On Nash Equilibria for a Network Creation Game

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Abstract

We study a network creation game recently proposed by Fabrikant, Luthra, Maneva, Papadimitriou and Shenker. In this game, each player (vertex) can create links (edges) to other players at a cost of α per edge. The goal of every player is to minimize the sum consisting of (a) the cost of the links he has created and (b) the sum of the distances to all other players.

Fabrikant et al. conjectured that there exists a constant A such that, for any $\alpha > A$, all non-transient Nash equilibria graphs are trees. They showed that if a Nash equilibrium is a tree, the price of anarchy is constant. In this paper we disprove the tree conjecture. More precisely, we show that for any positive integer n_0, there exists a graph built by $n \geq n_0$ players which contains cycles and forms a non-transient Nash equilibrium, for any α with $1 < \alpha \leq \sqrt{n/2}$. Our construction makes use of some interesting results on finite affine planes. On the other hand we show that, for $\alpha \geq 12n\lceil \log n \rceil$, every Nash equilibrium forms a tree.

Without relying on the tree conjecture, Fabrikant et al. proved an upper bound on the price of anarchy of $O(\sqrt{\alpha})$, where $\alpha \in [2, n^2]$. We improve this bound. Specifically, we derive a constant upper bound for $\alpha \in O(\sqrt{n})$ and for $\alpha \geq 12n\lceil \log n \rceil$. For the intermediate values we derive an improved bound of $O(1 + (\min\{\frac{\alpha^2}{n}, \frac{n^2}{\alpha}\})^{1/3})$.

Additionally, we develop characterizations of Nash equilibria and extend our results to a weighted network creation game as well as to scenarios with cost sharing.

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1 Introduction

Network design is a fundamental problem in computer science and operations research. This line of research assumes a central authority that constructs the network and has various optimization criteria to fulfill. In practice, however, many networks are actually formed by selfish players who are motivated by their own interests and their own objective function. For instance, the Internet, networks for exchanging goods and social networks are all formed by many players and not by a single authority. This motivates the research of network creation by multiple selfish players.

In this work we focus on the later model and allow individual users to decide which edges to buy. The appropriate concept for studying such a scenario is that of Nash equilibria [19], where no user has the incentive to deviate from his strategy. We analyze the performance of the resulting network architectures using the *price of anarchy*, introduced by Koutsoupias and Papadimitriou in their seminal paper [17]. Recently, Nash equilibria and their associated price of anarchy have been studied for a wide range of classical computer problems such as job scheduling, routing, facility location and, last but not least, network design and creation, see e.g. [1, 2, 3, 7, 6, 8, 11, 10, 13, 15, 17, 21]. This also includes variants of the price of anarchy, called the price of stability [1, 2, 6].

In this paper we study a network creation game introduced by Fabrikant et al. [10]. The game is defined as follows, there are n players, each of which is associated with a separate network vertex. These players have to build a connected, undirected graph. Each player may lay down edges to other players. Once the edges are installed, they are regarded as undirected and may be used in both directions. The resulting network is the set of players (vertices) and the union of all edges laid out. The cost of each player consists of two components. Firstly, a player pays an edge building cost equal to α times the number of edges laid out by him, for some $\alpha > 0$. Secondly, the player incurs a connection cost equal to the sum of the shortest path distances to other players. This game models scenarios in which peers wish to communicate and transfer data. Each peer incurs a hardware cost and pays for the communication delays to other players.

Formally, we represent the set of players by a vertex set $V = \{1, \ldots, n\}$. A *strategy*, for a player $v \in V$, is a set of vertices $S_v \subseteq V \setminus \{v\}$ such that v creates an edge to every $w \in S_v$. (Note that we consider only pure strategies of the players.) Given a joint strategy $\vec{S} = (S_1, \ldots, S_n)$, the resulting graph $G(\vec{S}) = (V, E)$ consists of the edge set $E = \bigcup_{v \in V} \bigcup_{w \in S_v} \{v, w\}$. In our analysis it will sometimes be convenient to assume that the edges have a direction. A directed edge (v, w) indicates that the player v built an edge to w. The cost of a player v under $\vec{S}$ is $Cost(v, \vec{S}) = \alpha|S_v| + \sum_{w \in V, w \neq v} \delta(v, w)$, where $\delta(v, w)$ is the length of the shortest path between v and w in $G(\vec{S})$.

