Research Article

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On multivalued Suzuki-type θ-contractions and related applications

https://doi.org/10.1515/math-2020-0139
received September 30, 2019; accepted March 11, 2020

Abstract: In this study, we develop the concept of multivalued Suzuki-type θ-contractions via a gauge function and established two new related fixed point theorems on metric spaces. We also discuss an example to validate our results.

Keywords: fixed point, θ-contraction, gauge function, metric space, multivalued mapping

MSC 2010: 47H10, 54H25

1 Introduction and preliminaries

Let $(\Bbbk, \widehat{d})$ be a metric space. For $\upsilon \in \Bbbk$ and $A \subseteq \Bbbk$, let $\widehat{d}(\upsilon_1, A) = \inf\{\widehat{d}(\upsilon_1, \upsilon_2): \upsilon_2 \in A\}$. Denote $N(\Bbbk)$, $CL(\Bbbk)$, $CB(\Bbbk)$, and $K(\Bbbk)$ by the class of all nonempty subsets of $\Bbbk$, the class of all nonempty closed subsets of $\Bbbk$, the class of all nonempty bounded closed subsets of $\Bbbk$, and the class of all nonempty compact subsets of $\Bbbk$, respectively. The Pompeiu-Hausdorff metric $\hat{H}$ induced by $\widehat{d}$ on $CL(\Bbbk)$ is defined as follows:

$$\hat{H}(A, B) = \max\left\{\sup_{\upsilon_1 \in A} \widehat{d}(\upsilon_1, B), \sup_{\upsilon_2 \in B} \widehat{d}(\upsilon_2, A)\right\},$$

for all $A, B \in CL(\Bbbk)$.

A point $\upsilon \in \Bbbk$ is said to be a fixed point of $\widetilde{T}: \Bbbk \to CL(\Bbbk)$, if $\upsilon \in \widetilde{T}\upsilon$. If, for $\upsilon_0 \in \Bbbk$, there exists a sequence $\{\upsilon_n\}$ in $\Bbbk$ such that $\upsilon_n \in \widetilde{T}\upsilon_{n-1}$, then $O(\widetilde{T}, \upsilon_0) = \{\upsilon_0, \upsilon_1, \upsilon_2, \ldots\}$ is said to be an orbit of $\widetilde{T}: \Bbbk \to CL(\Bbbk)$. A mapping $f: \Bbbk \to \mathbb{R}$ is said to be $\widetilde{T}$-orbitally lower semicontinuous, if $\{\upsilon_n\}$ is a sequence in $O(\widetilde{T}, \upsilon_0)$ and $\upsilon_n \to \rho$ implies $f(\rho) \leq \liminf_n f(\upsilon_n)$.

A multivalued mapping $\widetilde{T}: \Bbbk \to CL(\Bbbk)$ is called a Nadler contraction, if there is $\gamma \in (0, 1)$ such that

$$\hat{H}(\widetilde{T}\upsilon_1, \widetilde{T}\upsilon_2) \leq \gamma \widehat{d}(\upsilon_1, \upsilon_2), \quad \text{for all } \upsilon_1, \upsilon_2 \in \Bbbk.$$

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Nadler [1] acquired the variety of multivalued Banach contraction principle. Let $(\Bbbk, \widehat{d})$ be a complete metric space and $\widetilde{T}: \Bbbk \to CL(\Bbbk)$ be a Nadler contraction. Then, $\widetilde{T}$ possesses at least one fixed point. Thereafter, many researchers worked on existence of fixed point results for multivalued mappings satisfying different classes of contractive conditions [2–15]. Among them, Vetro [16] recently proved the following result.

Theorem 1.1. *Let $(\Bbbk, \widehat{d})$ be a complete metric space and $\widetilde{T}: \Bbbk \to CB(\Bbbk)$ be a multivalued mapping. Suppose that there exist $\theta \in \Xi$ and $k \in (0, 1)$ such that*

$$\upsilon_1, \upsilon_2 \in \Bbbk, \hat{H}(\widetilde{T}\upsilon_1, \widetilde{T}\upsilon_2) > 0 \Rightarrow \theta[\hat{H}(\widetilde{T}\upsilon_1, \widetilde{T}\upsilon_2)] \leq [\theta(\widehat{d}(\upsilon_1, \upsilon_2))]^k, \tag{1}$$

where Ξ is the set of functions $\theta: (0, \infty) \to (1, \infty)$ satisfying the following conditions:
(θ_{i}) *θ is nondecreasing and right continuous;*
(θ_{ii}) *for every $\{s_n\}$ in $(0, \infty)$, $\lim_{n\to\infty}\theta(s_n) = 1$ if and only if $\lim_{n\to\infty}s_n = 0$;*
(θ_{iii}) *there exist $r \in (0, 1)$ and $\ell \in (0, +\infty]$ such that $\lim_{s\to 0^+}\frac{\theta(s)-1}{s^r} = \ell$.*
Then, $\widetilde{T}$ possesses at least one fixed point.

Remark 1.2. Let $(\Bbbk, \widehat{d})$ be a metric space. If $\widetilde{T}: \Bbbk \to CL(\Bbbk)$ is a multivalued mapping satisfying (1.1), then

$$\ln\theta(\hat{H}(\widetilde{T}\upsilon_1, \widetilde{T}\upsilon_2)) \leq k\ln\theta(\widehat{d}(\upsilon_1, \upsilon_2)) < \ln\theta(\widehat{d}(\upsilon_1, \upsilon_2)).$$

Since θ is nondecreasing, we obtain

$$\hat{H}(\widetilde{T}\upsilon_1, \widetilde{T}\upsilon_2) < \widehat{d}(\upsilon_1, \upsilon_2), \quad \text{for all } \upsilon_1, \upsilon_2 \in \Bbbk, \widetilde{T}\upsilon_1 \neq \widetilde{T}\upsilon_2.$$

Example 1.3. The following functions $\theta_1, \theta_2: (0, \infty) \to (1, \infty)$ defined by $\theta_1(r) = e^{\sqrt{r}}$ and $\theta_2(r) = 1 + \sqrt{r}$ are in Ξ.

Lemma 1.4. [16]. *Let $(\Bbbk, \widehat{d})$ be a metric space and $A, B \in CL(\Bbbk)$ with $\hat{H}(A, B) > 0$. Then, for every $h > 1$ and $\upsilon \in A$, there exists $v = v(\upsilon) \in B$ such that*

$$\widehat{d}(\upsilon, v) < h\hat{H}(A, B).$$

Lemma 1.5. *Let $(\Bbbk, \widehat{d})$ be a metric space, $B \in CL(\Bbbk)$ and $\upsilon \in \Bbbk$. Then, for each $\varepsilon > 0$, there exists $v \in B$ such that*

$$\widehat{d}(\upsilon, v) \leq \widehat{d}(\upsilon, B) + \varepsilon.$$

In [17], the following family of mappings is considered:

$$\Phi = \left\{\psi: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} | \psi \text{ satisfies } \psi(r_1, r_2) \leq \frac{1}{2}r_1 - r_2\right\}.$$

The following functions ψ_1 and ψ_2 are elements of Φ:
(i) $\psi_1: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $\psi_1(r_1, r_2) = v(r_1) - u(r_2)$, where $v, u: \mathbb{R}^+ \to \mathbb{R}^+$ are given as $v(r_1) = \frac{r_1}{2}$ and $u(r_2) = r_2$.
(ii) $\psi_2: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $\psi_2(r_1, r_2) = \frac{r_1}{2} - \frac{v(r_1, r_2)}{u(r_1, r_2)}r_2$, where $v, u: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are given as $v(r_1, r_2) = r_1r_2$ and $u(r_1, r_2) = r_1r_2 + r_2$ for all $r_1, r_2 > 0$.

Recently, Khojasteh et al. [18] investigated the notion of a simulation function (see also [19–21]).

