this is only possible if exactly one of j, j' is congruent to 2 or 3 modulo 4, and if $n - 2 \equiv 1 \pmod 4$, this is only possible if exactly one of j, j' is congruent to 1 or 2 modulo 4.

COROLLARY 6. Let $n \geqq 5$ be odd, and suppose that ϕ from $C_n(C_k)$ has no j-points with $j \neq 0, 2$. Then ϕ has an even number of unfilled 2-points on C_k.

Proof: Assume that $n \equiv 1 \pmod 4$ and consider the subgraph S of C_k consisting of all the edges of C_k with lengths (with respect to ϕ) congruent to 2 or 3 modulo 4. Then (a) implies that each unfilled 2-point of ϕ has degree 1 in S, while the degree of any filled 2-point, or of any 0-point, is even. Hence the number of unfilled 2-points is the number of odd degree vertices of S, and thus even. When $n \equiv 3 \pmod 4$, we use (b) in place of (a).

LEMMA 7. Let $n \geqq 5$ be odd. If $G \not\to C_n$ and if ϕ is a non-isolated vertex of $C_n(G)$, then ϕ has an even number of 2-points on some odd cycle C_k of G.

Proof: Let $\phi\phi'$ be an edge of $C_n(G)$, and let X be the set of vertices $v \in V(G)$ with $\phi(v)\phi(w) \notin E(C_n)$ for some neighbour w of v in G. It can be seen that X contains an odd cycle C_k of G. (If X induced a bipartite subgraph of G, then the mapping equal to ϕ on $(G-X) \cup$ (one part of X) and equal to ϕ' on the other part of X, would be a homomorphism $G \to C_n$, cf. Proposition 4.1 in [4].) The restriction of ϕ to C_k is not isolated in $C_n(C_k)$ - the restriction of ϕ' is adjacent to it - and so Corollary 6 applies, yielding the following conclusion: If ϕ has an odd number of 2-points on C_k then it has an odd number of filled 2-points, and thus at least one, say v. It would follow that $\phi'(v) = \phi(v)$ and so $\phi'(v)\phi(w) \notin E(C_n)$ for some neighbour w of v in G - a contradiction.

THEOREM 3. Each cycle C_n is multiplicative.

Proof: In view of Example 2 and [4] we may assume that n is odd and $n \geqq 5$. The following claim will establish the Theorem (cf. Lemma 4(f)):

CLAIM 3. If $G \not\to C_n$ then $C_n(G) \to C_n$.

Thus let $G \not\to C_n$ and let F be any non-trivial component of the graph $C_n(G)$. Let ϕ_0 be a vertex of F and let C_k be an odd cycle of G on which ϕ_0 has an even number of 2-points (cf. Lemma 7). We now show that all $\phi \in F$ have the same parity of the number of 2-points on C_k. We may assume without loss of generality that ϕ and ϕ' are adjacent in F. The homomorphism $C_k \times C_n(C_k) \to C_n$ from Lemma 4(b) restricted to $C_k \times \{\phi, \phi'\}$ is then a homomorphism $C_k \times K_2 \cong C_{2k} \to C_n$, and any such homomorphism has an even number of 2-points. (This is easily seen by considering how the homomorphism must "wind" the even cycle C_{2k} around the odd cycle C_n, or proved by induction as in Lemma 3.2 of [4].) Since the homomorphism of Lemma 4(b) (cf. its proof) maps $C_k \times \{\phi, \phi'\}$ to C_n by taking (v, ϕ) to $\phi(v)$ and (v, ϕ') to $\phi'(v)$, each of its 2-points is either some (v, ϕ) where v is a 2-point of ϕ' on C_k, or some (v, ϕ') where v is a 2-point of ϕ on C_k. Thus the number of 2-points of ϕ plus the number of 2-points of ϕ' is the number of 2-points of a homomorphism $C_{2k} \to C_n$, and hence an even number. Therefore the parity of the number of 2-points of ϕ and ϕ' is the same.

Since we assumed that ϕ_0 has an even number of 2-points on the odd cycle C_k of G, we know now that each element ϕ of F has also an even number of such points. Let H be the subgraph of $C_n(C_k)$ whose vertices are the elements ϕ of F restricted to C_k.

