ON n-DIMENSIONAL RIEMANNIAN SPACES ADMITTING A GROUP OF MOTIONS OF ORDER $n(n-1)/2+1$

BY

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1. As is well known [2; 4; 11](1), we have the following theorem.

THEOREM A. *If an n-dimensional Riemannian space admits a group of motions of the maximum order $n(n+1)/2$, then, the space is of constant curvature.*

Thus, it might be interesting to ask whether an n-dimensional Riemannian space can admit complete groups of motions of order $n(n+1)/2-1$, $n(n+1)/2-2, \cdots$ or not, and if so, then what the structure of the corresponding space is.

In this connection, we have a very suggestive theorem due to G. Fubini [4, p. 229; 5]:

THEOREM B. *An n-dimensional Riemannian space for $n>2$ cannot admit a complete group of motions of order $n(n+1)/2-1$.*

On the other hand, it was an open problem to determine the n-dimensional Finslerian space which admits a group of motions of the maximum order $n(n+1)/2$. Recently, H. C. Wang [8] gave the answer to this problem by proving the following beautiful theorem.

THEOREM C. *If an n-dimensional Finslerian space for $n>2$, $n\neq 4$, admits a group of motions of order greater than $n(n-1)/2+1$, then the space is Riemannian and of constant curvature*(2).

To prove this theorem, Wang used, among others, the first of the following theorems due to D. Montgomery and H. Samelson [6].

THEOREM D. *In an n-dimensional Euclidean space for $n\neq 4$, there exists no proper subgroup of the rotation group of order greater than $(n-1)(n-2)/2$.*

THEOREM E. *In an n-dimensional Euclidean space for $n\neq 4$, $n\neq 8$, any subgroup of the rotation group of order $(n-1)(n-2)/2$ fixes one and only one direction.*

Wang's Theorem C not only generalizes Theorem A, but also gives the following interesting

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(1) The numbers between brackets refer to the bibliography at the end of the paper.

(2) Professor H. C. Wang pointed out to the author that Theorem D below, and consequently this Theorem C, are not true for $n=4$.

THEOREM F. *An n-dimensional Riemannian space for $n>2, n\neq 4$, which is not of constant curvature cannot admit a group of motions of order greater than $n(n-1)/2+1$.*

On the other hand, by studying the integrability conditions of the so-called Killing equations, I. P. Egorov [3] has proved recently the following two theorems.

THEOREM G. *The maximum order of the complete groups of motions in n-dimensional Riemannian spaces which are not Einstein spaces is $n(n-1)/2+1$.*

THEOREM H. *The order of complete groups of motions of those n-dimensional Riemannian spaces which are different from spaces of constant curvature is not larger than $n(n-1)/2+2$.*

According to Theorem C, if an n-dimensional Riemannian space for $n>2$, $n\neq 4$, admits a group of motions G_r of order $r>n(n-1)/2+1$, then the space is of constant curvature. The largest group of motions in a space of constant curvature being of order $n(n+1)/2$, if we denote it by G, then G_r must be a subgroup of G.

But, as will be seen in the next section, by exactly the same method as that used by Wang to prove Theorem C, we can prove that the group G_r coincides with the largest group G, that is to say:

THEOREM I. *In an n-dimensional Riemannian space for $n\neq 4$, there exists no group of motions of order r such that*

$$\frac{1}{2}n(n+1) > r > \frac{1}{2}n(n-1)+1.$$

Thus, it might not be useless to study the n-dimensional Riemannian spaces which admit a group of motions of order $n(n-1)/2+1$. This is the main purpose of the present paper, in which Theorem E plays an important rôle. The main result appears in the last Theorem 9.

2. We begin with a sketch of the proof of Theorem I. We consider an n-dimensional Riemannian space V_n with positive definite fundamental metric form $ds^2=g_{jk}(x)dx^jdx^k$ $(i, j, k, l, m=1, 2, \cdots, n)$ and assume that the space V_n admits a continuous group G_r of motions of order $r>n(n-1)/2+1$.

We take an arbitrary point $P_0(x_0^i)$ in the space V_n and consider all the motions of G_r leaving this point P_0 fixed. These motions constitute a subgroup $G(P_0)$ of G_r consisting of the motions:

$$T_\alpha:\quad \bar{x}^i = f^i(x;\alpha) \tag{1}$$

with the property

$$x_0^i = f^i(x_0;\alpha),$$

α being $r_0 \geqq r-n$ essential parameters [4, pp. 64–65]. The subgroup $G(P_0)$ is called a subgroup of stability of G_r at the point P_0.

To each transformation T_α of $G(P_0)$ corresponds a linear transformation S_α defined by

$$S_\alpha: \quad d\bar{x}^i = \frac{\partial f^i(x_0; \alpha)}{\partial x_0^j} dx^j. \tag{2}$$

It is easily seen that if $T_\alpha \to S_\alpha$ and $T_{\alpha'} \to S_{\alpha'}$, then $T_\alpha T_{\alpha'} \to S_\alpha S_{\alpha'}$, where $T_\alpha T_{\alpha'}$ is a product of two transformations while $S_\alpha S_{\alpha'}$ is a product of two matrices.

Thus, all the S's forming a linear group $L(P_0)$, it is readily proved [8] that the correspondence $T_\alpha \to S_\alpha$ is an isomorphism between $G(P_0)$ and $L(P_0)$ in the sense of topological groups, and consequently that $G(P_0)$ and $L(P_0)$ are of the same order.

The group $G(P_0)$ being a group of motions fixing the point P_0 in an n-dimensional Riemannian space, the group $L(P_0)$ is a rotation group in an n-dimensional Euclidean space.

