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ON MULTIPLY TRANSITIVE PERMUTATION GROUPS II

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Introduction

This note is a continuation of [1]. The purpose of the present note is to prove the following theorem.

Main Theorem. *Let p be an odd prime. Let G be a $2p$-ply transitive permutation group on a set $\Omega=\{1, 2, \cdots, n\}$. If the order of $G_{1,2,\cdots,2p}$ is not divisible by p, then G must be one of $S_n(2p\leqslant n\leqslant 3p-1)$ and $A_n(2p+2\leqslant n\leqslant 3p-1)$ on their natural action.*

Some parts of the main theorem were already proved in Miyamoto [4] and Bannai [1]. Namely, Miyamoto proved in [4] that if G satisfies the assumption of Main Theorem and if the order of $G_{1,2,\cdots,p}$ is divisible by p only to the first power, then G must be one of the groups listed in the conclusion of Main Theorem. On the other hand, Theorem 1 in [1] asserts that if G satisfies the assumption of Main Theorem and if the order of $G_{1,2,\cdots,p}$ is divisible by p^2, then $n\equiv 0$ or 1 (mod p). Therefore, in order to complete the proof of Main Theorem, we have only to prove the following theorem.

Theorem 1. *Let p be an odd prime. Then there exists no permutation group G on a set Ω which satisfies the following three conditions:*
(i) *G is $2p$-ply transitive on Ω, and $G\not\geq A^{\Omega}$,*
(ii) *the order of $G_{1,2,\cdots,2p}$ is not divisible by p, and*
(iii) *$n\equiv 0$ or 1 (mod p).*

The idea of the proof of Theorem 1 is due to Livingston and Wagner [3] and Oyama [5, 6].

1. Proof of Theorem 1

Let us assume that G satisfies the assumptions of Theorem 1. Let R be a Sylow 2 subgroup of $G_{1,2,\cdots,2p}$, and let Δ be one of the orbits of R on $\Omega-I(R)$ of minimal length, where $I(R)$ denotes the set of the elements of Ω which are

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fixed by any element of R. (From the assumption of $G\not\geq A^{\Omega}$ and from the $2p$-ply ($\geqslant 6$-ply) transitivity of G, we obtain $\Omega\neq I(R)$ by applying a result of Hall [2].) Let t be an element of Δ.

(1) $N_G(R_t)^{I(R_t)}$ satisfies the following condition: let $i_1, i_2, \cdots, i_{2p}$ be any (distinct) $2p$ elements in $I(R_t)$. Let S be a Sylow 2 subgroup of $N_G(R_t)^{I(R_t)}_{i_1,i_2,\cdots,i_{2p}}$. Then S fixes $|I(R)|$ (set$=2p+r$) elements on $I(R_t)$, and is semiregular on the remaining elements of $I(R_t)$.

This is essentially proved in Oyama [5, Lemma 1].

(2) $r=0$ or 1. If $r=0$, then $|I(R_t)|=2p+2$ or $2p+4$. If $r=1$, then $|I(R_2)|=2p+3$ or $2p+5$.

Since G is $2p(\geqslant 6)$-ply transitive, we obtain that $r=0$ or 1, by applying a result of Hall [2] (since M_{12} has no transitive extension). If $r=0$, then $|I(R_t)|= 2p+2$, $2p+4$ or $2p+8$, and if $r=1$, then $|I(R_t)|=2p+3$, $2p+5$ or $2p+9$, by Theorem 1 in Oyama [6]. (Notice that $N_G(R_t)^{I(R_t)-\{1,2,\cdots 2p-4\}}_{1,2,\cdots,2p-4}$ satisfies the assumptions of Theorem 1 in Oyama [6].) But the two cases $|I(R_t)|=2p+8$ and $2p+9$ are impossible, because otherwise $N_G(R_t)^{I(R_t)-\{1,2,\cdots,2p-5\}}_{1,2,\cdots,2p-5}$ is one of $S_1\times M_{12}$. $S_1\times S_1\times M_{12}$ and $S_2\times M_{12}$ (since M_{12} has no transitive extension), and this contradicts (1). That is, take $i_1=1, \cdots, i_{2p-5}=2p-5$ and $i_{2p-4}, i_{2p-3}, i_{2p-2}, i_{2p-1}$ and i_{2p} among the orbit corresponding to M_{12}.