CLAIM 4. There exists a non-isolated vertex ψ of $C_n(H)$ which has an odd number of 2-points on each odd cycle of H.

First we note that Claim 4, taken together with Lemma 7 establish that $F \to H \to C_n$. (to obtain a homomorphism $F \to H$, associate with each element of F its restriction to C_k.) This is all that is needed to complete the proof of Claim 3 (and so of the Theorem), because we can repeat this argument for each non-trivial component of $C_n(G)$; the trivial components admit a homomorphism to C_n because of Lemma 4(a).

Let v be a fixed vertex of C_k and let ψ be the mapping of $V(H)$ to $V(C_n)$ which assigns to each ϕ in H the value $\phi(v)$. Note that ψ is a non-isolated vertex of $C_n(H)$; it is adjacent to the mapping resulting from having fixed some v' adjacent to v. Let C_m by any odd cycle of H. Then $C_k \times C_m$ is a subgraph of $C_k \times C_n(C_k)$ and hence there is a homomorphism

$f: C_k \times C_m \to C_n$; in fact $f(v, \phi) = \phi(v)$, cf. Lemma 4(b). Thus each mapping $f_\phi: V(C_k) \to V(C_n)$ induced (cf. the proof of Lemma 4(e) for the definition) from f by $\phi \in C_m$ is just ϕ, and the mapping $f_v: V(C_m) \to V(C_n)$ induced from f by $v \in C_k$ is just ψ restricted to C_m. We now have a homomorphism $f: C_k \times C_m \to C_n$ such that all induced f_ϕ, $\phi \in C_m$, have an even number of 2-points (cf. the discussion preceeding Claim 4); similarly, all unduced f_u, $u \in C_k$, also have the same parity of their number of 2-points. We wish to conclude that this parity is odd, i.e., that all f_u, $u \in C_k$, (and in particular f_v) have an odd number of 2-points on C_m. This can be proved as in [4, Proposition 3.4], and we only give a sketch of the proof here. The quadrilaterals of $C_k \times C_m$ (i.e., the four-cycles (ϕ^-, u), (ϕ, u^+), (ϕ^+, u), (ϕ, u^-), where u^- is the predecessor and u^+ the successor of u on C_m and ϕ^- the predecessor, ϕ^+ the successor of ϕ on C_k) can only have the following values of f:

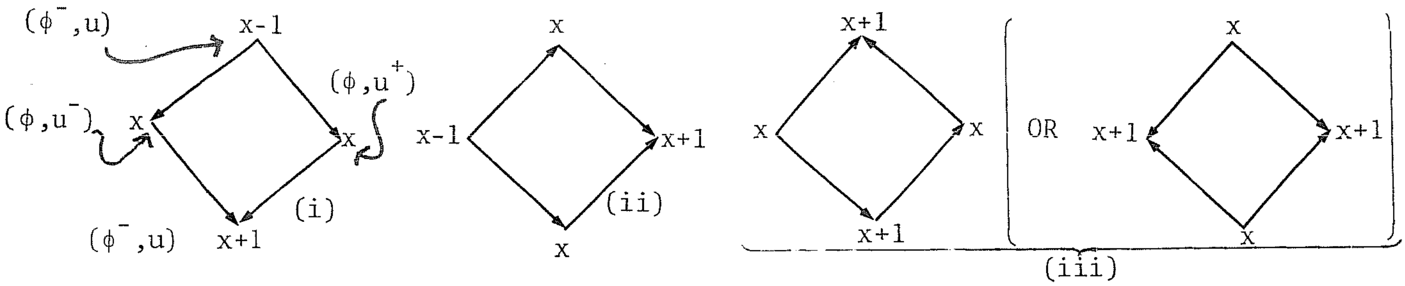

FIGURE 2.