On the other hand, we know that the order r_0 of $G(P_0)$ or of $L(P_0)$ satisfies the inequality

$$r_0 \geqq r - n.$$

But, we are assuming that

$$r > \frac{1}{2} n(n-1) + 1,$$

and consequently we have

$$r_0 > \frac{1}{2} n(n-1) + 1 - n = \frac{1}{2}(n-1)(n-2).$$

Thus, from Theorem D, we must have, for $n \neq 4$,

$$r_0 = \frac{1}{2} n(n-1),$$

and consequently the group $L(P_0)$ coincides with rotation group $O(n)$. Thus, $G(P_0)$ contains a motion which carries any given direction at P_0 into any given direction at P_0. We note here that, in the above discussion, the point P_0 was an arbitrary point.

Now, we take two arbitrary points P_1 and P_2 in V_n such that they are sufficiently near to each other and consequently they can be joined by a geodesic. We consider a midpoint M of this geodesic segment and a direction at M tangent to this geodesic. Then, in the group of stability $G(M)$, there exists a motion which changes the direction of this tangent into the opposite direction. Since a motion does not change the length of a curve and carries a geodesic into a geodesic, this motion carries P_1 into P_2.

If there are given two arbitrary points A and B in V_n, then we join these points by a curve, and choose a series of points on it $P_1, P_2, \cdots, P_N$ in such a way that A and P_1, P_1 and P_2, P_2 and $P_3, \cdots, P_{N-1}$ and P_N, P_N and B can be joined by geodesics. If we denote the midpoints of the geodesic segments $AP_1, P_1P_2, P_2P_3, \cdots, P_{N-1}P_N, P_NB$ by $M_0, M_1, \cdots, M_N$ respectively, then, applying suitable motions belonging to $G(M_0), G(M_1), \cdots, G(M_N)$ successively, we can carry the point A into the point B. The points A and B being any points in V_n, this means that our group G_r is transitive and consequently that

$$r = r_0 + n = n(n+1)/2.$$

Thus, Theorem I is proved.

3. Next, we assume that the space V_n admits a continuous group G_r of motions of order $r=n(n-1)/2+1$.

If we denote r infinitesimal operators of the group G_r by

$$X_Af = \overset{i}{\xi}_A(x)\frac{\partial f}{\partial x^i} \qquad (A = 1, 2, \cdots, r)$$

and the rank of the matrix $(\xi^i_A(x_0))$ at a point $P_0(x_0^i)$ by q_0, then $n \geqq q_0$, and the subgroup $G(P_0)$ of stability at P_0 is of order [4, p. 65]

$$r - q_0 = \frac{1}{2}n(n-1) + 1 - q_0.$$

Now, suppose that $n > q_0$, then we have

$$r - q_0 > \frac{1}{2}n(n-1) + 1 - n = \frac{1}{2}(n-1)(n-2).$$

Thus, the subgroup $G(P_0)$ of stability at P_0, and consequently the corresponding rotation group $L(P_0)$, is of order greater than $(n-1)(n-2)/2$. Thus, from Theorem D, we conclude that, for $n \neq 4$, the rotation group $L(P_0)$ coincides with $O(n)$.

Thus, if we denote the generic rank of the matrix $(\xi^i_A(x))$ by q and assume that $n > q$, then our Riemannian space admits free mobility around any point of the space, and, consequently, our group becoming transitive (see, for example, [2]), we have $n = q$, which contradicts our assumption.

Thus, we must have $n = q$ and consequently:

THEOREM 1. *If an n-dimensional Riemannian space for $n \neq 4$ admits a group of motions of order $n(n-1)/2+1$, then the group is transitive.*

In the following, since we need always Theorem E, we assume hereafter that $n \neq 4$, $n \neq 8$.

If we suppose that our V_n admits a group G_r of motions of order $r = n(n-1)/2+1$, and if we fix a point P_0 in V_n, then the above mentioned rotation group $L(P_0)$ is of order $(n-1)(n-2)/2$, and consequently, by Theorem E, it consists of all rotations around a fixed direction. Thus, with every point P of the space V_n, there is attached one and only one direction which is left invariant under the subgroup $G(P)$ of stability at the point P. We shall denote this direction by $\xi(P)$.

Now, we take two arbitrary points P and Q in our Riemannian space. Since the group G_r of motions is transitive, there exists a motion T carrying the point P into the point Q. If we denote an arbitrary motion fixing the point Q by T_Q, then the motion $T^{-1}T_QT$ fixes the point P. Thus, applying $T^{-1}T_QT$ to the direction $\xi(P)$, we obtain

$$T^{-1}T_QT\xi(P) = \xi(P),$$

$\xi(P)$ being invariant under $T^{-1}T_QT$. From the above equation, we have

$$T_QT\xi(P) = T\xi(P),$$

which shows that the direction $T\xi(P)$ at Q is invariant under any T_Q. Thus, we must have

$$T\xi(P) = \xi(Q),$$

and consequently:

THEOREM 2. *If an n-dimensional Riemannian space V_n for $n \neq 4$, $n \neq 8$ admits a group of motions of order $n(n-1)/2+1$, then there exists a field of directions such that the direction $\xi(P)$ at P is transformed into the direction $\xi(Q)$ at Q by any motion of the group carrying the point P into the point Q.*

Now, we consider the geodesic which is tangent to the above mentioned direction $\xi(P)$ at P. Since the group $G(P)$ of stability at P is a group of motions and fixes the point P and the direction $\xi(P)$, it fixes also this geodesic pointwise. Thus, if we take an arbitrary point Q on this geodesic, $G(P)$ fixes the point Q. Now, we consider an orthogonal frame at P whose first axis is in the direction of $\xi(P)$ and transport it parallelly along the geodesic to the point Q. Then we have, at Q, an orthogonal frame whose first axis is tangent to the geodesic. The parallelism of vectors along a curve being preserved by a motion, a motion belonging to $G(P)$ gives the same effect on the orthogonal frame at Q as on that at P. This shows that the group $G(P)$ behaves, at Q, as a group of motions fixing the point Q and of order $(n-1)(n-2)/2$, and consequently that $G(P) = G(Q)$. The group $G(Q)$ fixing the tangent to the geodesic and $\xi(Q)$, the tangent must coincide with $\xi(Q)$, and consequently the geodesic is a trajectory of the direction ξ.

