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On $N(\kappa)$-Contact Metric Manifolds Satisfying Certain Curvature Conditions

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Abstract. We consider pseudo-symmetric and Ricci generalized pseudo-symmetric $N(\kappa)$-contact metric manifolds. We also consider $N(\kappa)$-contact metric manifolds satisfying the condition $S \cdot R = 0$ where R and S denote the curvature tensor and the Ricci tensor respectively. Finally we give some examples.

1. Introduction

An n-dimensional Riemannian manifold (M, g) is called *locally symmetric* if the condition $\nabla R = 0$ holds on M, where ∇ denotes the Levi-Civita connection and R is the corresponding curvature tensor of M. The class of locally symmetric manifolds is a natural generalization of the class of manifolds of constant curvature. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be *semi-symmetric* [14] if its curvature tensor R satisfies $R \cdot R = 0$, where $R(X, Y)$ acts on R as a derivation. Several studies have been done in contact geometry related to semi-symmetry condition and its generalizations. In [15], S. Tanno showed that a semi-symmetric K-contact manifold M^{2n+1} is locally isometric to the unit sphere $S^{2n+1}(1)$. In [13], D. Perrone studied contact metric manifolds satisfying $R(\xi, X) \cdot R = 0$ and under additional assumptions, it was shown that the manifold is either a Sasakian manifold of constant curvature 1 or $R(X, \xi)\xi = 0$. In [10], it was proved that a Sasakian manifold M^{2n+1} satisfying $R(\xi, X) \cdot C = 0$ is

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locally isometric to $S^{2n+1}(1)$, where C is the Weyl conformal curvature tensor. As a generalization of this result, in [4], it was proved that if ξ belongs to the κ-nullity distribution and if $R(\xi, X)\cdot C = 0$, then the contact metric manifold M^{2n+1} is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. In [7], the concircular curvature tensor of $N(\kappa)$-contact metric manifold have been studied. These circumstances motivate us to study some pseudo-symmetry type conditions for a contact metric manifold, which are another generalizations of semi-symmetry type conditions. The paper is organized as follows: In section 2, we give a brief introduction for $N(\kappa)$-contact metric manifolds. In section 3, pseudo-symmetry type manifolds are introduced. In section 4, pseudo-symmetric and Ricci generalized pseudo-symmetric $N(\kappa)$-contact metric manifolds are studied. In section 5, we consider a $N(\kappa)$-contact metric manifold satisfying the condition $S \cdot R = 0$. Finally we give some examples.

2. $N(\kappa)$-contact metric manifolds

A $(2n+1)$-dimensional manifold M is said to admit an *almost contact structure* if it admits a tensor field φ of type (1,1), a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

An almost contact structure is said to be *normal* if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric with almost contact structure, that is,

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

then M becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). From (2.1), it can be easily seen that

$$g(X, \varphi Y) = -g(\varphi X, Y)$$

and

$$g(X, \xi) = \eta(X)$$

for all vector fields X and Y. An almost contact metric structure becomes a contact metric structure if

$$g(X, \varphi Y) = d\eta(X, Y),$$

for vector fields X and Y. The 1-form η is then a contact form and ξ is its characteristic vector field. We call the normal contact metric manifold as a *Sasakian manifold*. A contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \quad X, Y \in TM,$$

where ∇ is Levi-Civita connection of the Riemannian metric g.

The κ-nullity distribution $N(\kappa)$ of a Riemannian manifold M is defined by

$$N(\kappa) : p \to N_p(\kappa) = \{Z \in T_pM : R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y)\},$$

κ being a constant. If the characteristic vector field $\xi \in N(\kappa)$, then we call a contact metric manifold an $N(\kappa)$-*contact metric manifold*. If $\kappa = 1$, then an $N(\kappa)$-contact metric manifold is Sasakian and if $\kappa = 0$, then $N(\kappa)$-contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$. If $\kappa < 1$, the scalar curvature is $r = 2n(2n - 2 + \kappa)$ [8].