Definition 1.6. [18] A mapping $\Gamma: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a simulation function if:
(Γ1) $\Gamma(0, 0) = 0$;
(Γ2) $\Gamma(r, t) < t - r$ for all $r, t > 0$;
(Γ3) if $\{r_n\}$, $\{t_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} t_n > 0$, then

$$\limsup_{n\to\infty} \Gamma(r_n, t_n) < 0.$$

Due to the self-domain in (Γ2), we have $\Gamma(r, r) < 0$ for each $r > 0$. Denote by ∇ the set of all functions satisfying the conditions (Γ1)–(Γ3).

Example 1.7. [18–21] For $i = 1, 2$, let $\vartheta_i: \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions with $\vartheta_i(r) = 0$ if and only if $r = 0$. The following functions $\Gamma_j: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ ($j = 1, \ldots, 6$) are in ∇:
(i) $\Gamma_1(r, t) = \vartheta_1(t) - \vartheta_2(r)$ for all $r, t \geq 0$, where $\vartheta_1(r) \leq r \leq \vartheta_2(r)$ for all $r > 0$;
(ii) $\Gamma_2(r, t) = t - \frac{l_1(r,t)}{l_2(r,t)} r$ for all $r, t \geq 0$, where $l_1, l_2: \mathbb{R}^+ \times \mathbb{R}^+ \to (0, \infty)$ are continuous functions such that $l_1(r, t) > l_2(r, t)$ for all $r, t > 0$;
(iii) $\Gamma_3(r, t) = t - \vartheta_3(t) - r$ for all $r, t \geq 0$;
(iv) $\Gamma_4(r, t) = t\varphi(t) - r$ for all $r, t \geq 0$, where $\varphi: \mathbb{R}^+ \to (0, 1)$ is a function such that $\lim_{r\to s^+} \sup \varphi(r) < 1$ for all $s > 0$;
(v) $\Gamma_5(r, t) = \phi(t) - r$ for all $r, t \geq 0$, where $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is an upper semicontinuous function such that $\phi(r) < r$ for all $r > 0$ and $\phi(0) = 0$;
(vi) $\Gamma_6(r, t) = t - \int_0^r \varsigma(u)\mathrm{d}u$ for all $r, t \geq 0$, where $\varsigma: \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\int_0^\varepsilon \varsigma(u)\mathrm{d}u$ exists and $\int_0^\varepsilon \varsigma(u)\mathrm{d}u > \varepsilon$ for all $\varepsilon > 0$.

Let $(\Bbbk, \hat{d})$ be a metric space, $\tilde{T}$ be a self-mapping on $\Bbbk$, and $\Gamma \in \nabla$. $\tilde{T}$ is said to be a ∇-contraction w.r.t. Γ, if

$$\Gamma(\hat{d}(\tilde{T}\upsilon_1, \tilde{T}\upsilon_2), \hat{d}(\upsilon_1, \upsilon_2)) \geq 0, \quad \text{for all } \upsilon_1, \upsilon_2 \in \Bbbk.$$

Due to (Γ2), we have $\hat{d}(\tilde{T}\upsilon_1, \tilde{T}\upsilon_2) \neq \hat{d}(\upsilon_1, \upsilon_2)$ for all distinct points $\upsilon_1, \upsilon_2 \in \Bbbk$. Thus, $\tilde{T}$ is not an isometry, whenever $\tilde{T}$ is a ∇-contraction w.r.t. Γ. Conversely, if a ∇-contraction mapping $\tilde{T}$ on a metric space possesses a fixed point, then it is necessarily unique [18].

Theorem 1.8. [18] *Every ∇-contraction $\tilde{T}$ on a metric space possesses a fixed point υ^*. Also, every Picard sequence converges to υ^*.*

Samet et al. [22] and then Ali et al. [23] theorized a new type of contractions to integrate several existing theorems in the literature by classical functions.

Definition 1.9. [23] Let $(\Bbbk, \hat{d})$ be a metric space and Λ be a nonempty subset of $\Bbbk$. A map $\tilde{T}: \Lambda \to CL(\Bbbk)$ is called α-admissible, if there exists a mapping $\alpha: \Lambda \times \Lambda \to (0, \infty)$ such that

$$\alpha(a, b) \geq 1 \quad \Rightarrow \quad \alpha(\upsilon, \nu) \geq 1,$$

for all $\upsilon \in \tilde{T}a \cap \Lambda$ and $\nu \in \tilde{T}b \cap \Lambda$.

Throughout this article, E denotes an interval on $\mathbb{R}^+$ containing 0, that is, an interval of the form $[0, R]$, $[0, R)$, or $[0, \infty)$.

Lemma 1.10. [24] *Given $\upsilon_0 \in \Lambda$ (Λ is a closed subset of $\Bbbk$) such that*

$$\widehat{d}\,(\upsilon_0, \widetilde{T}\upsilon_0) \in E,$$

and $\upsilon_n \in \Lambda$ for some $n \geq 0$. Then, we have $\widehat{d}\,(\upsilon_n, \widetilde{T}\upsilon_n) \in E$.

Definition 1.11. [24] Suppose $\upsilon_0 \in \Lambda$ and $\widehat{d}\,(\upsilon_0, \widetilde{T}\upsilon_0) \in E$. Then, for every iterate $\upsilon_n (n \geq 0)$, which belongs to Λ, we define the closed ball $\overline{b}\,(\upsilon_n, \rho)$ with center υ_n and radius $\rho > 0$.

Lemma 1.12. [24] *If an element $\upsilon_0 \in \Lambda$ satisfies $\widehat{d}\,(\upsilon_0, \widetilde{T}\upsilon_0) \in E$ and $\overline{b}\,(\upsilon_n, \rho) \subset \Lambda$ for some $n \geq 0$, then $\upsilon_{n+1} \in \Lambda$ and $\overline{b}\,(\upsilon_{n+1}, \rho) \subset \overline{b}\,(\upsilon_n, \rho)$.*

Definition 1.13. [24] Let $i \geq 1$. A function $\xi\colon E \to E$ is said to be a gauge function of order i on E, if it satisfies the following conditions:
(i) $\xi\,(\lambda\upsilon) < \lambda^i \xi\,(\upsilon)$ for all $\lambda \in (0, 1)$ and $\upsilon \in E$;
(ii) $\xi\,(\upsilon) < \upsilon$ for all $\upsilon \in E - \{0\}$.

It is easy to see that the first condition of Definition 1.13 is equivalent to the following: $\xi\,(0) = 0$ and $\xi\,(\upsilon)/\upsilon^i$ is nondecreasing on $E - \{0\}$.

Definition 1.14. [24] A gauge function $\xi\colon E \to E$ is said to be a Bianchini-Grandolfi gauge function on E if

$$\sigma\,(\upsilon) = \sum_{i=0}^{\infty} \xi^i\,(\upsilon) < \infty, \quad \text{for all } \upsilon \in E. \tag{2}$$

Note that a Bianchini-Grandolfi gauge function also satisfies the following functional equation:

$$\sigma\,(\upsilon) = \sigma\,(\xi\,(\upsilon)) + \upsilon. \tag{3}$$

2 Multivalued Suzuki-type ($\theta - \xi$)-contractions

We start with the following.

Definition 2.1. Let $(\Bbbk, \widehat{d})$ be a metric space, Λ be a closed subset of $\Bbbk$, and ξ be a Bianchini-Grandolfi gauge function on an interval E. A mapping $\widetilde{T}\colon \Lambda \to CL(\Bbbk)$ is said to be a multivalued Suzuki-type $(\theta - \xi)$-contraction, if there exist $\psi \in \Phi$ and $\theta \in \Xi$ such that for $\widetilde{T}\upsilon \cap \Lambda \neq \varnothing$

$$\psi\,[\widehat{d}\,(\upsilon, \widetilde{T}\upsilon \cap \Lambda), \widehat{d}\,(\upsilon, \nu)] < 0,$$

implies that

$$\theta\,[\hat{H}\,(\widetilde{T}\upsilon \cap \Lambda, \widetilde{T}\nu \cap \Lambda)] \leq [\theta\,(\xi\,(\widehat{d}\,(\upsilon, \nu)))]^k, \tag{4}$$

for all $\upsilon \in \Lambda$, $\nu \in \widetilde{T}\upsilon \cap \Lambda$ with $\widehat{d}\,(\upsilon, \nu) \in E$, where $0 < k < 1$.