Since there is one and only one trajectory passing through an arbitrary point of the space, these geodesics depend on $n-1$ parameters, and are transformed into each other by a motion belonging to G_r. Thus we have [2, p. 294]:

THEOREM 3. *If an n-dimensional Riemannian space V_n for $n\neq 4$, $n\neq 8$ admits a group of motions of order $n(n-1)/2+1$, then there exists a family of geodesics such that, passing through a point of the space, there is one and only one geodesic of the family and the geodesic passing through P is transformed into the geodesic passing through Q by a motion of the group carrying the point P into the point Q.*

4. Now we take a point M in V_n; then there is associated a direction $\xi(M)$ with this point. We attach to this point an orthogonal frame of reference $[\mathbf{e}_i]$ in such a way that the first axis $\mathbf{e}_1$ is in the direction of $\xi(M)$ and we consider all the frames of reference which are obtainable from this original one by all the motions of the transitive group G_r. Such a family of orthogonal frames of reference is said to be adapted to the group of motions under consideration.

The frames of reference thus attached to different points of the space depend on $n(n-1)/2+1$ parameters, the first n of which are coordinates $x^1, x^2, \cdots, x^n$ of the origin M and the last $(n-1)(n-2)/2$ of which are parameters $v^1, v^2, \cdots, v^{(n-1)(n-2)/2}$ fixing the directions of the axes $\mathbf{e}_2, \cdots, \mathbf{e}_n$.

Now, with respect to these moving orthogonal frames of reference, we write down the formulas

$$dM = \omega_i \mathbf{e}_i, \qquad d\mathbf{e}_i = \omega_{ij}\mathbf{e}_j \tag{3}$$

defining the Euclidean connexion without torsion of the Riemannian space under consideration. Here the ω_i and ω_{ij} are Pfaffian forms with respect to x and v.

The frames of reference being orthogonal ones, we must have

$$\omega_{ij} + \omega_{ji} = 0. \tag{4}$$

Moreover, the frames of reference being adapted ones, the Pfaffian forms ω_i and ω_{ij} are invariant under the group [2, p. 274].

We can see that the forms ω_i are linear homogeneous in dx^i, because they must vanish when the point (x^i) is fixed, that is to say, when the dx^i vanish, and moreover that the forms ω_{1j} are also linear homogeneous in dx^i because the vector $\mathbf{e}_1$ must be invariant, say, $d\mathbf{e}_1 = \omega_{1j}\mathbf{e}_j = 0$, when the point (x^i) is fixed.

Thus, ω_i being n linearly independent forms, we must have relations of the form

$$\omega_{1j} = c_{jk}\omega_k, \tag{5}$$

c_{jk} being functions of x and v.

But, all the motions belonging to G_r leave ω_{1j} and ω_k invariant, and consequently they leave c_{jk} invariant. Thus, the group being transitive on all the frames, the c_{jk} are all constants.

To find the values of these constants c_{jk}, we shall follow a method given by E. Cartan [2, pp. 293–295].

At two infinitely nearby points M and M' of the space, we consider the orthogonal frames of reference (R_M) and $(R_{M'})$ both adapted to the group G_r. Next, we effect, on both of them, an infinitesimal rotation around the first axis, these rotations being defined respectively with respect to (R_M) and $(R_{M'})$ by a bivector having the same components ξ_{ij}. By the assumption, ξ_{ij} must satisfy

$$\xi_{1j} = \xi_{j1} = 0. \tag{6}$$

We denote the orthogonal frames of reference thus obtained from (R_M) and $(R_{M'})$ by $(\overline{R}_M)$ and $(\overline{R}_{M'})$ respectively. Then the figure consisting of (R_M) and $(\overline{R}_M)$ is congruent to the figure consisting of $(R_{M'})$ and $(\overline{R}_{M'})$, that is to say, there exists a displacement which carries at the same time (R_M) into $(R_{M'})$ and $(\overline{R}_M)$ into $(\overline{R}_{M'})$. This displacement is analytically represented with respect to (R_M) by the set of vector ω_i and bivector ω_{ij}. But, under the transformation of the orthogonal frames of reference which carries (R_M) into $(\overline{R}_M)$, these components ω_i and ω_{ij} will receive the variations

$$\delta\omega_i = \xi_{ik}\omega_k, \qquad \delta\omega_{ij} = \xi_{ik}\omega_{kj} + \xi_{jk}\omega_{ik}. \tag{7}$$

But, from (5), we have

$$\delta\omega_{1j} = c_{jk}\delta\omega_k.$$

Substituting (5) and (7) into this equation, we find

$$\xi_{1k}\omega_{kj} + \xi_{jk}c_{kl}\omega_l = c_{jk}\xi_{kl}\omega_l,$$

or

$$\xi_{jk}c_{kl} - c_{jk}\xi_{kl} = 0 \tag{8}$$

by virtue of $\xi_{1k}=0$ and the linear independence of ω_l.