In a $N(\kappa)$-contact metric manifold,

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y), \tag{2.2}$$

$$R(\xi,X)Y = \kappa(g(X,Y)\xi - \eta(Y)X) \tag{2.3}$$

and

$$S(X,\xi) = 2n\kappa\eta(X), \quad Q\xi = 2n\kappa\xi \tag{2.4}$$

hold [8], where Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

We also recall the notion of a $\mathcal{D}$-homothetic deformation. For a given contact metric structure (ϕ, ξ, η, g), this is, the structure defined by

$$\eta = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\eta \oplus \eta,$$

where a is a positive constant. While such a change preserves the state of being contact metric K-contact, Sasakian or strongly pseudo-convex CR-manifold, it destroys a condition like $R(X,Y)\xi = 0$ or $R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y)$. However the form of the $(k,\mu)-$nullity condition is preserved under a $\mathcal{D}$-homothetic deformation with

$$\bar{k} = \frac{k + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

Given a non-Sasakian $(k,\mu)-$manifold M, E.Boeckx [9] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-k}}$$

and showed that for two non-Sasakian $(k,\mu)-$manifolds $M_i(\phi_i, \xi_i, \eta_i, g_i), i = 1, 2$, we have $IM_1 = IM_2$ if and only if up to a $\mathcal{D}-$homothetic deformation, the two manifolds are locally isometric as contact metric manifolds. Thus we know all non-Sasakian $(k,\mu)-$ manifolds locally as soon as we have for every odd dimension $2n+1$ and for every possible value of the invariant I, one $(k,\mu)-$manifold $M(\phi,\xi,\eta,g)$ with $I_M = I$. For $I > 1$ such examples may be found from the standard contact matric structure on the tangent sphere bundle of a manifold of contact curvature c where we have $I = \frac{1+c}{|1-c|}$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I \leq -1$.

Using this invariant, we now construct an example of a $(2n+1)$-dimension $N(1-\frac{1}{n})-$contact metric manifold, $n > 1$.

Example 2.1. Since the Boeckx invariant for a $(1-\frac{1}{n},0)-$manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)-$dimensional manifold of constant curvature c so chosen that the resulting $\mathcal{D}-$homothetic deformation will be a $(1-\frac{1}{n},0)-$manifold. That is, for $k = c(2-c)$ and $\mu = -2c$ we solve

$$1-\frac{1}{n} = \frac{k+a^2-1}{a^2} \qquad 0 = \frac{\mu+2a-2}{a}$$

for a and c. The result is

$$c = \frac{(\sqrt{n}\pm 1)^2}{n-1}, \quad a = 1+c$$

and taking c and a to be these values we obtain an $N(1-\frac{1}{n})-$contact metric manifold.

The above example will be used in Theorem 5.1.

3. Pseudo-symmetry type manifolds

Let (M,g) be an $n(\geq 3)$-dimensional differentiable manifold of class C^∞. We define tensors $R\cdot R$ and $\bar{Q}(g,R)$ by

$$\begin{aligned}(R(X,Y)\cdot R)(X_1,X_2,X_3) &= R(X,Y)R(X_1,X_2)X_3 \\ -R(R(X,Y)X_1,X_2)X_3 - R(X_1,R(X,Y)X_2)X_3 &- R(X_1,X_2)R(X,Y)X_3 \qquad (3.1)\end{aligned}$$

and

$$\begin{aligned}\bar{Q}(g,R)(X_1,X_2,X_3;X,Y) &= (X\wedge Y)R(X_1,X_2)X_3 \\ -R((X\wedge Y)X_1,X_2)X_3 - R(X_1,(X\wedge Y)X_2)X_3 &- R(X_1,X_2)(X\wedge Y)X_3,\end{aligned}$$

respectively, where $X_1,X_2,X_3,X,Y \in TM$ and $X\wedge Y$ is an endomorphism [11] defined by