Our first main result is as follows.

Theorem 2.2. *Let $(\Bbbk, \widehat{d})$ be a complete metric space, Λ be a closed subset of $\Bbbk$, and $\widetilde{T}: \Lambda \to CB(\Bbbk)$ be a multivalued Suzuki-type $(\theta - \xi)$-contraction. In addition, suppose $\upsilon_0 \in \Lambda$ such that $\widehat{d}(\upsilon_0, c^\star) \in E$ for some $c^\star \in \widetilde{T}\upsilon_0 \cap \Lambda$. Then, the following assertions hold:*

(i) *there exist an orbit $\{\upsilon_n\}$ of $\widetilde{T}$ in Λ and $a^\star \in \Lambda$ such that $\lim_{n\to\infty}\upsilon_n = a^\star$;*

(ii) *$a^\star$ is a fixed point of $\widetilde{T}$ if and only if the function $g(\upsilon) := \widehat{d}(\upsilon, \widetilde{T}\upsilon \cap \Lambda)$ is $\widetilde{T}$-orbitally lower semicontinuous at $a^\star$.*

Proof. Choose $\upsilon_1 = c^\star \in \widetilde{T}\upsilon_0 \cap \Lambda$. In the case that $\widehat{d}(\upsilon_0, \upsilon_1) = 0$, then υ_0 is a fixed point of $\widetilde{T}$. Thus, we assume that $\widehat{d}(\upsilon_0, \upsilon_1) \neq 0$. Then,

$$\begin{aligned}\psi[\widehat{d}(\upsilon_0, \widetilde{T}\upsilon_0 \cap \Lambda), \widehat{d}(\upsilon_0, \upsilon_1)] &\le \frac{1}{2}\widehat{d}(\upsilon_0, \widetilde{T}\upsilon_0 \cap \Lambda) - \widehat{d}(\upsilon_0, \upsilon_1)\\ &\le \frac{1}{2}\widehat{d}(\upsilon_0, \widetilde{T}\upsilon_0) - \widehat{d}(\upsilon_0, \upsilon_1)\\ &< \widehat{d}(\upsilon_0, \widetilde{T}\upsilon_0) - \widehat{d}(\upsilon_0, \upsilon_1)\\ &\le \widehat{d}(\upsilon_0, \upsilon_1) - \widehat{d}(\upsilon_0, \upsilon_1)\\ &= 0.\end{aligned} \tag{5}$$

Define $\rho = \sigma(\widehat{d}(\upsilon_0, \upsilon_1))$. From (3), we have $\sigma(r) \ge r$. Hence, $\widehat{d}(\upsilon_0, \upsilon_1) \le \rho$ and so $\upsilon_1 \in \overline{b}(\upsilon_0, \rho)$. Since $\widehat{d}(\upsilon_0, \upsilon_1) \in E$, so that from (2.1) and (2.2) it follows that

$$\theta[\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda)] \le [\theta(\xi(\widehat{d}(\upsilon_0, \upsilon_1)))]^k < [\theta(\widehat{d}(\upsilon_0, \upsilon_1))]^k. \tag{6}$$

By the property of right continuity of θ, there exists a real number $h_1 > 1$ such that

$$\theta[h_1\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda)] \le [\theta(\widehat{d}(\upsilon_0, \upsilon_1))]^k. \tag{7}$$

From

$$\widehat{d}(\upsilon_1, \widetilde{T}\upsilon_1 \cap \Lambda) \le \hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda) < h_1\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda),$$

by Lemma 1.4, there exists $\upsilon_2 \in \widetilde{T}\upsilon_1 \cap \Lambda$ such that $\widehat{d}(\upsilon_1, \upsilon_2) \le h_1\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda)$. Since θ is nondecreasing, by (7), this inequality gives that

$$\theta(\widehat{d}(\upsilon_1, \upsilon_2)) \le \theta[h_1\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda)] \le [\theta(\widehat{d}(\upsilon_0, \upsilon_1))]^k. \tag{8}$$

We assume that $\widehat{d}(\upsilon_1, \upsilon_2) \neq 0$, otherwise υ_1 is a fixed point of $\widetilde{T}$. From Remark 1.2, we have $\widehat{d}(\upsilon_1, \upsilon_2) < \widehat{d}(\upsilon_0, \upsilon_1)$ and so $\widehat{d}(\upsilon_1, \upsilon_2) \in E$. Next, $\upsilon_2 \in \overline{b}(\upsilon_0, \rho)$ because that

$$\begin{aligned}\widehat{d}(\upsilon_0, \upsilon_2) &\le \widehat{d}(\upsilon_0, \upsilon_1) + \widehat{d}(\upsilon_1, \upsilon_2)\\ &\le \widehat{d}(\upsilon_0, \upsilon_1) + \xi(\widehat{d}(\upsilon_0, \upsilon_1))\\ &\le \widehat{d}(\upsilon_0, \upsilon_1) + \sigma(\xi(\widehat{d}(\upsilon_0, \upsilon_1)))\\ &= \sigma(\widehat{d}(\upsilon_0, \upsilon_1)) = \rho.\end{aligned}$$

Also, since

$$\begin{aligned}\psi[\hat{d}(\upsilon_1, \widetilde{T}\upsilon_1 \cap \Lambda), \hat{d}(\upsilon_1, \upsilon_2)] &\le \frac{1}{2}\hat{d}(\upsilon_1, \widetilde{T}\upsilon_1 \cap \Lambda) - \hat{d}(\upsilon_1, \upsilon_2)\\ &\le \frac{1}{2}\hat{d}(\upsilon_1, \widetilde{T}\upsilon_1) - \hat{d}(\upsilon_1, \upsilon_2)\\ &< \hat{d}(\upsilon_1, \widetilde{T}\upsilon_1) - \hat{d}(\upsilon_1, \upsilon_2))\\ &\le \hat{d}(\upsilon_1, \upsilon_2) - \hat{d}(\upsilon_1, \upsilon_2)\\ &= 0,\end{aligned}$$

from (4), we get

$$\theta[\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda)] \le [\theta(\xi(\hat{d}(\upsilon_1, \upsilon_2)))]^k < [\theta(\hat{d}(\upsilon_1, \upsilon_2))]^k. \tag{9}$$

Since θ is right continuous, there exists a real number $h_2 > 1$ such that

$$\theta[h_2\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda)] \le [\theta(\hat{d}(\upsilon_1, \upsilon_2))]^k. \tag{10}$$

Next, from

$$\hat{d}(\upsilon_2, \widetilde{T}\upsilon_2 \cap \Lambda) \le \hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda) < h_2\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda),$$

by Lemma 1.4, there exists $\upsilon_3 \in \widetilde{T}\upsilon_2 \cap \Lambda$ such that $\hat{d}(\upsilon_2, \upsilon_3) \le h_2\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda)$. By (10), this inequality gives that

$$\theta(\hat{d}(\upsilon_2, \upsilon_3)) \le \theta[h_2\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda)] \le [\theta(\hat{d}(\upsilon_1, \upsilon_2))]^k \le [\theta(\hat{d}(\upsilon_0, \upsilon_1))]^{k^2}.$$

We assume that $\hat{d}(\upsilon_2, \upsilon_3) \neq 0$, otherwise υ_2 is a fixed point of $\widetilde{T}$. From Remark 1.2, we have $\hat{d}(\upsilon_2, \upsilon_3) < \hat{d}(\upsilon_1, \upsilon_2)$ and so $\hat{d}(\upsilon_2, \upsilon_3) \in E$. Also, we have $\upsilon_3 \in \overline{b}(\upsilon_0, \rho)$, since

$$\begin{aligned}\hat{d}(\upsilon_0, \upsilon_3) &\le \hat{d}(\upsilon_0, \upsilon_1) + \hat{d}(\upsilon_1, \upsilon_2) + \hat{d}(\upsilon_2, \upsilon_3)\\ &\le \hat{d}(\upsilon_0, \upsilon_1) + \xi(\hat{d}(\upsilon_0, \upsilon_1)) + \xi^2(\hat{d}(\upsilon_0, \upsilon_1))\\ &\le \sum_{i=0}^{\infty} \xi^i(\hat{d}(\upsilon_0, \upsilon_1))\\ &= \sigma(\hat{d}(\upsilon_0, \upsilon_1)) = \rho.\end{aligned}$$