First putting $j=1$ in (5), we find

$$c_{1k} = 0. \tag{9}$$

Next, putting $l=1$ in (8), and taking account of $\xi_{k1}=0$, we get

$$\xi_{jk}c_{k1} = 0,$$

which must be satisfied by any ξ_{jk} $(=-\xi_{kj})$ satisfying (6), from which we conclude

$$c_{k1} = 0. \tag{10}$$

Finally, since $\xi_{1j}=\xi_{j1}=0$, we may consider that the summation index k in (8) takes the values $2, 3, \cdots, n$ only. Then, in (8), putting $j=r$, $l=t$ $(r, s, t, u, v=2, 3, \cdots, n)$, we obtain

$$\xi_{rs}c_{st} - c_{rs}\xi_{st} = 0,$$

which may be also written in the form

$$\xi_{uv}(\delta_{ur}c_{vt} - c_{ru}\delta_{vt}) = 0. \tag{11}$$

Equation (11) must be satisfied for any ξ_{uv} satisfying $\xi_{uv}+\xi_{vu}=0$, from which we get

$$(\delta_{ur}c_{vt} - c_{ru}\delta_{vt}) - (\delta_{vr}c_{ut} - c_{rv}\delta_{ut}) = 0.$$

Contracting, in this equation, with respect to r and v, we find

$$(n-2)c_{ut} + c_{tu} = c_{vv}\delta_{ut}. \tag{12}$$

If $n=3$, then we have

$$c_{ut} + c_{tu} = c_{vv}\delta_{ut}, \tag{13}$$

and consequently, we can conclude from (9), (10), and (13), that the matrix (c_{jk}) has the form

$$(c_{jk}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & \alpha \\ 0 & -\alpha & c \end{pmatrix}. \tag{14}$$

Since this case was throughly studied by E. Cartan [2, pp. 300–306], we assume hereafter $n>3$, $n\neq 4$, $n\neq 8$, that is to say, $n>4$, $n\neq 8$.

Then, taking the anti-symmetric part of both members of (12), we find $(n-3)(c_{ut}-c_{tu})=0$, from which

$$c_{ut} = c_{tu},$$

and consequently

$$c_{ut} = \frac{1}{n-1}c_{vv}\delta_{ut}.$$

Thus, in this case, the matrix (c_{jk}) has the form

$$(c_{jk}) = \begin{Bmatrix} 0 & 0 & 0\cdots 0 \\ 0 & c & 0\cdots 0 \\ 0 & 0 & c\cdots 0 \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot \\ 0 & 0 & 0\cdots c \end{Bmatrix}. \tag{15}$$

Consequently, we have, from (5),

$$\omega_{1s} = c\omega_s. \tag{16}$$

Thus, from the equations of structure

$$d\omega_i = [\omega_{ij}\omega_j],$$

we find

$$d\omega_1 = [\omega_{1s}\omega_s] = 0.$$

Thus, the Pfaffian form ω_1 is an exact differential:

$$\omega_1 = dg(x), \tag{17}$$

and, consequently, we can see that, in our space, there exists an ∞^1 family of hypersurfaces $g(x)=$constant along which we have $\omega_1=0$, or

$$dM = \omega_2\mathbf{e}_2 + \cdots + \omega_n\mathbf{e}_n.$$

Since vectors $\mathbf{e}_2, \cdots, \mathbf{e}_n$ are always tangent to one of these hypersurfaces, we can see that these hypersurfaces, regarded as $(n-1)$-dimensional Riemannian spaces, admit the free mobility. Thus, these hypersurfaces regarded as $(n-1)$-dimensional Riemannian spaces are all of constant curvature.

It is clear that the orthogonal trajectories of these hypersurfaces are geodesics referred to in Theorem 3.

Since any of these geodesics which are orthogonal trajectories of these hypersurfaces is transformed into any of these geodesics by a motion of G_r, we can see that any of these hypersurfaces is also transformed into any of these hypersurfaces by a motion of G_r. Thus, these hypersurfaces, regarded as $(n-1)$-dimensional Riemannian spaces, must be of *the same* constant curvature.

Now, we must distinguish here two cases: (I) $c=0$ and (II) $c\neq 0$.

We shall first assume that $c=0$ in (16). Then we have

$$\omega_{1s} = 0, \tag{18}$$

and consequently

$$d\mathbf{e}_1 = 0, \tag{19}$$

which shows that the $\mathbf{e}_1$ is a parallel vector field. Thus, the normal to the hypersurfaces referred to above being always parallel, the hypersurfaces must be totally geodesic, their orthogonal trajectories being geodesics.

We next assume that $c\neq 0$ in (16). Then we have

$$\omega_s = \frac{1}{c}\,\omega_{1s},$$

and consequently

$$
\begin{aligned}
dM &= \omega_1 e_1 + \omega_s e_s \\
&= \omega_1 e_1 + \frac{1}{c}\,\omega_{1s} e_s \\
&= \omega_1 e_1 + \frac{1}{c}\,de_1,
\end{aligned}
$$

from which

$$d\left(M - \frac{1}{c}\,e_1\right) = \omega_1 e_1, \tag{20}$$

which shows that

$$d\left(M - \frac{1}{c}\,e_1\right) = 0$$

along the hypersurfaces $\omega_1 = 0$, that is to say, the vector e_1 is a concurrent vector field [10] along the hypersurfaces referred to above. The normals to these hypersurfaces being concurrent along them, these hypersurfaces are totally umbilical hypersurfaces with constant mean curvature and their orthogonal trajectories are geodesic Ricci curves. Thus we have:

THEOREM 4. *If an n-dimensional Riemannian space V_n for $n>4$, $n\neq 8$ admits a group of motions of order $n(n-1)/2+1$; then* (I) *there exists an ∞^1 family of totally geodesic hypersurfaces whose orthogonal trajectories are geodesics, these hypersurfaces regarded as $(n-1)$-dimensional Riemannian spaces being of the same constant curvature, or* (II) *there exists an ∞^1 family of totally umbilical hypersurfaces with constant mean curvature whose orthogonal trajectories are geodesic Ricci curves, these hypersurfaces regarded as $(n-1)$-dimensional Riemannian spaces being of the same constant curvature. In both cases, the group leaves the family of geodesics and that of hypersurfaces invariant.*

5. We shall first study case (I). If case (I) in Theorem 4 occurs, then, the normals to these hypersurfaces being a parallel vector field, by a well known theorem [10], there exists a coordinate system in which the fundamental metric form of the space takes the form

$$ds^2 = (dx^1)^2 + g_{st}(x^r)dx^s dx^t, \tag{21}$$

the form $g_{st}(x^r)dx^s dx^t$ being the fundamental metric form of an $(n-1)$-dimensional Riemannian space V_{n-1} of constant curvature.

Conversely, if there exists a coordinate system in which the fundamental metric form of the space V_n takes the form (21), $g_{st}(x^r)dx^s dx^t$ being the fundamental metric form of an $(n-1)$-dimensional Riemannian space V_{n-1} of constant curvature, then it is evident that case (I) in Theorem 4 occurs and the space admits a group of motions G_r of order $n(n-1)/2+1$:

$$\bar{x}^1 = x^1 + t, \qquad \bar{x}^r = f^r(x; a),$$

where $\bar{x}^r = f^r(x; a)$ represent the group of motions of order $n(n-1)/2$ in the $(n-1)$-dimensional Riemannian space V_{n-1} of constant curvature. Thus we have:

THEOREM 5. *A necessary and sufficient condition that case* (I) *in Theorem* 4 *occur is that there exist a coordinate system in which the fundamental metric form of the space takes the form* (21), $g_{st}(x^r)dx^s dx^t$ *being the fundamental metric form of an* $(n-1)$*-dimensional Riemannian space of constant curvature.*

In this coordinate system, the fundamental tensors being of the form

$$(g_{jk}) = \begin{pmatrix} 1 & 0 \\ 0 & g_{st}(x^r) \end{pmatrix}, \qquad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & g^{rs}(x^t) \end{pmatrix},$$

if we calculate the Christoffel symbols of V_n:

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{1}{2} g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right),$$

we then find

$$\begin{Bmatrix} r \\ st \end{Bmatrix} = \begin{Bmatrix} r \\ st \end{Bmatrix}^*, \tag{22}$$

the other

$$\begin{Bmatrix} i \\ jk \end{Bmatrix}$$

being zero, where

$$\begin{Bmatrix} r \\ st \end{Bmatrix}^*$$

denotes the Christoffel symbols of V_{n-1}:

$$\begin{Bmatrix} r \\ st \end{Bmatrix}^* = \frac{1}{2} g^{ru}\left(\frac{\partial g_{us}}{\partial x^t} + \frac{\partial g_{ut}}{\partial x^s} - \frac{\partial g_{st}}{\partial x^u}\right).$$

Next, calculating the Riemann-Christoffel curvature tensor of V_n:

$$R^i{}_{jkl} = \frac{\partial \begin{Bmatrix} i \\ jk \end{Bmatrix}}{\partial x^l} - \frac{\partial \begin{Bmatrix} i \\ jl \end{Bmatrix}}{\partial x^k} + \begin{Bmatrix} m \\ jk \end{Bmatrix}\begin{Bmatrix} i \\ ml \end{Bmatrix} - \begin{Bmatrix} m \\ jl \end{Bmatrix}\begin{Bmatrix} i \\ mk \end{Bmatrix},$$

we find

$$R^r{}_{stu} = R^{*r}{}_{stu}, \tag{23}$$

the other $R^i{}_{jkl}$ being zero, where $R^{*r}{}_{stu}$ denotes the Riemann-Christoffel curvature tensor of V_{n-1}:

$$R^{*r}{}_{stu} = \frac{\partial \left\{ {r \atop st} \right\}^*}{\partial x^u} - \frac{\partial \left\{ {r \atop su} \right\}^*}{\partial x^t} + \left\{ {v \atop st} \right\}^* \left\{ {r \atop vu} \right\}^* - \left\{ {v \atop su} \right\}^* \left\{ {r \atop vt} \right\}^*.$$

But, we know that

$$R^{*r}{}_{stu} = \frac{R^*}{(n-1)(n-2)} (g_{st}\delta^r_u - g_{su}\delta^r_t), \tag{24}$$

R^* being an absolute constant, and consequently we have, for the Ricci tensor $R_{jk} = R^i{}_{jki}$ of V_n,

$$R_{st} = \frac{R^*}{n-1} g_{st}, \tag{25}$$

the other R_{jk} being zero. From (25), we obtain, for the scalar curvature $R = g^{jk}R_{jk}$ of V_n,

$$R = R^*. \tag{26}$$

Thus if we put

$$\pi_{jk} = -\frac{R_{jk}}{n-2} + \frac{Rg_{jk}}{2(n-1)(n-2)}, \qquad \pi^i{}_k = g^{ij}\pi_{jk}, \tag{27}$$

we then find

$$\begin{aligned} \pi_{11} &= \frac{R^*}{2(n-1)(n-2)}, & \pi_{st} &= -\frac{R^* g_{st}}{2(n-1)(n-2)}, \\ \pi^1{}_1 &= \frac{R^*}{2(n-1)(n-2)}, & \pi^r{}_t &= -\frac{R^*\delta^r_t}{2(n-1)(n-2)}, \end{aligned} \tag{28}$$

the other π's being zero.