A joint strategy $\vec{S}$ forms a Nash equilibrium if, for any player $v \in V$ and any other joint strategy $\vec{U}$ that differ from $\vec{S}$ only in v's strategy, $Cost(v, \vec{S}) \leq Cost(v, \vec{U})$. The induced graph $G(\vec{S})$ is called the equilibrium graph. $\vec{S}$ is a *strong* Nash equilibrium if, for every player v, strict inequality $Cost(v, \vec{S}) < Cost(v, \vec{U})$ holds. Otherwise, it is a *weak* Nash equilibrium. In a weak Nash equilibrium at least one player can change its strategy without affecting its cost. We will also use the notion of *transient* Nash equilibria [10]. A transient Nash equilibrium is a weak equilibrium from which there exists a sequence of single-player strategy changes, which do not change the deviator's cost, leading to a non-equilibrium position.

For a joint strategy $\vec{S}$, let $Cost(\vec{S}) = \sum_{v \in V} Cost(v, \vec{S})$ be the total cost of all players. Let $Cost(OPT)$ be the cost of the social optimum that achieves the smallest possible value. The price of anarchy is the worst-case ratio $Cost(\vec{S})/Cost(OPT)$, taken over all Nash equilibria $\vec{S}$.

The main interest of Fabrikant et al. [10] was to analyze the price of anarchy of the game. They observed that, for $\alpha < 2$ and $\alpha > n^2$, it is constant. Their main contribution is an upper bound of $O(\sqrt{\alpha})$ for $\alpha \in [2, n^2]$. This upper bound can be as large as $O(n)$ when $\alpha = n^2$. Fabrikant et al. pointed out that in their constructions as well as in experiments they performed they only found tree Nash equilibria. The only exception was the Petersen graph that represents a transient Nash equilibrium. This fact motivated them to formulate a *tree conjecture* stating that there exists a constant A such that, for any $\alpha > A$, all non-transient Nash equilibria are trees. In other words, every Nash equilibrium that has a cycle in the underlying graph is transient and, in particular, weak. They proved that if the tree conjecture holds, the price of anarchy is constant, for any α.

Our contribution: In this paper we first show that the tree conjecture is incorrect, and show that the possible resulting equilibria can have a rich and involve structure. We prove that, for any positive integer n_0, there exists a graph built by $n \geq n_0$ players that contains cycles and forms a strong Nash equilibrium, for any α with $1 < \alpha \leq \sqrt{n/2}$. The graphs we construct are *geodetic*, i.e. the shortest path between any two vertices is unique, and have a diameter of 2. These properties are crucial in showing that the Nash equilibrium is indeed strong. If a player deviates from its original strategy and builds less edges or edges to different players, then — since the original graph was geodetic — the shortest path distance cost increases substantially. If a player decides to build more edges, then — since the graph has diameter 2 — the cost saving is negligible. Our construction resorts to some concepts from graph theory and geometry. In particular, we use results on finite affine planes. To the best of our knowledge, these concepts have never been used in game theoretic investigations and might be helpful when studying other graph oriented games.

We proceed and give improved upper bounds on the price of anarchy. Our main result here is a constant upper bound on the price of anarchy for both $\alpha \in O(\sqrt{n})$ and $\alpha \geq 12n\lceil\log n\rceil$ and a worst case bound of $O(n^{1/3})$ instead of $O(n)$. More precisely, we prove that if $\alpha \geq 12n\lceil\log n\rceil$, the price of anarchy is not larger than 1.5 and goes to 1 as α increases. Interestingly, the proof shows that if $\alpha \geq 12n\lceil\log n\rceil$, any Nash equilibrium is indeed a tree. For any α, we prove an upper bound of $O(1 + (\min\{\frac{\alpha^2}{n}, \frac{n^2}{\alpha}\})^{1/3})$. Thus, if $\alpha \in O(\sqrt{n})$, the price of anarchy is again constant. For $\alpha \in [\sqrt{n}, n]$ the value increases, reaching a maximum of $O(n^{1/3})$ at $\alpha = n$. For $\alpha > n$, the price of anarchy is decreasing.

Furthermore, we analyze the structure of Nash equilibria, investigating solutions with short induced cycles. We prove that any Nash equilibrium that forms a chordal graph having induced cycles of length three is indeed transient. We show that such equilibria do exist for all n. Furthermore, we show that if $\alpha < n/2$, then the only tree that forms an equilibrium is the star and that there exists Nash equilibria graphs of n vertices which are not trees.