(3) $N_G(R_t)^{I(R_t)}$ contains an element of order p which fixes more than $p+1$ points in $I(R_t)$.

Proof. If $r=0$, then $N_G(R_t)^{I(R_t)}$ is $2p$-ply transitive by Livingston and Wagner [3, Lemma 6]. Therefore, $N_G(R_t)^{I(R_t)}=S_{2p+2}$ or A_{2p+4}, and so we obtain the assertion. Now, let $r=1$. If $N_G(R_t)^{I(R_t)}$ is primitive, then it contains $A^{I(R_t)}$ and we obtain the assertion. Otherwise, since $N_G(R_t)$ contains an element of degree at most 4, we obtain $|I(R_t)|\leqslant 8$, by a well known and easily verified result about primitive groups of class ($=$minimal degree) 4 which was proved first by C. Jordan, and this is impossible. $N_G(R_t)^{I(R_t)}$ is not imprimitive. Otherwise, there exists a system of imprimitivity $\Pi_1, \Pi_2, \cdots, \Pi_u$ ($u\geqslant 3$, since $|I(R_t)|=ub$ is odd, where $|\Pi_i|=b$.) Now, if we take $2p$ elements $i_1, i_2, \cdots, i_{2p}$ in such a way that just $(b-1)$ elements of Π_1 and Π_2 are contained in the set respectively, we have a contradiction to (1). Now, let us assume that $N_G(R_t)^{I(R_t)}$ is intransitive on $I(R_t)$. Let Σ be an orbit of $N_G(R_t)^{I(R_t)}$ of the minimal length (including 1). Then the setwise stabilizer of Σ in $N_G(R_t)^{I(R_t)}$ contains $A^{I(R_t)-\Sigma}$ (when restricted to $I(R_t)-\Sigma$) because of Livingston and Wagner [3, Lemma 4]. Therefore, since $|\Sigma|<|I(R_t)-\Sigma|$ (because $|I(R_t)|$ is odd), we obtain that the pointwise stabilizer of Σ in $N_G(R_t)$ also contains $A^{I(R_t)-\Sigma}$. Thus we immediately obtain the assertion.

(4) $N_G(R_t)$ (hence G) contains an element of order a power of p which fixes more than $p+1$ points in Ω.

This is obvious from (3).

But, the assertion (4) clearly contradicts the assumption of Theorem 1. Thus we have completed the proof of Theorem 1.

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Appendix (Added in January 1974)

As an application of the method which is used in the proof of Theorem 1, we prove the following result about 4-ply transitive permutation groups.

Theorem 2[1]**.** *Let G be a 4-ply transitive permutation group on a set $\Omega=\{1, 2,\cdots, n\}$. Let us assume that*
(i) $n\not\equiv 0 \pmod 6$, *and*
(ii) *the order of $G_{1,2,3,4}$ is not divisible by* 3.
Then G must be one of the groups S_4, S_5 and M_{11}.

Proof of Theorem 2. (a) First let us assume that $G_{1,3,2,4}$ is of odd order. Then by Hall [2], G must be one of the groups S_4, S_5, A_6, A_7 and M_{11}. Among them only S_4, S_5 and M_{11} satisfy the assumptions of Theorem 2.

(b) Next we assume that $G_{1,2,3,4}$ is of even order. Let R be a Sylow 2 subgroup of $G_{1,2,3,4}$, and let Δ be a minimal orbit of R on $\Omega-I(R)$. Moreover, let t be a point in Δ. By Oyama [5, 5′] $|I(R)|$ is one of 4, 5 and 7. We treat these three cases separately

(b–1) Let us assume that $|I(R)|=4$. Then, by a result of Oyama ([6, Theorem 1]), $N_G(R_t)^{I(R_t)}$ must be isomorphic to one of S_6, A_8 and M_{12}. From the assumption (i) that $n\not\equiv 0 \pmod 6$ (only here we use this assumption), we have $|I(x)|\leqslant 2$ for any element x of G whose order is a power of 3. While, S_6, A_8 and M_{12} contain an element y of order 3 such that the number of the fixed points of y on $I(R_t)$ is $\geqslant 3$. But, this is a contradiction.