Thus, for the Weyl conformal curvature tensor:

$$C^i{}_{jkl} = R^i{}_{jkl} + \pi_{jk}\delta^i_l - \pi_{jl}\delta^i_k + g_{jk}\pi^i{}_l - g_{jl}\pi^i{}_k, \tag{29}$$

we find

$$C^i{}_{jkl} = 0. \tag{30}$$

Thus, since we are assuming $n > 4$, our space must be conformally flat.

Conversely, if we assume that our space is conformally flat and admits a parallel vector field, then there exists a coordinate system in which

$$ds^2 = (dx^1)^2 + g_{st}(x^r)dx^s dx^t$$

and

$$\left\{ {r \atop st} \right\} = \left\{ {r \atop st} \right\}^*, \qquad R^r{}_{stu} = R^{*r}{}_{stu}, \qquad R_{st} = R^*{}_{st}, \qquad R = R^*,$$

the other

$$\left\{ {i \atop jk} \right\},$$

$R^i{}_{jkl}$ and R_{jk} being zero. From these we have

$$\pi_{11} = \frac{R^*}{2(n-1)(n-2)}, \qquad \pi_{st} = -\frac{R^*{}_{st}}{n-2} + \frac{R^* g_{st}}{2(n-1)(n-2)},$$

$$\pi^1{}_1 = \frac{R^*}{2(n-1)(n-2)}, \qquad \pi^r{}_t = -\frac{R^{*r}{}_t}{n-2} + \frac{R^* \delta^r_t}{2(n-1)(n-2)},$$

the other π's being zero.

First, from

$$C^1{}_{st1} = \pi_{st} + g_{st}\pi^1{}_1 = 0,$$

we find

$$R^*{}_{st} = \frac{R^*}{n-1} g_{st},$$

and consequently

$$\pi_{st} = -\frac{R^* g_{st}}{2(n-1)(n-2)}, \qquad \pi^r{}_t = -\frac{R^* \delta^r_t}{2(n-1)(n-2)}.$$

Next, from

$$C^r{}_{stu} = R^r{}_{stu} + \pi_{st}\delta^r_u - \pi_{su}\delta^r_t + g_{st}\pi^r{}_u - g_{su}\pi^r{}_t = 0,$$

we find

$$R^{*r}{}_{stu} = \frac{R^*}{(n-1)(n-2)} (g_{st}\delta^r_u - g_{su}\delta^r_t),$$

which shows that the hypersurfaces $x^1 = \text{const.}$ regarded as $(n-1)$-dimensional Riemannian spaces are of the same constant curvature. Thus we have:

THEOREM 6. *A necessary and sufficient condition that case* (I) *in Theorem* 4 *occur is that the space be conformally flat and admit a parallel vector field.*

T. Adati and the present author [12] proved that a necessary and sufficient condition that a space be Kagan's subprojective space is that the space be conformally flat and admit a concircular vector field. Thus Theorem 6 shows that the space under consideration is Kagan's subprojective space.

Next, we shall try to get a characterization by curvature tensor of the space referred to above.

First of all, there exists, in our space, a parallel vector field ξ^i:

$$\xi_{j;k} = 0, \tag{31}$$

semi-colon denoting the covariant differentiation.

We assume that ξ_i is a unit vector field and, ξ_i being a gradient field, we put

$$\xi_i = \frac{\partial g(x)}{\partial x^i}. \tag{32}$$

First, from (31), we find

$$R_{ijkl}\xi^i = 0, \qquad (R_{ijkl} = g_{im}R^m{}_{jkl}). \tag{33}$$

The sectional curvature at a point of the space determined by a 2-plane containing the unit vector ξ^i and an arbitrary unit vector η^i orthogonal to ξ^i is given by $-R_{ijkl}\xi^i\eta^j\xi^k\eta^l$. But, the space admitting a transitive group of motions which carry the field ξ^i into itself and any vector orthogonal to ξ^i into any vector orthogonal to ξ^i, this sectional curvature must be an absolute constant. But, from (33), we have

$$-R_{ijkl}\xi^i\eta^j\xi^k\eta^l = 0 \tag{34}$$

for any η^i, which shows that this sectional curvature is always zero.

On the other hand, we know that the hypersurfaces given by

$$g(x) = \text{constant} \tag{35}$$

are totally geodesic and are of the same constant curvature. Thus, representing one of (35) by parametric equations:

$$x^i = x^i(u^r),$$

and putting

$$\eta_r{}^i = \frac{\partial x^i}{\partial u^r},$$

we have, from the equation of Gauss,

$$R^*{}_{rstu} = R_{ijkl}\eta_r{}^i\eta_s{}^j\eta_t{}^k\eta_u{}^l, \tag{36}$$

where $R^*{}_{rstu}$ are components of curvature tensor of the hypersurface, and consequently have the form

(37) $$R^*_{rstu} = K(\overset{*}{g}_{st}\overset{*}{g}_{ru} - \overset{*}{g}_{su}\overset{*}{g}_{rt}),$$

g^*_{st} being the fundamental tensor of the hypersurface: $g^*_{st} = g_{jk}\eta_s{}^j\eta_t{}^k$.

$$\overset{*}{g}_{st} = g_{jk}\eta_s{}^j\eta_t{}^k.$$

The K in (37) represents the sectional curvature determined by a 2-plane orthogonal to ξ^i. The space admitting a transitive group of motions fixing ξ^i invariant, K must be an absolute constant.