In the figures (i - iii), x denotes any value $0, 1, \ldots, n-1$ and the operations are modulo n; the directions of the edges are explained below. Note that in (i) ϕ is a 2-point of f_u, in (ii) u is a 2-point of f_ϕ, and in (iii) neither is a 2-point. Hence the sum of all numbers of 2-points of all f_u and f_ϕ is the number of quadrilaterals of type (i) and (ii). Next we direct the edges of $C_k \times C_m$ so that edges joining vertices with values x and $x+1$ are directed from x to x+1, as the figures show. Note that opposite pairs of edges in the quadrilaterals (i) and (ii) are directed in the same direction, while opposite edges in (iii) have opposite directions. Hence in each diagonal band of quadrilaterals in $C_k \times C_m$ there is an even number of quadrilaterals of type (iii); therefore the total number of quadrilaterals of type (iii) is even. Thus the sum of all numbers of 2-points of all f_u and all f_ϕ is odd, and consequently each f_u, $u \in C_k$, has an odd number of 2-points on C_m.

5. FURTHER REMARKS

Because they may be of independent interest, we restate Claims 1 and 2:

PROPOSITION 1. Any product of oriented paths (respectively of oriented cycles) of net length k contains an oriented path (respectively cycle) of net length k.

We have so far only admitted finite graphs. For infinite graphs, Hajnal has studied the multiplicativity of complete graphs, [9]. Undirected one-way and two-way infinite paths are multiplicative for the reason explain in Example 2. Directed one-way and two-way infinite paths and countable tournaments are also multiplicative. Let $\vec{P}_\omega$ denote the digraph with vertices 0, 1, 2, ..., and arcs $\overrightarrow{i(i+1)}$ (i=0,1,...); let $\vec{C}_\omega$ denote the digraph with vertices ..., -2, -1, 0, 1, 2, ... and arce $\overrightarrow{i(i+1)}$ ($i \in Z$); let TT_ω denote the digraph with vertices 0, 1, 2, ..., and all arcs $\overrightarrow{ij}$ with $i < j$. Let S_ω denote the digraph obtained from $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$, ..., by identifying their endpoints to a common vertex, and let $\mathcal{U}$ be the class of all digraphs obtained the same way from any family of oriented paths P(1), P(2), P(3), ..., with the property that the net length of P(i) is i.

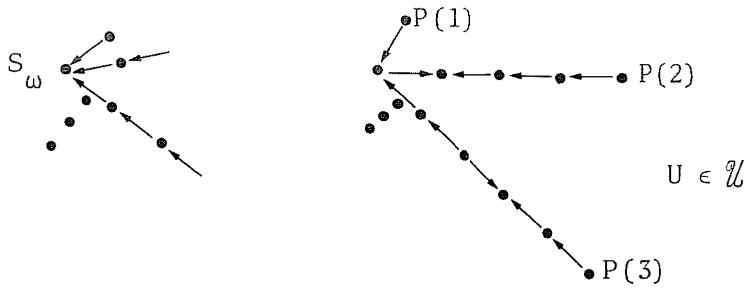

FIGURE 3.

PROPOSITION 2. (a) $D \not\to \vec{P}_\omega$ if and only if $U \to D$ for some $U \in \mathcal{U}$.

(b) $D \not\to \vec{C}_\omega$ if and only if $C \to D$ for some oriented cycle C of non-zero net length.

(c) $D \not\to TT_\omega$ if and only if $S_\omega \to D$.

The proofs are very similar to those of Lemma 5(d, f) and Example 1. For instance to prove (b) we first note that every closed walk of $\vec{C}_\omega$ has net length 0 and so $C \not\to \vec{C}_\omega$ for any oriented cycle C of non-zero net length. Thus $C \to D$ implies $D \not\to \vec{C}_\omega$. On the other hand, if no oriented cycle C of non-zero net length has $C \to D$ then D contains no closed walks of non-zero net length and any two walks from v to w have the same net length. We may assume that D is connected and define $f(v) = 0$ for some vertex v of D, and $f(w)$ = the net length of any path from v to w, for all other vertices w of D. Then f is a homomorphism $D \to \vec{C}_\omega$.

Proposition 2 yields obvious complete sets of obstructions for $\vec{P}_\omega$, $\vec{C}_\omega$, and TT_ω. In the product of U and U' from $\mathcal{U}$, there is a special vertex at which end oriented walks of net lengths 1, 2, ... (cf. Proposition 1); thus some $U^* \in \mathcal{U}$ has $U^* \to U'$. If C and C' are oriented cycles of non-zero net lengths k and k', then there exist oriented cycles $\tilde{C}$ and $\tilde{C}'$ of net length kk' such that $\tilde{C} \to C$ and $\tilde{C}' \to C'$, and some C^* of net length $kk' \neq 0$ has $C^* \to \tilde{C} \to C$ and $C^* \to \tilde{C}' \to C'$ by Proposition 1. Therefore

COROLLARY. $\vec{P}_\omega$, $\vec{C}_\omega$, and TT_ω are multiplicative.