Additionally, we study a weighted network creation game in which player v wishes to send a certain amount of traffic to player u, for any v and u. In the cost of player v, the shortest path distance to u is multiplied by this traffic amount. We provide an upper bound on the price of anarchy. Our bound in the weighted case is such, that when the traffic amounts are uniform, the bound is asymptotically equal to that of the unweighted game.

Finally, we consider settings with cost sharing where players can pay for a fraction of an edge. The edge exists if the total contribution by all players is at least α. We show that in both the unweighted and weighted games part of our upper bounds on the price of anarchy carry over. We also prove that there exist strong Nash equilibria with cycles in which the cost is split evenly among players.

Due to lack of space some of our proofs are omitted and can be found in the complete version of the paper available on the web.

Related work: There exists a large body of previous work on other network design problems. Anshelevich et al. [1] investigate a network design problem where players, in a given graph, have to connect desired terminal pairs. They analyze the quality of the best Nash equilibrium under Shapley cost sharing. Anshelevich et al. [2] consider connection games where each player has to connect a set of terminals and present algorithms for computing approximate Nash equilibria. Further work on cost sharing in network design includes [12, 15, 20, 16]. Bala and Goyal [3] study a network formation problem in which players incur cost but also benefit from building edges to other players. They tradeoff the costs of forming links against the potential reward from doing so. Haller and Sarangi [13] build on this work and allow player heterogeneity.

In a recent work Corbo and Parkes [5] study the price of anarchy in the model introduced by Fabrikant et al. with a (crucial) variation that the edges are not bought by a single player but by both players at the end points of the edge. In a recent unpublished note, independent of our work, Lin [18] shows that for $\alpha = O(\sqrt{n})$ and $\alpha = \Omega(n^{3/2})$ the price of anarchy is constant.

Social and economic networks in which each player is a different vertex in the graph play a major role in the economic literature. For a recent and detailed review of social and economics models see [14].

2 Disproving the tree conjecture

We will present a family of graphs that form strong Nash equilibria and have induced cycles of length three and five. To construct these graphs, we have to define affine planes, see e.g. Mac Williams and Sloane [22].

DEFINITION 1. *An affine plane is a pair $(A, \mathcal{L})$, where A is a set (of points) and $\mathcal{L}$ is a family of subsets of A (of lines) satisfying the following four conditions.*

- *For any two points, there is a unique line containing these points.*
- *Each line contains at least two points.*
- *Given a point x and a line L that does not contain x, there is a unique line L' that contains x and is disjoint from L.*
- *There exists a triangle, i.e. there are three distinct points which do not lie on a line.*

If A is finite, then the affine plane is called finite.

Two lines are *parallel*, in signs $\|$, if the lines are disjoint or if they are equal. Given a point x and a line L, we denote by $(x\|L)$ the unique line that is parallel to L and contains x. Parallelism defines an equivalence relation on the lines, and the equivalence class of L is denoted by $[L]$.

If q is a prime power, then for the field $F = GF(q)$ the sets $A = F^2$ and $\mathcal{L} = \{a + bF \mid a, b \in A, b \neq 0\}$ are an affine plane of order q, denoted by $AG(2,q)$. The plane contains q^2 points and $\binom{q^2}{2}/\binom{q}{2} = q(q+1)$ lines. There are $q+1$ equivalence classes ($q-1$ real slopes, horizontal and vertical lines). Each class has q lines and each such line contains q points.

We are now ready to describe the graphs representing strong Nash equilibria. The graphs were also constructed by Blokhuis and Brouwer [4] as instances of geodetic graphs. For an affine plane $AG(2,q)$ we define a graph $G = (V, E)$ with $V = A \cup \mathcal{L}$. In the following, when we refer to a point or a line, we often mean the corresponding vertex or player. The edge set E is specified as follows.

- A point and a line are connected by an edge if and only if the line contains the point.
- Two lines are connected by an edge if and only if they are parallel.
- No two points are connected by an edge.