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1) The author heard from Professor N. Ito that he had also proved some part of Theorem 2(i.e., under the assumption $n\equiv 0 \pmod 3$)) by a different method.

(b–2) Let us assume that $|I(R)|=5$. In this case we have $|I(x)|\leqslant 3$ for any element x of G whose order is a power of 3. Thus, in order to derive a contradiction, we have only to show that $N_G(R_t)^{I(R_t)}$ contains an element y of order 3 such that y fixes more than 3 points on $I(R_t)$. By Oyama [6, Theorem 1(II)], $|I(R_t)|$ is one of 7, 9 and 13. (b–2–1) If $|I(R_t)|=13$, then $N_G(R_t)^{I(R_t)}$ is $S_1\times M_{12}$ by the result of Oyama, and so we clearly have the assertion. (b–2–2) Next let us assume that $|I(R_t)|=9$. Then $N_G(R_t)^{I(R_t)}$ contains an element of degree 4. Therefore, if it is primitive then it contains $A^{I(R_t)}$, and we have the assertion. If $N_G(R_t)^{I(R_t)}$ is transitive and imprimitive, then Π_1, Π_2 and Π_3 is a system of imprimitivity, where $|\Pi_i|=3$ $(i=1, 2, 3)$. If we take two points i and j from Π_1 and two points k and l from Π_2, then $N_G(R_t)^{I(R_t)}_{i,j,k,l}$ fixes Π_1 and Π_2 pointwisely, hence fixes at least 6 points. This is a contradiction. Let us assume that $N_G(R_t)^{I(R_t)}$ is intransitive. Let Ω_1 be an orbit of $N_G(R_t)$ on $I(R_t)$ of minimal (including 1) length. If $|\Omega_1|=1$, then $N_G(R_t)^{I(R_t)-\Omega_1}$ is 4-ply transitive (by Livingston and Wagner [3, Lemma 6]), and it contains A_8. Thus we have the assertion immediately. If $|\Omega_1|=2$, then $N_G(R_t)^{I(R_t)-\Omega_1}$ is 3-ply transitive, and so it contains A_7. Thus we have the assertion immediately. Next let $|\Omega_1|=3$. We may assume without loss of generality that $\Omega_1=\{1, 2, 3\}$ and $I(R_t)-\Omega_1=\{4, 5, 6, 7, 8, 9\}$. Moreover, we may assume that $N_G(R_t)^{I(R_t)}$ contains an element a of order 2 such that

$$a=(1)\,(2, 3)\,(4)\,(5)\,(6)\,(7)\,(8, 9)\,.$$

If we take an element b in $N_G(R_t)^{I(R_t)}$ of order 2 which fixes 2, 7, 8 and 9, then $(ab)^2$ is of order 3 and fixes 6 points. Thus we have the assertion. Let $|\Omega_1|=4$. Also, we may assume that $\Omega_1=\{1, 2, 3, 4\}$ and $I(R_t)-\Omega_1=\{5, 6, 7, 8, 9\}$, and that $N_G(R_t)^{I(R_t)}$ contains an element a such that

$$a=(1, 2)\,(3)\,(4)\,(5)\,(6)\,(7)\,(8, 9)\,.$$

If we take an element b of order 2 of $N_G(R_t)^{I(R_t)}$ which fixes 1, 4, 8 and 9, then $(ab)^2$ is of order 3 and fixes 6 points. Thus we have the assertion, and we have completed the proof of the case (b–2–2). (b–2–3) Let $|I(R_t)|=7$. If $N_G(R_t)^{I(R_t)}$ is transitive, then we immediately have that it is isomorphic to S_7 and we have the assertion. On the other hand, when it is transitive, we obtain the assertion by using the similar (but more elementary) argument as in the proof in the case (b–2–2).

(b–3) Finally let us assume that $|I(R)|=7$. Then $N_G(R)^{I(R)}=A_7$ contains an element of order 3 which fixes 4 points in $I(R)$. This is a contradiction.

Thus, we have completed the proof of Theorem 2.

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