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y. \qquad (3.2)$$

If the tensors $R\cdot R$ and $\bar{Q}(g,R)$ are linearly dependent, then M is called *pseudo-symmetric* which is introduced by R. Deszcz [11] as a generalization of the semi-symmetry. This is equivalent to

$$R\cdot R = L_R\bar{Q}(g,R)$$

holding on the set $U_R = \{x\in M : \bar{Q}(g,R)\neq 0 \text{ at } x\}$, where L_R is some function on U_R [11]. In particular, if L_R is constant, M is called a *pseudo-symmetric manifold of constant type* [3]. A pseudo-symmetric manifold is said to be *proper* if it is not semi-symmetric. Every semi-symmetric manifold is pseudo-symmetric, but the converse statement is not true. It is trivial that if M is locally symmetric, then it is semi-symmetric.

If the tensors $R\cdot R$ and $\bar{Q}(S,R)$ are linearly dependent, then M is called *Ricci generalized pseudo-symmetric* [11]. This is equivalent to

$$R\cdot R = LQ(S,R)$$

holding on the set $U = \{x\in M : Q(S,R)\neq 0 \text{ at } x\}$, where L is some function on U. The tensors $Q(S,R)$ and $X\wedge_S Y$ are defined [11] by

$$Q(S,R)(X_1,X_2,X_3;X,Y) = (X\wedge_S Y)R(X_1,X_2)X_3 \tag{3.3}$$

$$-R((X\wedge_S Y)X_1,X_2)X_3 - R(X_1,(X\wedge_S Y)X_2)X_3 - R(X_1,X_2)(X\wedge_S Y)X_3$$

and

$$(X\wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y, \tag{3.4}$$

respectively. Moreover for a non-flat Riemannian manifold (M,g), the tensor $S\cdot R$ is defined by

$$(S\cdot R)(X_1,X_2,X_3,X_4) = -R(QX_1,X_2,X_3,X_4) - R(X_1,QX_2,X_3,X_4) \tag{3.5}$$

$$-\ R(X_1,X_2,QX_3,X_4) - R(X_1,X_2,X_3,QX_4),$$

where $X_1,X_2,X_3,X_4,X,Y,Z\in TM$. Semi-Riemannian manifolds satisfying the condition $S\cdot R = 0$ were investigated in [1] and [2].

4. Pseudo-symmertic and Ricci generalized pseudo-symmetric $N(\kappa)$-contact metric manifolds

We know from [6] that a contact metric manifold of constant curvature is necessarily a Sasakian manifold of constant curvature $+1$ or is 3-dimensional and flat, and a contact metric manifold M^{2n+1} satisfying $R(X,Y)\xi = 0$ is locally isometric to $E^{n+1}\times S^n(4)$ for $n>1$ and flat in dimension 3.

Now, we begin with the following:

Theorem 4.1. *Let M be a non-flat $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold. If M is a proper pseudo-symmetric manifold, then the manifold is a pseudosymmetric manifold of constant type.*

Proof. We assume that M is a proper pseudo-symmetric $N(\kappa)-$contact metric manifold. From (2.3) and (3.2), since

$$R(\xi, X)Y = \kappa(\xi \wedge X)Y,$$

it is easy to see that

$$R(\xi, X) \cdot R = \kappa(\xi \wedge X) \cdot R,$$

which implies that the pseudo-symmetry function $L_R = \kappa$. Hence the manifold is a pseudo-symmetric manifold of constant type. This completes the proof. □

Let us suppose that the $N(\kappa)-$contact metric manifold is semi-symmetric, that is,

$$(R(U, X) \cdot R)(Y, Z)W = 0,$$

which implies

$$(R(\xi, X) \cdot R)(Y, Z)W = 0. \tag{4.1}$$

From (4.1) we can write

$$\begin{aligned} R(\xi, X)R(Y, Z)W &- R(R(\xi, X)Y, Z)W - R(Y, R(\xi, X)Z)W \\ &- R(Y, Z)R(\xi, X)W = 0. \end{aligned} \tag{4.2}$$