Before leaving the subject of infinite graphs we note that if infinite products are allowed then even K_2 is not multiplicative - it was observed in [15] that $\prod_{k\geq 1} C_{2k+1} \to K_2$ while of course each $C_{2k+1} \not\to K_2$.

Returning now to the multiplicativity of finite graphs, note that Lemma 2(b) allows us to restrict out attention to graphs which have no proper retracts; we shall use the term core to describe such a graph. Equivalently, a core is a graph H with $H \not\to G$ for any proper subgraph G, or a graph H in which each homomorphism $H \to H$ is an isomorphism (and hence the set of all homomorphisms $H \to H$ forms a group under composition, cf. [14]). Cores were studied in [14] (although the term core was not used in the published version) and in [5] (where they were termed "minimal graphs"). Let W be a core and W_1, W_2 two subgraphs of G isomorphic to W: Since any homomorphism $W_1 \to W_2$ is an isomorphism, W_1 is a retract of G if and only if W_2 is. We let $W \mathrel{\triangleleft\!\!|} G$ denote the fact that W is a subgraph of G but not a retract of G. Note that $W \mathrel{\triangleleft\!\!|} G$ implies $G \not\to W$ because any homomorphism $G \to W$ restricted to W is a homomorphism $W \to W$ and hence an automorphism a of W; then $a^{-1}f$ would be a retraction of G to W. The following observation is due to Emo Welzl (personal communication):

PROPOSITION 3. Let W be a core. Then W is multiplicative if and only if

(*) $\quad W \mathrel{\triangleleft\!\!|} G$ and $W \mathrel{\triangleleft\!\!|} H$ imply that $W \mathrel{\triangleleft\!\!|} G \times H$.

Proof: If W is multiplicative and if $W \mathrel{\triangleleft\!\!|} G$, $W \mathrel{\triangleleft\!\!|} H$, then $G \times H \not\to W$; at the same time W is a subgraph of $W \times W$ which is a subgraph of $G \times H$. Thus $W \mathrel{\triangleleft\!\!|} G \times H$. On the other hand, if $G \not\to W$ and $H \not\to W$ then $W \mathrel{\triangleleft\!\!|} G \cup W$ and

$W \nleftarrow H \cup W$. If this implies that $W \nleftarrow (G \cup W) \times (H \cup W) = (G \times H) \cup (G \times W) \cup (W \times H) \cup (W \times W)$, then $G \times H \nrightarrow W$ as above.

It is interesting to note that while connectivity of G and H makes no difference in the definition of multiplicativity of W, it seems to make things easier in (*): In [3] and in [18] it was shown that $K_n \nleftarrow G$ and $K_n \nleftarrow H$ for connected graphs G and H implies that $K_n \nleftarrow G \times H$.

In a recent paper of G. Bloom and S. Burr [2] the results in part of Example 1 and in Lemma 5(d, f) are independently derived and used to give polynomial algorithms for testing if $D \to TT_n$, $D \to \vec{P}_n$, and $D \to \vec{C}_n$ for an input digraph D.

Since the preliminary version of our paper we have become aware of the reference [19]. There the authors also study, among other topics, the multiplicativity of graphs (they use the term "productivity"). While they also obtained Example 1, and Lemma 5(d-f), the only significant overlap is our Corollary 1 - the multiplicativity of directed paths. Corollary 2, the multiplicativity of directed cycles of prime power length is also discussed, but only proved in the prime case; thus our Corollary 2 answers the general case conjectured there. Two other problems posed in [19] are solved by our Theorem 3; namely (1) are there infinitely many undirected multiplicative graphs, and (2) is there a non-complete undirected multiplicative graph. Some other results in [19] are related to complete sets of obstructions - e.g., they imply that no undirected graph other than K_1 has a <u>finite</u> complete set of obstructions.

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