Continuing in this manner, we build two sequences $\{\upsilon_n\} \subset \overline{b}(\upsilon_0, \rho)$ and $\{h_n\} \subset (1, \infty)$ such that $\upsilon_{n+1} \in \widetilde{T}\upsilon_n \cap \Lambda$, $\upsilon_n \neq \upsilon_{n+1}$ with $\hat{d}(\upsilon_n, \upsilon_{n+1}) \in E$ and

$$1 < \theta(\hat{d}(\upsilon_n, \upsilon_{n+1})) \le \theta(h_n\hat{H}(\widetilde{T}\upsilon_{n-1} \cap \Lambda, \widetilde{T}\upsilon_n \cap \Lambda)) \le [\theta(\hat{d}(\upsilon_{n-1}, \upsilon_n))]^k,$$

for all $n \in \mathbb{N}$. Then,

$$1 < \theta(\hat{d}(\upsilon_n, \upsilon_{n+1})) \le [\theta(\hat{d}(\upsilon_0, \upsilon_1))]^{k^n}, \quad \text{for all } n \in \mathbb{N}, \tag{11}$$

which gives that

$$\lim_{n\to\infty} \theta(\hat{d}(\upsilon_n, \upsilon_{n+1})) = 1,$$

and by (θ_{ii}), we have

$$\lim_{n\to\infty} \hat{d}(\upsilon_n, \upsilon_{n+1}) = 0. \tag{12}$$

Next, we prove that $\{\upsilon_n\}$ is a Cauchy sequence in $\Bbbk$. Setting $\delta_n: =\widehat{d}\,(\upsilon_n, \upsilon_{n+1})$, from (θ_{iii}), there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\delta_n) - 1}{(\delta_n)^r} = \ell.$$

Take $\lambda \in (0, \ell)$. From the definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$[\delta_n]^r \le \lambda^{-1}[\theta(\delta_n) - 1], \quad \text{for all } n > n_0.$$

Using (11) and the above inequality, we deduce

$$n[\delta_n]^r \le \lambda^{-1} n\left([\theta(\delta_0)]^{k^n} - 1\right), \quad \text{for all } n > n_0.$$

This implies that

$$\lim_{n\to\infty} n[\delta_n]^r = \lim_{n\to\infty} n[d(\upsilon_n, \upsilon_{n+1})]^r = 0.$$

Thence, there exists $n_1 \in \mathbb{N}$ such that

$$d(\upsilon_n, \upsilon_{n+1}) \le \frac{1}{n^{1/r}}, \quad \text{for all } n > n_1. \tag{13}$$

Let $p > n > n_1$. Then, using the triangular inequality and (13), we get

$$\widehat{d}\,(\upsilon_n, \upsilon_p) \le \sum_{j=n}^{p-1} \widehat{d}\,(\upsilon_j, \upsilon_{j+1}) \le \sum_{j=n}^{p-1} \frac{1}{j^{1/r}} < \sum_{j=n}^{\infty} \frac{1}{j^{1/r}}.$$

Due to the convergence of the series $\sum_{j=n}^{\infty} \frac{1}{j^{1/r}}$, we deduce that $\{\upsilon_n\}$ is a Cauchy sequence in the closed ball $\overline{b}\,(\upsilon_0, \rho)$. Since $\overline{b}\,(\upsilon_0, \rho)$ is closed in $\Bbbk$, there exists an $a^\star \in \overline{b}\,(\upsilon_0, \rho)$ such that $\upsilon_n \to a^\star$. Note that $a^\star \in \Lambda$, because $\upsilon_{n+1} \in \widetilde{T}\upsilon_n \cap \Lambda$. Obviously,

$$\frac{1}{2}\widehat{d}\,(\upsilon_n, \widetilde{T}\upsilon_n \cap \Lambda) < \widehat{d}\,(\upsilon_n, \widetilde{T}\upsilon_n) \le \widehat{d}\,(\upsilon_n, \upsilon_{n+1}),$$

which implies that

$$\psi[\widehat{d}\,(\upsilon_n, \widetilde{T}\upsilon_n \cap \Lambda), \widehat{d}\,(\upsilon_n, \upsilon_{n+1})] < 0.$$

Also, we know that $\widehat{d}\,(\upsilon_n, \upsilon_{n+1}) \in E$ for all n. Thus, from (4), we have

$$\begin{aligned}\theta(\widehat{d}\,(\upsilon_{n+1}, \widetilde{T}\upsilon_{n+1} \cap \Lambda)) &\le \theta[\hat{H}(\widetilde{T}\upsilon_n \cap \Lambda, \widetilde{T}\upsilon_{n+1} \cap \Lambda)]\\ &\le [\theta(\xi(\widehat{d}\,(\upsilon_n, \upsilon_{n+1})))]^k\\ &< \theta[(\widehat{d}\,(\upsilon_n, \upsilon_{n+1}))]^k.\end{aligned}$$

From Remark 1.2, we deduce that

$$\widehat{d}\,(\upsilon_{n+1}, \widetilde{T}\upsilon_{n+1} \cap \Lambda) < \widehat{d}\,(\upsilon_n, \upsilon_{n+1}). \tag{14}$$

Taking limit $n \to \infty$ in (14), we obtain

$$\lim_{n\to\infty} \widehat{d}\,(\upsilon_{n+1}, \widetilde{T}\upsilon_{n+1} \cap \Lambda) = 0.$$

Since $g(\upsilon) = \widehat{d}\,(\upsilon, \widetilde{T}\upsilon \cap \Lambda)$ is $\widetilde{T}$-orbitally lower semicontinuous at $a^\star$, then

$$\widehat{d}\,(a^\star, \widetilde{T}a^\star \cap \Lambda) = g(a^\star) \le \liminf_n g(\upsilon_{n+1}) = \liminf_n \widehat{d}\,(\upsilon_{n+1}, \widetilde{T}\upsilon_{n+1} \cap \Lambda) = 0.$$

Since $\widetilde{T}a^\star$ is closed, we have $a^\star \in \widetilde{T}a^\star$. Conversely, if $a^\star$ is a fixed point of $\widetilde{T}$, then $g(a^\star) = 0 \le \liminf_n g(\upsilon_n)$, since $a^\star \in \Lambda$. □

Corollary 2.3. *Let $(\Bbbk, \widehat{d})$ be a complete metric space, Λ be a closed subset of $\Bbbk$, ξ be a Bianchini-Grandolfi gauge function on an interval E, and $\widetilde{T}\colon \Lambda \to CL(\Bbbk)$ be a given multivalued mapping. If there exist $\psi \in \Phi$ and $k \in (0, 1)$ such that for $\widetilde{T}\upsilon \cap \Lambda \neq \varnothing$*

$$\psi[\widehat{d}\,(\upsilon, \widetilde{T}\upsilon \cap \Lambda), \widehat{d}\,(\upsilon, \nu)] < 0,$$

implies that

$$\sqrt{\hat{H}(\widetilde{T}\upsilon \cap \Lambda, \widetilde{T}\nu \cap \Lambda)} \le k\sqrt{\xi(\widehat{d}\,(\upsilon, \nu))},$$

for all $\upsilon \in \Lambda$, $\nu \in \widetilde{T}\upsilon \cap \Lambda$ with $\widehat{d}\,(\upsilon, \nu) \in E$. In addition, suppose $\upsilon_0 \in \Lambda$ such that $\widehat{d}\,(\upsilon_0, c^\star) \in E$ for some $c^\star \in \widetilde{T}\upsilon_0 \cap \Lambda$. Then, the following assertions hold:

(i) *there exist an orbit $\{\upsilon_n\}$ of $\widetilde{T}$ in Λ and $a^\star \in \Lambda$ such that $\lim_{n\to\infty}\upsilon_n = a^\star$;*

(ii) *$a^\star$ is a fixed point of $\widetilde{T}$ if and only if the function $g(\upsilon) \coloneqq \widehat{d}\,(\upsilon, \widetilde{T}\upsilon \cap \Lambda)$ is $\widetilde{T}$-orbitally lower semicontinuous at $a^\star$.*

Corollary 2.4. *Let $(\Bbbk, \widehat{d})$ be a complete metric space, ξ be a Bianchini-Grandolfi gauge function on an interval E, and $\widetilde{T}\colon \Bbbk \to CB(\Bbbk)$ be a given multivalued mapping. If there exist $\psi \in \Phi$, $\theta \in \Xi$, and $k \in (0, 1)$ such that*

$$\psi[\widehat{d}\,(\upsilon, \widetilde{T}\upsilon), \widehat{d}\,(\upsilon, \nu)] < 0 \Rightarrow \theta[\hat{H}(\widetilde{T}\upsilon, \widetilde{T}\nu)] \le [\theta(\xi(\widehat{d}\,(\upsilon, \nu)))]^k, \tag{15}$$

for all $\upsilon \in \Bbbk$, $\nu \in \widetilde{T}\upsilon$ with $\widehat{d}\,(\upsilon, \nu) \in E$. Suppose that $\upsilon_0 \in \Bbbk$ such that $\widehat{d}\,(\upsilon_0, c^\star) \in E$ for some $c^\star \in \widetilde{T}\upsilon_0$. Then, there exists an orbit $\{\upsilon_n\}$ of $\widetilde{T}$ in $\Bbbk$ that converges to the fixed point $a^\star \in \mathcal{F} = \{\upsilon \in \Bbbk\colon \widehat{d}\,(\upsilon, a^\star) \in E\}$ of $\widetilde{T}$.

Example 2.5. Let $\Bbbk = [-10, \infty)$ be endowed with the usual metric $\widehat{d}$ and let $E = [0, \infty)$. Consider the mapping $\widetilde{T}\colon \Bbbk \to CB(\Bbbk)$ defined by

$$\widetilde{T}(\upsilon) = \begin{cases} \left[0, \dfrac{\upsilon}{8}\right], & \upsilon \in [0,4], \\ \{0, \upsilon\}, & \upsilon \in [-10,0) \cup (4,\infty). \end{cases}$$

Let $\psi(r, s) = \frac{r}{2} - s$, if $r, s \in [0, 4]$ and $\psi(r, s) = 2s$, otherwise. Clearly, $\psi[\widehat{d}\,(\upsilon, \widetilde{T}\upsilon), \widehat{d}\,(\upsilon, \nu)] < 0$ if and only if $\upsilon, \nu \in [0, 4]$. Let $\upsilon_0 = 4$, then we have $c^\star = \frac{1}{2} \in \widetilde{T}\upsilon_0$ such that $\widehat{d}\,(\upsilon_0, c^\star) \in E$. First, we examine that $\widetilde{T}$ satisfies the inequality (15) with $\theta(r) = e^{\sqrt{re^r}}$, $\xi(r) = \frac{r}{2}$, and $k = \frac{1}{2}$. For $\upsilon \in [0, 4]$ and $\nu \in \widetilde{T}\upsilon$, we get

$$\begin{aligned}\theta[\hat{H}(\widetilde{T}\upsilon, \widetilde{T}\nu)] &= \theta\left(\frac{|\upsilon-\nu|}{8}\right)\\ &= e^{\sqrt{\frac{|\upsilon-\nu|}{8}e^{\frac{|\upsilon-\nu|}{8}}}}\\ &\leq e^{\frac{1}{2}\sqrt{\frac{|\upsilon-\nu|}{2}e^{\frac{|\upsilon-\nu|}{2}}}}\\ &= e^{\frac{1}{2}\sqrt{\xi(\widehat{d}(\upsilon,\nu))e^{\xi(\widehat{d}(\upsilon,\nu))}}}\\ &= [\theta(\xi(\widehat{d}(\upsilon,\nu)))]^{k}.\end{aligned}$$

Consequently, all the conditions of Corollary 2.4 are fulfilled and 0 is a fixed point of $\widetilde{T}$. Next, observe that for $\upsilon = 0$ and $\nu = 5$

$$\theta[\hat{H}(\widetilde{T}\upsilon, \widetilde{T}\nu)] = \theta[\hat{H}(\widetilde{T}0, \widetilde{T}5)] = \theta(5) > [\theta(5)]^{k} = [\theta(\widehat{d}(\upsilon,\nu))]^{k},$$

for all $\theta \in \Xi$ and $k \in (0,1)$. Therefore, Theorem 1.1 cannot applied to this example.

3 Multivalued Suzuki-type ($\alpha - \nabla$)-contractions

Definition 3.1. Let $(\Bbbk, \widehat{d})$ be a metric space, Λ be a closed subset of $\Bbbk$, and ξ be a Bianchini-Grandolfi gauge function on an interval E. A mapping $\widetilde{T}: \Lambda \to CL(\Bbbk)$ is said to be a multivalued Suzuki-type $(\alpha - \nabla)$-contraction, if there exist $\psi \in \Phi$ and $\Gamma \in \nabla$ such that for $\widetilde{T}\upsilon \cap \Lambda \neq \varnothing$

$$\psi[\widehat{d}(\upsilon, \widetilde{T}\upsilon \cap \Lambda), \widehat{d}(\upsilon,\nu)] < 0,$$

implies that

$$\Gamma[\alpha(\upsilon,\nu)\hat{H}(\widetilde{T}\upsilon \cap \Lambda, \widetilde{T}\nu \cap \Lambda), \xi(\widehat{d}(\upsilon,\nu))] \geq 0, \tag{16}$$

for all $\upsilon \in \Lambda$, $\nu \in \widetilde{T}\upsilon \cap \Lambda$ with $\widehat{d}(\upsilon,\nu) \in E$.

The second one of our results is as follows.

Theorem 3.2. *Let $(\Bbbk, \widehat{d})$ be a complete metric space, Λ be a closed subset of $\Bbbk$, and $\widetilde{T}: \Lambda \to CL(\Bbbk)$ be a multivalued Suzuki-type $(\alpha - \nabla)$-contraction. Suppose that the following conditions are satisfied:*
(i) *$\widetilde{T}$ is α-admissible;*
(ii) *there exists $\upsilon_0 \in \Lambda$ with $\widehat{d}(\upsilon_0, \upsilon_1) \in E$ for some $\upsilon_1 \in \widetilde{T}\upsilon_0 \cap \Lambda$ such that $\alpha(\upsilon_0, \upsilon_1) \geq 1$.*

Then,
(a) *there exist an orbit $\{\upsilon_n\}$ of $\widetilde{T}$ in Λ and $a^* \in \Lambda$ such that $\lim_{n\to\infty}\upsilon_n = a^*$;*
(b) *a^* is a fixed point of $\widetilde{T}$ if and only if the function $g(\upsilon) := \widehat{d}(\upsilon, \widetilde{T}\upsilon \cap \Lambda)$ is $\widetilde{T}$-orbitally lower semi-continuous at a^*.*