Then using (2.3), the equation (4.2) can be written as

$$\begin{aligned} \kappa[R(Y, Z, W, X)\xi &- \eta(R(Y, Z)W)X - g(X, Y)R(\xi, Z)W + \eta(Y)R(X, Z)W \\ &- g(X, Z)R(Y, \xi)W + \eta(Z)R(Y, X)W - g(X, W)R(Y, Z)\xi \\ &+ \eta(W)R(Y, Z)X] = 0. \end{aligned} \tag{4.3}$$

Hence taking the inner product of (4.3) with ξ we get

$$\begin{aligned} \kappa[R(Y, Z, W, X) &- \eta(R(Y, Z)W)\eta(X) - g(X, Y)\eta(R(\xi, Z)W) + \eta(Y)\eta(R(X, Z)W) \\ &- g(X, Z)\eta(R(Y, \xi)W) + \eta(Z)\eta(R(Y, X)W) + \eta(W)\eta(R(Y, Z)X)] \\ &= 0. \end{aligned} \tag{4.4}$$

So in view of (2.2) and (2.3), the equation (4.4) turns into the form

$$\kappa[R(Y, Z, W, X) + \kappa(g(X, Z)g(Y, W) - g(X, Y)g(Z, W))] = 0,$$

which gives either $\kappa = 0$, or $R(Y, Z, W, X) = \kappa(g(X, Y)g(Z, W) - g(X, Z)g(Y, W))$.

For $\kappa = 0$, we get the manifold is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ [6]. On the other hand, if $R(Y,Z,W,X) = \kappa(g(X,Y)g(Z,W) - g(X,Z)g(Y,W))$, then M is a space of constant curvature κ. So from [6], it is necessarily a Sasakian manifold of constant curvature $+1$ for $n > 1$.

Thus we have the following:

Corollary 4.1. *Let M be a semisymmetric $N(\kappa)-$contact metric manifold, then the manifold is either locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$, or a Sasakian manifold of constant curvature $+1$.*

Theorem 4.2. *Let M be a non-flat $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold. Then M satisfies the condition $R(\xi, X) \cdot R = L(\xi \wedge_S X) \cdot R$ if and only if it is a Sasakian manifold of constant curvature $+1$ or locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$.*

Proof. By the use of (3.3), we can write

$$
\begin{aligned}
(4.5)\quad ((\xi \wedge_S X) \cdot R)(Y,Z,W) &= \bar{Q}(S,R)(Y,Z,W;\xi,X) \\
&= (\xi \wedge_S X)R(Y,Z)W \\
&- R((\xi \wedge_S X)Y,Z)W - R(Y,(\xi \wedge_S X)Z)W \\
&- R(Y,Z)(\xi \wedge_S X)W.
\end{aligned}
$$

So we get by using (3.4) from the above that

$$
\begin{aligned}
(4.6)\quad \bar{Q}(S,R)(Y,Z,W;\xi,X) &= S(R(Y,Z)W,X)\xi - S(\xi,R(Y,Z)W)X \\
&- S(X,Y)R(\xi,Z)W + S(\xi,Y)R(X,Z)W \\
&- S(X,Z)R(Y,\xi)W + S(\xi,Z)R(Y,X)W \\
&- S(X,W)R(Y,Z)\xi + S(\xi,W)R(Y,Z)X.
\end{aligned}
$$

Taking the inner product of (4.6) with ξ, we have

$$
\begin{aligned}
(4.7)\; g(\bar{Q}(S,R)(Y,Z,W;\xi,X),\xi) &= S(R(Y,Z)W,X) - S(\xi,R(Y,Z)W)\eta(X) \\
&- S(X,Y)\eta(R(\xi,Z)W) + S(\xi,Y)\eta(R(X,Z)W) \\
&- S(X,Z)\eta(R(Y,\xi)W) + S(\xi,Z)\eta(R(Y,X)W) \\
&+ S(\xi,W)\eta(R(Y,Z)X).
\end{aligned}
$$