Now, putting $\eta^s{}_j = g_{ij}g^{*rs}\eta_r{}^i$, we have

(38) $$\eta_r{}^i\eta^r{}_j = \delta^i_j - \xi^i\xi_j, \qquad \overset{*}{g}_{st}\eta^s{}_j\eta^t{}_k = g_{jk} - \xi_j\xi_k.$$

Multiplying both members of (36) by $\eta^r{}_a\eta^s{}_b\eta^t{}_c\eta^u{}_d$ $(a, b, c, d = 1, 2, \cdots, n)$ and contracting, we have, by virtue of (37) and (38),

$$K(\overset{*}{g}_{st}\overset{*}{g}_{ru} - \overset{*}{g}_{su}\overset{*}{g}_{rt})\eta^r{}_a\eta^s{}_b\eta^t{}_c\eta^u{}_d$$
$$= R_{ijkl}(\delta^i_a - \xi^i\xi_a)(\delta^j_b - \xi^j\xi_b)(\delta^k_c - \xi^k\xi_c)(\delta^l_d - \xi^l\xi_d),$$

or, by virtue of (33) and (38),

$$K[(g_{bc} - \xi_b\xi_c)(g_{ad} - \xi_a\xi_d) - (g_{bd} - \xi_b\xi_d)(g_{ac} - \xi_a\xi_c)] = R_{abcd},$$

from which

(39) $$R_{ijkl} = K[(g_{jk}g_{il} - g_{jl}g_{ik}) - (\xi_i g_{jk} - \xi_j g_{ik})\xi_l + (\xi_i g_{jl} - \xi_j g_{il})\xi_k].$$

Conversely, suppose that the curvature tensor of the space has the form (39) where K is a constant and ξ_i is a unit parallel vector field. The vector ξ_i being a gradient, if we put $\xi_i = \partial g/\partial x^i$, then the hypersurfaces $g(x)$ = constant are totally geodesic and their orthogonal trajectories are geodesics.

Representing one of these hypersurfaces by $x^i = x^i(u^r)$, we have, from (39) and the equation of Gauss,

$$R^*_{rstu} = K(\overset{*}{g}_{st}\overset{*}{g}_{ru} - \overset{*}{g}_{su}\overset{*}{g}_{rt}),$$

where R^*_{rstu} is curvature tensor of the hypersurface. This equation shows that the hypersurfaces regarded as $(n-1)$-dimensional Riemannian spaces are of the same constant curvature. Thus we have:

THEOREM 7. *A necessary and sufficient condition that case* (I) *in Theorem* 4 *occur is that the curvature tensor of the space have the form* (39) *where K is a constant and ξ_i is a unit parallel vector field.*

From (39), we have

(40) $$R_{ijkl;m} = 0.$$

Thus, we can see that our space is symmetric in the sense of E. Cartan [2].

6. We shall next study case (II). If case (II) in Theorem 4 occurs, then

the normals to the hypersurfaces being Ricci directions, by a well known theorem [9], the space admits a so-called concircular transformation [9] and consequently there exists a coordinate system in which the fundamental metric form of the space takes the form [9]:

$$ds^2 = (dx^1)^2 + f(x^1)f_{st}(x^r)dx^s dx^t, \tag{41}$$

the form $g_{st}dx^s dx^t = f(x^1)f_{st}(x^r)dx^s dx^t$ being that of $(n-1)$-dimensional Riemannian spaces V_{n-1} of the same constant curvature.

Here, if the function $f(x^1)$ reduces to a constant, then our case reduces to case (I). Consequently, in this case (II), we assume that $f(x^1)$ is not a constant.

Calculating the Christoffel symbols of V_n, we find

$$\left\{ {1 \atop st} \right\} = -\frac{1}{2}\frac{f'}{f} g_{st}, \quad \left\{ {r \atop 1t} \right\} = \left\{ {r \atop t1} \right\} = +\frac{1}{2}\frac{f'}{f}\delta^r_t, \quad \left\{ {r \atop st} \right\} = \left\{ {r \atop st} \right\}^*, \tag{42}$$

the other

$$\left\{ {i \atop jk} \right\}$$

being zero, where $f' = df/dx^1$ and the

$$\left\{ {r \atop st} \right\}^*$$

denote Christoffel symbols formed from $g_{st} = f(x^1)f_{st}(x^r)$ or, what amounts to the same thing, from $f_{st}(x^r)$.

Next, calculating the Riemann-Christoffel curvature tensor $R^i{}_{jkl}$ of V_n, we find

$$\begin{aligned} R^1{}_{s1u} &= -R^1{}_{su1} = +\left(\frac{1}{2}\frac{f''}{f} - \frac{1}{4}\frac{f'^2}{f^2}\right) g_{su}, \\ R^r{}_{11u} &= -R^r{}_{1u1} = -\left(\frac{1}{2}\frac{f''}{f} - \frac{1}{4}\frac{f'^2}{f^2}\right) \delta^r_u, \\ R^r{}_{stu} &= -R^r{}_{sut} = R^{*r}{}_{stu} - \frac{1}{4}\frac{f'^2}{f^2}(g_{st}\delta^r_u - g_{su}\delta^r_t), \end{aligned} \tag{43}$$

the other $R^i{}_{jkl}$ being zero, where $R^{*r}{}_{stu}$ denotes Riemann-Christoffel curvature tensor of V_{n-1}.

From (43), we get

$$\begin{aligned} R_{1s1u} &= \left(\frac{1}{2}\frac{f''}{f} - \frac{1}{4}\frac{f'^2}{f^2}\right) g_{su}, \\ R_{rstu} &= R^*{}_{rstu} - \frac{1}{4}\frac{f'^2}{f^2}(g_{st}g_{ru} - g_{su}g_{rt}), \end{aligned} \tag{44}$$

the other R_{ijkl} not related to these being zero.