Proof. By the hypothesis, there exists $\upsilon_0 \in \Lambda$ with $\widehat{d}(\upsilon_0, \upsilon_1) \in E$ for some $\upsilon_1 \in \widetilde{T}\upsilon_0 \cap \Lambda$ such that $\alpha(\upsilon_0, \upsilon_1) \geq 1$. On the other hand, we have

$$\begin{aligned}\psi[\widehat{d}(\upsilon_0, \widetilde{T}\upsilon_0 \cap \Lambda), \widehat{d}(\upsilon_0, \upsilon_1)] &\le \frac{1}{2}\widehat{d}(\upsilon_0, \widetilde{T}\upsilon_0 \cap \Lambda) - \widehat{d}(\upsilon_0, \upsilon_1)\\ &\le \frac{1}{2}\widehat{d}(\upsilon_0, \widetilde{T}\upsilon_0) - \widehat{d}(\upsilon_0, \upsilon_1)\\ &< \widehat{d}(\upsilon_0, \widetilde{T}\upsilon_0) - \widehat{d}(\upsilon_0, \upsilon_1))\\ &\le \widehat{d}(\upsilon_0, \upsilon_1) - \widehat{d}(\upsilon_0, \upsilon_1)\\ &= 0.\end{aligned} \tag{17}$$

In the case that $\widehat{d}(\upsilon_0, \upsilon_1) = 0$, then υ_0 is a fixed point of $\widetilde{T}$. Thus, we assume that $\widehat{d}(\upsilon_0, \upsilon_1) \neq 0$. Define $\rho = \sigma(\widehat{d}(\upsilon_0, \upsilon_1))$. From (3), we have $\sigma(r) \ge r$. Hence $\widehat{d}(\upsilon_0, \upsilon_1) \le \rho$ and so $\upsilon_1 \in \overline{b}(\upsilon_0, \rho)$. Since $\alpha(\upsilon_0, \upsilon_1) \ge 1$ and $\widehat{d}(\upsilon_0, \upsilon_1) \in E$, so that from (3.1) and (3.2) it follows that

$$\begin{aligned}0 &\le \Gamma[\alpha(\upsilon_0, \upsilon_1)\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda), \xi(\widehat{d}(\upsilon_0, \upsilon_1))]\\ &< \xi(\widehat{d}(\upsilon_0, \upsilon_1)) - \alpha(\upsilon_0, \upsilon_1)\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda),\end{aligned}$$

which implies that

$$\alpha(\upsilon_0, \upsilon_1)\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda) < \xi(\widehat{d}(\upsilon_0, \upsilon_1)).$$

We can choose an $\varepsilon_1 > 0$ such that

$$\alpha(\upsilon_0, \upsilon_1)\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda) + \varepsilon_1 \le \xi(\widehat{d}(\upsilon_0, \upsilon_1)).$$

Thus,

$$\begin{aligned}\widehat{d}(\upsilon_1, \widetilde{T}\upsilon_1 \cap \Lambda) + \varepsilon_1 &\le \hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda) + \varepsilon_1\\ &\le \alpha(\upsilon_0, \upsilon_1)\hat{H}(\widetilde{T}\upsilon_0 \cap \Lambda, \widetilde{T}\upsilon_1 \cap \Lambda) + \varepsilon_1\\ &\le \xi(\widehat{d}(\upsilon_0, \upsilon_1)).\end{aligned} \tag{18}$$

It follows from Lemma 1.5 that there exists $\upsilon_2 \in \widetilde{T}\upsilon_1 \cap \Lambda$ such that

$$\widehat{d}(\upsilon_1, \upsilon_2) \le \widehat{d}(\upsilon_1, \widetilde{T}\upsilon_1 \cap \Lambda) + \varepsilon_1. \tag{19}$$

From (18) and (19), we infer

$$\widehat{d}(\upsilon_1, \upsilon_2) \le \xi(\widehat{d}(\upsilon_0, \upsilon_1)). \tag{20}$$

We assume that $\widehat{d}(\upsilon_1, \upsilon_2) \neq 0$, otherwise υ_1 is a fixed point of $\widetilde{T}$. Since $\widehat{d}(\upsilon_1, \upsilon_2) \le \xi(\widehat{d}(\upsilon_0, \upsilon_1)) < \widehat{d}(\upsilon_0, \upsilon_1)$, we deduce that $\widehat{d}(\upsilon_1, \upsilon_2) \in E$. Next, $\upsilon_2 \in \overline{b}(\upsilon_0, \rho)$ because that

$$\begin{aligned}\widehat{d}(\upsilon_0, \upsilon_2) &\le \widehat{d}(\upsilon_0, \upsilon_1) + \widehat{d}(\upsilon_1, \upsilon_2)\\ &\le \widehat{d}(\upsilon_0, \upsilon_1) + \xi(\widehat{d}(\upsilon_0, \upsilon_1))\\ &\le \widehat{d}(\upsilon_0, \upsilon_1) + \sigma(\xi(\widehat{d}(\upsilon_0, \upsilon_1)))\\ &= \sigma(\widehat{d}(\upsilon_0, \upsilon_1)) = \rho.\end{aligned}$$

Since $\widetilde{T}$ is α-admissible, $\alpha(\upsilon_1, \upsilon_2) \ge 1$. Also, since

$$\begin{aligned}\psi[\hat{d}(\upsilon_1, \widetilde{T}\upsilon_1 \cap \Lambda), \hat{d}(\upsilon_1, \upsilon_2)] &\leq \frac{1}{2}\hat{d}(\upsilon_1, \widetilde{T}\upsilon_1 \cap \Lambda) - \hat{d}(\upsilon_1, \upsilon_2)\\ &\leq \frac{1}{2}\hat{d}(\upsilon_1, \widetilde{T}\upsilon_1) - \hat{d}(\upsilon_1, \upsilon_2)\\ &< \hat{d}(\upsilon_1, \widetilde{T}\upsilon_1) - \hat{d}(\upsilon_1, \upsilon_2))\\ &\leq \hat{d}(\upsilon_1, \upsilon_2) - \hat{d}(\upsilon_1, \upsilon_2)\\ &= 0,\end{aligned}$$

from (16), we get

$$\begin{aligned}0 &\leq \Gamma[\alpha(\upsilon_1, \upsilon_2)\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda), \xi(\hat{d}(\upsilon_1, \upsilon_2))]\\ &< \xi(\hat{d}(\upsilon_1, \upsilon_2)) - \alpha(\upsilon_1, \upsilon_2)\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda).\end{aligned}$$

This implies that

$$\alpha(\upsilon_1, \upsilon_2)\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda) < \xi(\hat{d}(\upsilon_1, \upsilon_2)).$$

Now choose an $\varepsilon_2 > 0$ such that

$$\alpha(\upsilon_1, \upsilon_2)\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda) + \varepsilon_2 \leq \xi(\hat{d}(\upsilon_1, \upsilon_2)).$$

Thus,

$$\begin{aligned}\hat{d}(\upsilon_2, \widetilde{T}\upsilon_2 \cap \Lambda) + \varepsilon_2 &\leq \hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda) + \varepsilon_2\\ &\leq \alpha(\upsilon_1, \upsilon_2)\hat{H}(\widetilde{T}\upsilon_1 \cap \Lambda, \widetilde{T}\upsilon_2 \cap \Lambda) + \varepsilon_2\\ &\leq \xi(\hat{d}(\upsilon_1, \upsilon_2)).\end{aligned} \tag{21}$$

It follows from Lemma 1.5 that there exists $\upsilon_3 \in \widetilde{T}\upsilon_2 \cap \Lambda$ such that

$$\hat{d}(\upsilon_2, \upsilon_3) \leq \hat{d}(\upsilon_2, \widetilde{T}\upsilon_2 \cap \Lambda) + \varepsilon_2. \tag{22}$$

From (21) and (22), we obtain

$$\hat{d}(\upsilon_2, \upsilon_3) \leq \xi^2(\hat{d}(\upsilon_0, \upsilon_1)). \tag{23}$$