There are no self-loops or multiple copies of an edge. We have to give orientations to these edges. Every equivalence class of a line L defines a complete subgraph K_q of G. Let $r(L)$ and $s(L)$ denote the indegree and outdegree of L in K_q, respectively. One can easily show by induction that there exists an orientation of the edges of K_q such that, for every line L in K_q, $|r(L) - s(L)| = 0$ if q is odd and $|r(L) - s(L)| = 1$ if q is even. In order to define an orientation for the edges between points and lines, we choose a representative line L^i, $0 \leq i \leq q$, for each of the $q+1$ equivalence classes. The lines of $[L^q] = \{L^q_0, \ldots, L^q_{q-1}\}$ do not build edges to their points; rather the existing edges are built by the points. As for the other equivalence classes, a line $L \in [L^i]$, $0 \leq i \leq q-1$, builds edges to the two points $L \cap L^q_i$ and $L \cap L^q_{i+1 (\mathrm{mod}\ q)}$. All the other edges are built by the points. Every point x is contained in a line $(x \parallel L^q) =: L^q_j$ and has exactly two incoming edges from the lines $(x \parallel L^j)$ and $(x \parallel L^{j-1 (\mathrm{mod}\ q)})$. For $q = 2$, we obtain the Petersen graph.

Figure 1 shows the graph structure relative to a line $L \notin [L^q]$. Let $x_1, \ldots, x_q$ be the q points contained in L. We number these points such that L builds edges to x_1 and x_2. Let $L_1, \ldots, L_{q-1}$ be the $q-1$ lines parallel to L. We number these lines such that the first $r = r(L)$ lines build edges to L while L builds edges to the remaining $q - 1 - r$ lines. For any point x_i, $1 \leq i \leq q$, we denote by $L^{x_i}_1, \ldots, L^{x_i}_q$ the other q lines that contain x_i. These sets of q lines are disjoint for different x_i since for every pair of points there is a unique line containing this pair. Furthermore these lines are different from $L_1, \ldots, L_{q-1}$. For any line L_i, $1 \leq i \leq q-1$, let $x^i_1, \ldots x^i_q$ be the q points contained in L_i. Again these point sets are disjoint for different L_i and are also different from $x_1, \ldots, x_q$ since the lines L and $L_1, \ldots, L_{q-1}$ are parallel. If $L \in [L^q]$, then the structure of the graph is the same except that the edges between L and its points are all built by the points. If $L \notin [L^q]$ then the cost of the player representing L is $(2+s)\alpha + (2q-1) + 2(2q-1)q = (s+2)\alpha + 4q^2 - 1$,

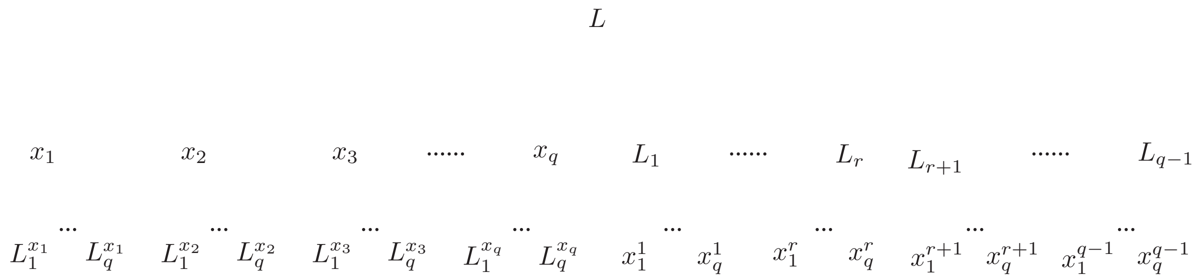

Figure 1: The distances with respect to a line L.

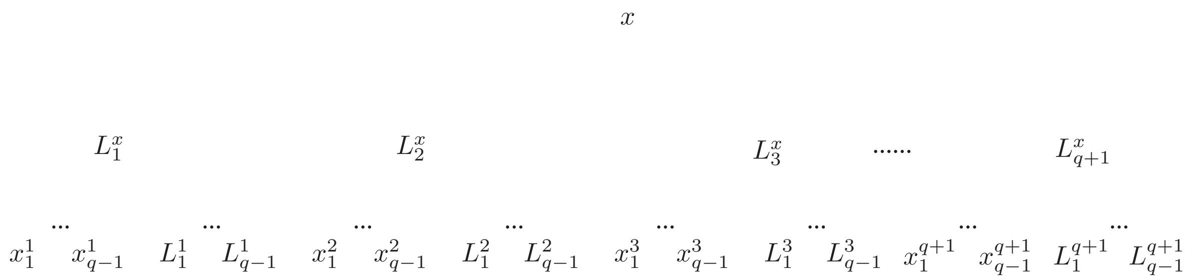

Figure 2: The distances with respect to a point x.

where $s = s(L) = q - 1 - r$. If $L \in [L^q]$, then the cost is $s\alpha + 4q^2 - 1$.