Hence making use of (2.2), (2.3) and (2.4) in the last equation (4.7), we obtain

$$
\begin{aligned}
& g(((\xi \wedge_S X) \cdot R)(Y,Z,W),\xi) = g(\bar{Q}(S,R)(Y,Z,W;\xi,X),\xi) \\
= \;& S(R(Y,Z)W,X) + 2n\kappa^2[g(X,Z)\eta(Y)\eta(W) - g(X,Y)\eta(Z)\eta(W)] \\
- \;& \kappa\{g(Z,W)S(X,Y) - S(X,Y)\eta(Z)\eta(W) \\
- \;& g(Y,W)S(X,Z) + S(X,Z)\eta(Y)\eta(W)\}.
\end{aligned}
$$

Also we can deduce

$$
\begin{aligned}
(4.8)\qquad & g(R(\xi,X).R)(Y,Z,W),\xi) \\
&= k[R(Y,Z,W,X) + k(g(X,Z)g(Y,W) - g(X,Y)g(Z,W))].
\end{aligned}
$$

Since the condition $R(\xi, X) \cdot R = L(\xi \wedge_S X) \cdot R$ holds on M, by (4.5) and (4.8), we can write as

$$\begin{aligned} \text{(4.9)} \qquad & \kappa[R(Y,Z,W,X) + \kappa(g(X,Z)g(Y,W) - g(X,Y)g(Z,W))] \\ &= L\{2n\kappa^2[g(X,Z)\eta(Y)\eta(W) - g(X,Y)\eta(Z)\eta(W)] \\ &+ S(R(Y,Z)W,X) - \kappa[g(Z,W)S(X,Y) - S(X,Y)\eta(Z)\eta(W) \\ &- g(Y,W)S(X,Z) + S(X,Z)\eta(Y)\eta(W)]\}. \end{aligned}$$

Taking $Y = \xi$ in (4.9) and using (2.3) and (2.4), we have

$$L\{2n\kappa(1-\kappa)g(Z,W)\eta(X) - S(X,Z)\eta(W) + 2n\kappa^2 g(X,Z)\eta(W)\} = 0.$$

Hence either $L = 0$ or

$$\text{(4.10)} \qquad 2n\kappa(1-\kappa)g(Z,W)\eta(X) - S(X,Z)\eta(W) + 2n\kappa^2 g(X,Z)\eta(W) = 0.$$

If $L = 0$, then $R(\xi, X) \cdot R = 0$ holds on M. Then by the proof of Corollary 4.1, it is a Sasakian manifold of constant curvature $+1$ or locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$.

Next, if the condition (4.10) holds on M, then contracting (4.10) over Z and W we get

$$4n^2\kappa(1-\kappa) = 0,$$

which gives us either $\kappa = 0$ or $\kappa = 1$. Hence in the case of $\kappa = 0$, M is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and in the case of $\kappa = 1$, M is a Sasakian manifold of constant curvature 1.

The converse statement is trivial. This proves the theorem. □

5. $N(\kappa)$-contact metric manifolds satisfying the condition $S \cdot R = 0$

In this section we prove the following:

Theorem 5.1. *Let M be a $(2n+1)$-dimensional non-flat and non-Sasakian $N(\kappa)$-contact metric manifold. If M satisfies the condition $S \cdot R = 0$, then M is either locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ or the manifold is locally isometric to the Example 2.1.*

Proof. From (3.5) we know

$$\begin{aligned} \text{(5.1)} \qquad (S \cdot R)(X,Y,Z,W) &= -R(QX,Y,Z,W) - R(X,QY,Z,W) \\ &- R(X,Y,QZ,W) - R(X,Y,Z,QW). \end{aligned}$$

Now assume that the condition $S \cdot R = 0$ holds on M. Then taking $X = \xi$ in (5.1), we get

$$\begin{aligned} \text{(5.2)} \qquad & R(Q\xi,Y,Z,W) + R(\xi,QY,Z,W) \\ &+ R(\xi,Y,QZ,W) + R(\xi,Y,Z,QW) = 0. \end{aligned}$$