We assume that $\hat{d}(\upsilon_2, \upsilon_3) \neq 0$, otherwise υ_2 is a fixed point of $\widetilde{T}$. From (23), we have $\hat{d}(\upsilon_2, \upsilon_3) < \hat{d}(\upsilon_1, \upsilon_2)$ and so $\hat{d}(\upsilon_2, \upsilon_3) \in E$. Also, we have $\upsilon_3 \in \overline{b}(\upsilon_0, \rho)$, since

$$\begin{aligned}\hat{d}(\upsilon_0, \upsilon_3) &\leq \hat{d}(\upsilon_0, \upsilon_1) + \hat{d}(\upsilon_1, \upsilon_2) + \hat{d}(\upsilon_2, \upsilon_3)\\ &\leq \hat{d}(\upsilon_0, \upsilon_1) + \xi(\hat{d}(\upsilon_0, \upsilon_1)) + \xi^2(\hat{d}(\upsilon_0, \upsilon_1))\\ &\leq \sum_{i=0}^{\infty} \xi^i(\hat{d}(\upsilon_0, \upsilon_1))\\ &= \sigma(\hat{d}(\upsilon_0, \upsilon_1)) = \rho.\end{aligned}$$

Continuing in this manner, we obtain a sequence $\{\upsilon_n\} \subset \overline{b}(\upsilon_0, \rho)$ such that $\upsilon_{n+1} \in \widetilde{T}\upsilon_n \cap \Lambda$, $\upsilon_n \neq \upsilon_{n+1}$ with $\alpha(\upsilon_n, \upsilon_{n+1}) \geq 1$ and $\hat{d}(\upsilon_n, \upsilon_{n+1}) \in E$ and

$$\hat{d}(\upsilon_n, \upsilon_{n+1}) \leq \xi^n(\hat{d}(\upsilon_0, \upsilon_1)), \quad \text{for all } n \in \mathbb{N}. \tag{24}$$

For $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality and (24), we get

$$\begin{aligned}\widehat{d}(\upsilon_n, \upsilon_m) &\le \widehat{d}(\upsilon_n, \upsilon_{n+1}) + \widehat{d}(\upsilon_{n+1}, \upsilon_{n+2}) + \cdots + \widehat{d}(\upsilon_{m-1}, \upsilon_m)\\ &\le \xi^n(\widehat{d}(\upsilon_0, \upsilon_1)) + \xi^{n+1}(\widehat{d}(\upsilon_0, \upsilon_1)) + \cdots + \xi^{m-1}(\widehat{d}(\upsilon_0, \upsilon_1))\\ &\le \sum_{j=n}^{\infty} \xi^j(\widehat{d}(\upsilon_0, \upsilon_1)) < \infty,\end{aligned}$$

which shows that $\{\upsilon_n\}$ is a Cauchy sequence in the closed ball $\overline{b}(\upsilon_0, \rho)$. Since $\overline{b}(\upsilon_0, \rho)$ is closed in $\Bbbk$, there exists an $a^\star \in \overline{b}(\upsilon_0, \rho)$ such that $\upsilon_n \to a^\star$. Note that $a^\star \in \Lambda$, because $\upsilon_{n+1} \in \widetilde{T}\upsilon_n \cap \Lambda$. Obviously,

$$\frac{1}{2}\widehat{d}(\upsilon_n, \widetilde{T}\upsilon_n \cap \Lambda) < \widehat{d}(\upsilon_n, \widetilde{T}\upsilon_n) \le \widehat{d}(\upsilon_n, \upsilon_{n+1}),$$

which implies that

$$\psi[\widehat{d}(\upsilon_n, \widetilde{T}\upsilon_n \cap \Lambda), \widehat{d}(\upsilon_n, \upsilon_{n+1})] < 0.$$

Also, we know that $\alpha(\upsilon_n, \upsilon_{n+1}) \ge 1$ and $\widehat{d}(\upsilon_n, \upsilon_{n+1}) \in E$ for all n. Thus, from (16), we have

$$\begin{aligned}0 &\le \Gamma[\alpha(\upsilon_n, \upsilon_{n+1})\hat{H}(\widetilde{T}\upsilon_n \cap \Lambda, \widetilde{T}\upsilon_{n+1} \cap \Lambda), \xi(\widehat{d}(\upsilon_n, \upsilon_{n+1}))]\\ &< \xi(\widehat{d}(\upsilon_n, \upsilon_{n+1})) - \alpha(\upsilon_n, \upsilon_{n+1})\hat{H}(\widetilde{T}\upsilon_n \cap \Lambda, \widetilde{T}\upsilon_{n+1} \cap \Lambda),\end{aligned}$$

which gives that

$$\alpha(\upsilon_n, \upsilon_{n+1})\hat{H}(\widetilde{T}\upsilon_n \cap \Lambda, \widetilde{T}\upsilon_{n+1} \cap \Lambda) < \xi(\widehat{d}(\upsilon_n, \upsilon_{n+1})).$$

Since $\upsilon_{n+1} \in \widetilde{T}\upsilon_n \cap \Lambda$, from (24), we get

$$\begin{aligned}\widehat{d}(\upsilon_{n+1}, \widetilde{T}\upsilon_{n+1} \cap \Lambda) &\le \alpha(\upsilon_n, \upsilon_{n+1})\hat{H}(\widetilde{T}\upsilon_n \cap \Lambda, \widetilde{T}\upsilon_{n+1} \cap \Lambda)\\ &< \xi(\widehat{d}(\upsilon_n, \upsilon_{n+1}))\\ &\le \xi^{n+1}(\widehat{d}(\upsilon_0, \upsilon_1)).\end{aligned} \tag{25}$$

Taking limit $n \to \infty$ in (25), we obtain

$$\lim_{n\to\infty} \widehat{d}(\upsilon_{n+1}, \widetilde{T}\upsilon_{n+1} \cap \Lambda) = 0.$$

Since $g(\upsilon) = \widehat{d}(\upsilon, \widetilde{T}\upsilon \cap \Lambda)$ is $\widetilde{T}$-orbitally lower semicontinuous at $a^\star$, then

$$\widehat{d}(a^\star, \widetilde{T}a^\star \cap \Lambda) = g(a^\star) \le \liminf_n g(\upsilon_{n+1}) = \liminf_n \widehat{d}(\upsilon_{n+1}, \widetilde{T}\upsilon_{n+1} \cap \Lambda) = 0.$$

Since $\widetilde{T}a^\star$ is closed, we have $a^\star \in \widetilde{T}a^\star$. Conversely, if $a^\star$ is a fixed point of $\widetilde{T}$, then $g(a^\star) = 0 \le \liminf_n g(\upsilon_n)$, since $a^\star \in \Lambda$. □

Taking $\Gamma(r, s) = s - \int_0^r \varsigma(t)\mathrm{d}t$ for all $r, s \ge 0$, in Theorem 3.2, we get the following result.

Corollary 3.3. *Let $(\Bbbk, \widehat{d})$ be a complete metric space, Λ be a closed subset of $\Bbbk$, ξ be a Bianchini-Grandolfi gauge function on an interval E, and $\widetilde{T}: \Lambda \to CL(\Bbbk)$ be a given multivalued mapping. If there exists $\psi \in \Phi$ such that for $\widetilde{T}\upsilon \cap \Lambda \ne \varnothing$*

$$\psi[\widehat{d}(\upsilon, \widetilde{T}\upsilon \cap \Lambda), \widehat{d}(\upsilon, \nu)] < 0,$$

implies that

$$\int_0^{\alpha(\upsilon,\nu)\hat{H}(\widetilde{T}\upsilon\cap\Lambda,\widetilde{T}\nu\cap\Lambda)} \varsigma(t)\mathrm{d}t \le \xi(\hat{d}(\upsilon,\nu)),$$

for all $\upsilon \in \Lambda$, $\nu \in \widetilde{T}\upsilon \cap \Lambda$ *with* $\hat{d}(\upsilon,\nu) \in E$, *where* $\varsigma: \mathbb{R}^+ \to \mathbb{R}^+$ *is a function such that* $\int_0^{\varepsilon} \varsigma(t)\mathrm{d}t$ *exists and* $\int_0^{\varepsilon} \varsigma(t)\mathrm{d}t > \varepsilon$ *for all* $\varepsilon > 0$. *Suppose that the following conditions are satisfied:*

(i) $\widetilde{T}$ *is* α*-admissible;*

(ii) *there exists* $\upsilon_0 \in \Lambda$ *with* $\hat{d}(\upsilon_0,\upsilon_1) \in E$ *for some* $\upsilon_1 \in \widetilde{T}\upsilon_0 \cap \Lambda$ *such that* $\alpha(\upsilon_0,\upsilon_1) \ge 1$.