From the first equation of (44), we see that the sectional curvature determined by two unit orthogonal vectors

$$(1, 0, 0, \cdots, 0), \qquad (0, \eta^2, \eta^3, \cdots, \eta^n)$$

is

$$- R_{1s1u}\eta^s\eta^u = -\left(\frac{1}{2}\frac{f''}{f} - \frac{1}{4}\frac{f'^2}{f^2}\right) \tag{45}$$

and does not depend on $(0, \eta^2, \cdots, \eta^n)$. But, the space admitting a transitive group of motions which carry the field $(1, 0, 0, \cdots, 0)$ into itself and any vector orthogonal to it into any vector orthogonal to it, this sectional curvature must be an absolute constant.

From the second equation of (44), we see that the sectional curvature determined by two mutually orthogonal unit vectors

$$(0, \eta^2, \eta^3, \cdots, \eta^n), \qquad (0, \zeta^2, \zeta^3, \cdots, \zeta^n)$$

is

$$- R_{rstu}\eta^r\zeta^s\eta^t\zeta^u = -\left(R^*_{rstu}\eta^r\zeta^s\eta^t\zeta^u + \frac{1}{4}\frac{f'^2}{f^2}\right).$$

This having to be independent of the choice of η^r and ζ^r, we must have

$$R^*_{rstu} = K^*(g_{st}g_{ru} - g_{su}g_{rt}), \tag{46}$$

and consequently

$$- R_{rstu}\eta^r\zeta^s\eta^t\zeta^u = K^* - \frac{1}{4}\frac{f'^2}{f^2}. \tag{47}$$

The group being transitive, this scalar must be also an absolute constant.

Equation (46) shows that the hypersurfaces $x^1 = \text{const.}$, regarded as $(n-1)$-dimensional Riemannian spaces, are of constant curvature. But we know that these must be of *the same* constant curvature. Thus, K^* is also an absolute constant.

On the other hand, we have

$$g_{st} = f(x^1)f_{st}(x^r),$$

$$\left\{ {r \atop st} \right\}^* = \frac{1}{2} f^{ru}\left(\frac{\partial f_{us}}{\partial x^t} + \frac{\partial f_{ut}}{\partial x^s} - \frac{\partial f_{st}}{\partial x^u}\right),$$

and consequently

$$R^{*r}{}_{stu} = F^r{}_{stu},$$

where $F^r{}_{stu}$ is the Riemann-Christoffel curvature tensor formed with $f_{st}(x^r)$.

Thus,

$$R^*_{rstu} = f(x^1)F_{rstu}. \tag{48}$$

From (46) and (48), we obtain

$$f(x^1)F_{rstu} = f(x^1)^2K^*(f_{st}f_{ru} - f_{su}f_{rt}),$$

or

$$F_{rstu} = F(f_{st}f_{ru} - f_{su}f_{rt}), \tag{49}$$

where

$$F = fK^* \tag{50}$$

is an absolute constant.

Now, we know that F and K^* are both absolute constants. But, we are assuming that the function $f(x^1)$ is not constant. Thus, we must have

$$F = 0, \qquad K^* = 0,$$

from which

$$F_{rstu} = 0, \qquad R^*_{rstu} = 0. \tag{51}$$

Moreover, the right-hand side of (47) being a constant, we put

$$\frac{1}{4}\frac{f'^2}{f^2} = k^2,$$

k being a constant different from zero, from which we get

$$f = a^2e^{2kx^1}, \tag{52}$$

a^2 being an arbitrary positive constant.

Thus, the fundamental metric form (41) takes the form

$$ds^2 = (dx^1)^2 + a^2e^{2kx^1}f_{st}(x^r)dx^sdx^t, \tag{53}$$

where the form $f_{st}(x^r)dx^sdx^t$ is, as equation (51) shows, the fundamental metric form of an $(n-1)$-dimensional Euclidean space.

Moreover, substituting (52) into (44), we get

$$R_{1s1u} = +k^2g_{su}, \qquad R_{rstu} = -k^2(g_{st}g_{ru} - g_{su}g_{rt}),$$

which may be also written as

$$R_{ijkl} = -k^2(g_{jk}g_{il} - g_{jl}g_{ik}). \tag{54}$$

Thus, the space is of negative constant curvature.

Conversely, if an n-dimensional Riemannian space is of negative constant curvature $-k^2$, then it is well known [1] that its metric can be written in the form (53), or

$$ds^2 = (dx^1)^2 + a^2 e^{2kx^1}[(dx^2)^2 + \cdots + (dx^n)^2], \tag{55}$$

or, on putting

$$ae^{kx^1} = \frac{1}{ku},$$

in the form

$$ds^2 = \frac{du^2 + (dx^2)^2 + \cdots + (dx^n)^2}{k^2 u^2}. \tag{56}$$

Thus, the space admits a group of motions of order $n(n-1)/2+1$ given by

$$\bar{u} = \alpha u, \qquad \bar{\overset{r}{x}} = \alpha(\overset{r}{a}_s \overset{s}{x} + \overset{r}{a}), \tag{57}$$

where α is a parameter and

$$\overset{*r}{x} = \overset{r}{a}_s \overset{s}{x} + \overset{r}{a}$$

represents a general motion in an $(n-1)$-dimensional Euclidean space. Thus we have:

THEOREM 8. *A necessary and sufficient condition that case* (II) *in Theorem* 4 *occur is that the space be of negative constant curvature.*

7. Gathering all the results, we can state the following:

THEOREM 9. *A necessary and sufficient condition that an n-dimensional Riemannian space V_n for $n>4$, $n\neq 8$ admit a group G_r of motions of order $r=n(n-1)/2+1$ is that the space be the product space of a straight line and an $(n-1)$-dimensional Riemannian space of constant curvature* (*this is equivalent to the fact that the space is conformally flat and admits a parallel vector field*) *or that the space be of negative constant curvature.*

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