So by the use of (2.4), the equation (5.2) turns into the form

$$\begin{aligned} 2n\kappa R(\xi, Y, Z, W) + R(\xi, QY, Z, W) \\ +R(\xi, Y, QZ, W) + R(\xi, Y, Z, QW) = 0. \end{aligned} \tag{5.3}$$

Moreover, by the use of (2.3) in (5.3), we have

$$\begin{aligned} & 2n\kappa^2[g(Y, Z)\eta(W) - g(Y, W)\eta(Z)] \\ + & \kappa[S(Y, Z)\eta(W) - S(Y, W)\eta(Z)] \\ + & \kappa[S(Y, Z)\eta(W) - 2n\kappa g(Y, W)\eta(Z)] \\ + & \kappa[2n\kappa g(Y, Z)\eta(W) - S(Y, W)\eta(Z)] = 0. \end{aligned} \tag{5.4}$$

Then putting $W = \xi$ in (5.4) we have

$$2\kappa[-S(Y, Z) - 2n\kappa g(Y, Z) + 4n\kappa\eta(Y)\eta(Z)] = 0,$$

which gives us either $\kappa = 0$ or the condition

$$S(Y, Z) - 2n\kappa[-g(Y, Z) + 2\eta(Y)\eta(Z)] = 0 \tag{5.5}$$

holds on M. In the case of $\kappa = 0$, M is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ [6]. If the condition (5.5) holds on M, then contracting the equation (5.5) over Y and Z, we get $r = 2n\kappa(1 - 2n)$. For a non-Sasakian $N(\kappa)$-contact metric manifold, since $r = 2n(2n - 2 + \kappa)$ [8], we obtain $\kappa = \frac{1}{n} - 1$. In this case, if $n = 1$, then $\kappa = 0$, hence M is flat [6]. But this case can not occur because of the non-flatness assumption. Thus the manifold is locally isometric to the Example 2.1. □

6. Examples

Example 6.1. In [12] J. Milnor gave a complete classification of three dimensional manifolds admitting the Lie algebra structure

$$[e_2, e_3] = c_1 e_1, \;\; [e_3, e_1] = c_2 e_2, \;\; [e_1, e_2] = c_3 e_3. \tag{6.1}$$

As in the case of the given example of [7], let us consider η be the dual 1-form to the vector field e_1. Using (6.1) we get

$$d\eta(e_2, e_3) = -d\eta(e_3, e_2) = \frac{c_1}{2} \neq 0$$

and $d\eta(e_i, e_j) = 0$ for $(i, j) \neq (2, 3), (3, 2)$. It is easy to check that η is a contact form and e_1 is the characteristic vector field. Defining a Riemannian metric g by $g(e_i, e_j) = \delta_{ij}$, then, because we must have $d\eta(e_i, e_j) = g(e_i, \phi e_j)$, ϕ has the same metric as $d\eta$ with respect to the basis e_i. Moreover, for g to be an associated metric, we must have $\phi^2 = -I + \eta \otimes e_1$. So for (ϕ, e_1, η, g) to be a contact metric structure we must have $c_1 = 2$. The unique Riemannian connection ∇ coresponding to g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) \\ &- g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

So using $c_1 = 2$ we easily get

$$\nabla_{e_1} e_1 = 0, \;\; \nabla_{e_2} e_2 = 0, \;\; \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = \frac{1}{2}(c_2 + c_3 - 2)e_3, \;\; \nabla_{e_2} e_1 = \frac{1}{2}(c_2 - c_3 - 2)e_3,$$

$$\nabla_{e_1} e_3 = -\frac{1}{2}(c_2 + c_3 - 2)e_2, \;\; \nabla_{e_3} e_1 = \frac{1}{2}(2 + c_2 - c_3)e_2.$$