Then,

(a) *there exist an orbit* $\{\upsilon_n\}$ *of* $\widetilde{T}$ *in* Λ *and* $a^\star \in \Lambda$ *such that* $\lim_{n\to\infty}\upsilon_n = a^\star$;

(b) $a^\star$ *is a fixed point of* $\widetilde{T}$ *if and only if the function* $g(\upsilon) := \hat{d}(\upsilon, \widetilde{T}\upsilon \cap \Lambda)$ is $\widetilde{T}$*-orbitally lower semicontinuous at* $a^\star$.

Corollary 3.4. *Let* $(\Bbbk, \hat{d})$ *be a complete metric space,* ξ *be a Bianchini-Grandolfi gauge function on an interval* E, *and* $\widetilde{T}: \Bbbk \to CL(\Bbbk)$ *be a given multivalued mapping. If there exist* $\psi \in \Phi$ *and* $\Gamma \in \nabla$ *such that*

$$\psi[\hat{d}(\upsilon,\widetilde{T}\upsilon),\hat{d}(\upsilon,\nu)]<0 \Rightarrow \Gamma[\alpha(\upsilon,\nu)\hat{H}(\widetilde{T}\upsilon,\widetilde{T}\nu),\xi(\hat{d}(\upsilon,\nu))] \ge 0, \tag{26}$$

for all $\upsilon \in \Bbbk$, $\nu \in \widetilde{T}\upsilon$ *with* $\hat{d}(\upsilon,\nu) \in E$. *Suppose that the following conditions are satisfied:*

(i) $\widetilde{T}$ *is* α*-admissible;*

(ii) *there exists* $\upsilon_0 \in \Bbbk$ *with* $\hat{d}(\upsilon_0,\upsilon_1) \in E$ *for some* $\upsilon_1 \in \widetilde{T}\upsilon_0$ *such that* $\alpha(\upsilon_0,\upsilon_1) \ge 1$.

Then, there exists an orbit $\{\upsilon_n\}$ *of* $\widetilde{T}$ *in* $\Bbbk$ *that converges to the fixed point* $a^\star \in \mathcal{F} = \{\upsilon \in \Bbbk: \hat{d}(\upsilon, a^\star) \in E\}$ of $\widetilde{T}$.

4 Conclusion

The study deals with the achievement of introducing the notion of a wider new class of multivalued Suzuki-type θ-contractions via a gauge function. Within this framework, we introduced two related fixed point results in metric spaces. A nontrivial example was constructed to support our main results. Herein, the presented theorems and corollaries cannot be directly procured from the correlative metric space version.

References

[1] Sam B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.

[2] Hojat Afshari and Hassen Aydi, *On the extended multivalued Suzuki type contractions via a topological property*, Fixed Point Theory **20** (2019), no. 2, 407–416.

[3] Nayab Alamgir, Quanita Kiran, Hüseyin Işık, and Hassen Aydi, *Fixed point results via a Hausdorff controlled type metric*, Adv. Differ. Equ. **2020** (2020), 24, DOI: 10.1186/s13662-020-2491-8.

[4] Aiman Mukheimer, Jelena Vujakovic, Aftab Hussain, Hassen Aydi, Stojan Radenoviv, and Saman Yaqoob, *A new approach to multivalued nonlinear weakly Picard operators*, J. Ineq. Appl. **2019** (2019), 288, DOI: 10.1186/s13660-019-2244-y.

[5] Pradip Patle, Deepesh Patel, Hassen Aydi, and Stojan Radenović, *On H^+-type multivalued contractions and applications in symmetric and probabilistic spaces*, Mathematics **7** (2019), 144, DOI: 10.3390/math7020144.

[6] Haitham Qawaqneh, Mohd S. Noorani, Wasfi Shatanawi, Hassen Aydi and Habes Alsamir, *Fixed point results for multivalued contractions in b-metric spaces*, Mathematics **7** (2019), 132, DOI: 10.3390/math7020132.

[7] Nedal Tahat, Hassen Aydi, Erdal Karapinar, and Wasfi Shatanawi, *Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces*, Fixed Point Theory Appl. **2012** (2012), 48, DOI: 10.1186/1687-1812-2012-48.
[8] Amjad Ali, Fahim Uddin, Muhammad Arshad, and Maliha Rashid, *Hybrid fixed point results via generalized dynamic process for F-HRS type contractions with application*, Physica A **538** (2020), 122669, DOI: 10.1016/j.physa.2019.122669.
[9] Muhammad U. Ali, Quanita Kiran, and Naseer Shahzad, *Fixed point theorems for multivalued mappings involving α-function*, Abstr. Appl. Anal. **2014** (2014), 409467, DOI: 10.1155/2014/409467.
[10] Maria Samreen, Khansa Waheed, and Quanita Kiran, *Multivalued φ-contractions and fixed point theorems*, Filomat **32** (2018), no. 4, 1209–1220.
[11] Hüseyin Işık and Cristiana Ionescu, *New type of multivalued contractions with related results and applications*, U.P.B. Sci. Bull. Series A **80** (2018), no. 2, 13–22.
[12] Heddi Kaddouri, Hüseyin Işık, and Said Beloul, *On new extensions of F-contraction with application to integral inclusions*, U.P.B. Sci. Bull. Series A **81** (2019), no. 3, 31–42.
[13] Hüseyin Işık, Vahid Parvaneh, Babak Mohammadi, and Ishak Altun, *Common fixed point results for generalized Wardowski type contractive multi-valued mappings*, Mathematics **7** (2019), 1130, DOI: 10.3390/math7111130.
[14] Hüseyin Işık, Babak Mohammadi, Choonkil Park and Vahid Parvaneh, *Common fixed point and endpoint theorems for a countable family of multi-valued mappings*, Mathematics **8** (2020), 292, DOI: 10.3390/math8020292.
[15] Mujahid Abbas, Hira Iqbal, and Adrian Petrusel, *Fixed points for multivalued Suzuki type (θ,R)-contraction mapping with applications*, J. Funct. Spaces **2019** (2019), 9565804, DOI: 10.1155/2019/9565804.
[16] Francesca Vetro, *A generalization of Nadler fixed point theorem*, Carpathian J. Math. **31** (2015), no. 3, 403–410.
[17] Hanan A. Alolaiyan, Mujahid Abbas, and Basit Ali, *Fixed point results of Edelstein-Suzuki type multivalued mappings on b-metric spaces with applications*, J. Nonlinear Sci. Appl. **10** (2017), 1201–1214.
[18] Farshid Khojasteh, Satish Shukla, and Stojan Radenović, *A new approach to the study of fixed point theorems via simulation functions*, Filomat **29** (2015), no. 6, 1189–1194.
[19] Ankush Chanda, Lakshmi K. Dey, and Stojan Radenović, *Simulation functions: a survey of recent results*, RACSAM **113** (2019), 2923–2957.
[20] Hüseyin Işık, Nurcan B. Gungor, Choonkil Park, and Sun Y. Jang, *Fixed point theorems for almost Z-contractions with an application*, Mathematics **6** (2018), 37, DOI: 10.3390/math6030037.
[21] Antonio-Francisco Roldán-López-de-Hierro, Erdal Karapınar, Concepción Roldan-Lopez-de-Hierro, and Juan Martınez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. **275** (2015), 345–355.
[22] Bessem Samet, Calogero Vetro, and Pascal Vetro, *Fixed point theorem for α-ψ-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.
[23] Muhammad U. Ali, Tayyab Kamran, and Erdal Karapinar, *A new approach to α-ψ-contractive nonself multivalued mappings*, J. Ineq. Appl. **2014** (2014), 71, DOI: 10.1186/1029-242X-2014-71.
[24] Petko D. Proinov, *A generalization of the Banach contraction principle with high order of convergence of successive approximations*, Nonlinear Anal. **67** (2007), 2361–2369.