But we also know that

$$\nabla_{e_2} e_1 = -\phi e_2 - \phi h e_2.$$

Comparing now those two relations of $\nabla_{e_2} e_1$ and using $\phi e_1 = 0$, $\phi e_3 = -e_2$ we conclude that

$$he_2 = \frac{c_3 - c_2}{2} e_2.$$

And hence

$$he_3 = -\frac{c_3 - c_2}{2} e_3.$$

Thus e_i are eigenvectors of h with corresponding eigenvalues $(0, \lambda, -\lambda)$ where $\lambda = \frac{c_3 - c_2}{2} e_2$. Moreover by direct calculation we have

$$R(e_2, e_1)e_1 = [1 - \frac{(c_3 - c_2)^2}{4}]e_2 + [2 - c_2 - c_3]he_2,$$

$$R(e_3, e_1)e_1 = [1 - \frac{(c_3 - c_2)^2}{4}]e_3 + [2 - c_2 - c_3]he_3.$$

$$R(e_2, e_3)e_1 = 0.$$

Putting $k = 1 - \frac{(c_3 - c_2)^2}{4}$ and $\mu = 2 - c_2 - c_3$ we conclude, from these relations that e_1 belongs to the (k, μ)-nullity distribution, for any c_2, c_3.

Now putting $c_2 = c_3 = 1$, we get the manifold is a $N(\kappa)-$contact metric manifold with $\kappa = 1$. Thus the manifold is a Sasakian manifold. Also from the expressions of the curvature tensor, for $c_2 = c_3 = 1$, we get that the manifold is a manifold of constant curvature. Thus the Theorem 4.2 is verified.

Example 6.2([5]). Every Sasakian space form $M^{2n+1}(c)$ is pseudo-symmetric. Also the Sasakian space form $M^{2n+1}(-3)$ is pseudo-symmetric but not semi-symmetric.

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References

[1] K. Arslan, R. Deszcz, R. Ezentaş and M. Hotlos, *On A certain class of conformally flat manifolds*, Bull. Inst. Math.Acad. Sinica, **26**(1998), 183-199.

[2] K. Arslan, R. Deszcz and R. Ezentaş, *On a certain class of hypersurfaces in semi-Euclidean spaces*, Soochow J. Math., **25**(1999), 223-236.

[3] N. Hashimoto and M. Sekizawa, *Three dimensional conformally flat pseudo-symmetric spaces of constant type*, Arch. Math.(Brno), **36**(2000), 279-286.

[4] C. Baikoussis and T. Koufogiorgos, *On a type of contact manifolds*, J. Geom., **46(1-2)**(1993), 1-9.

[5] M. Belkhelfa, R. Deszcz and L. Verstraelen, *Symmetry Properties of Sasakian Space Forms*, Soochow J. of Math., **31(4)**(2005), 611-616.

[6] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, 203. Birkhauser Boston, Inc., Boston, MA, USA, 2002.

[7] D. E. Blair, J. S. Kim and M. M. Tripathi, *On the concircular curvature tensor of a contact metric manifold*, J. Korean Math. Soc., **42(5)**(2005), 883-892.

[8] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., **91(1-3)**(1995), 189-214.

[9] E. Boeckx, *A full classification of contact metric $(\kappa,\mu)-$spaces*, Illinois J. of Math., **44**(2000), 212-219.

[10] M.C.Chaki and M. Tarafdar, *On a type of Sasakian manifold*, Soochow J. Math., **16(1)**(1990), 23-28.

[11] R. Deszcz, *On pseudosymmetric spaces*, Bull. Belgian Math. Soc., Ser. A, **44**(1992), 1-34.

[12] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Advances in Mathematics, **21**(1976), 293-311.

[13] D. Perrone, *Contact Riemannian manifolds satisfying $R(X,\xi)\cdot R = 0$*, Yokohama Math. J., **39(2)**(1992), 141-149.

[14] Z. I. Szabo, *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R = 0$, The local version*, J. Diff. Geom., **17**(1982), 531-582.

[15] S. Tanno, *Isometric immersions of Sasakian manifolds in spheres*, Kodai Math. J., **21**(1969), 448-458.

[16] K. Yano and M. Kon, Structures on manifolds, Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, 1984.