Figure 2 depicts the graph structure relative to a point x. Lines $L_1^x, \dots, L_{q+1}^x$ are the $q+1$ lines containing x. For a line L_i^x, $1 \le i \le q+1$, let $x_1^i, \dots, x_{q-1}^i$ be the other $q-1$ points of L_i^x and let $L_1^i, \dots, L_{q-1}^i$ be the $q-1$ lines parallel to L_i^x. These sets of $q-1$ points and lines are disjoint for different i. Thus the cost of the player representing x is $(q-1)\alpha + (q+1) + 2(q+1)(2(q-1)) = (q-1)\alpha + 4q^2 + q - 3$.

LEMMA 2.1. *Let $q > 10$. For α in the range $1 < \alpha < q+1$, no player associated with a line L has a different strategy that achieves a cost equal to or smaller than that of L's original one. For α in the range $1 \le \alpha \le q+1$, L has no strategy with a smaller cost.*

Proof. We prove the lemma for a line $L \notin [L^q]$, which builds two edges to points. This implies that the lemma also holds for lines $L' \in [L^q]$ which do not build edges to points. For, if a line $L' \in [L^q]$ had a different strategy with the same or a smaller cost, then any line $L \notin [L^q]$ could adopt the same strategy change while maintaining the two edges built to points. This would result in the same or a smaller cost, respectively. As we will show in the following, this is impossible.

Fix a line $L \notin [L^q]$. We consider all possible strategy changes. First, if L builds $l > s + 2$ edges, then at best there are $l - s - 2 + 2q - 1$ vertices at distance 1 while the other vertices are at distance 2 from L. In L's original strategy there are $2q - 1$ vertices at distance 1 while all other vertices are at distance 2. Thus, L's original strategy has a cost which is at least $\alpha(l - s - 2) - (l - s - 2)$ smaller than that of S, and this expression is strictly positive for $\alpha > 1$. Thus buying more than $s + 2$ edges does not pay off.

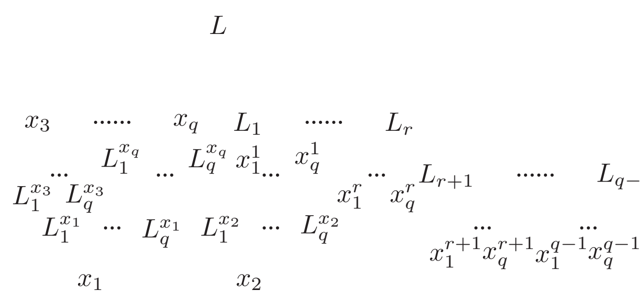

Figure 3: Strategy change S_0.

In the remainder of this proof we study the case that L builds at most $s + 2$ edges and start with the strategy S_0 in which L does not build any edges at all. The resulting shortest path tree of L is given in Figure 3. Lines $L_{r+1}, \dots, L_{q-1}$ are a distance of 2 away from L since these lines are connected to $L_1, \dots, L_r$. Lines $L_i^{x_1}$ and $L_i^{x_2}$, $1 \le i \le q$, are a distance of 3 away from L because they do not contain $x_3, \dots, x_q$ and are not parallel to $L_1, \dots, L_r$ but are connected to one line from $L_1^{x_j}, \dots, L_q^{x_j}$, for any j with $3 \le j \le q$, and are also connected to one point from $x_1^j, \dots x_q^j$, for any j with $1 \le j \le r$. Points $x_1^i, \dots, x_r^i$, with $r + 1 \le i \le q - 1$, are a distance of 3 away because they are not contained in $L_1, \dots, L_r$ but are connected to one line from $L_1^{x_j}, \dots, L_q^{x_j}$, for any $3 \le j \le q$. Finally points x_1

and x_2 are a distance of 4 away from L because these points are only contained in lines $L_1^{x_1}, \ldots, L_q^{x_1}$ and $L_1^{x_2}, \ldots, L_q^{x_2}$, respectively, at distance 3. The cost difference between S_0 and L's original strategy is $-(s+2)\alpha + s(q+1) + 2q + 6 = (q+1-\alpha)(s+2) + 4 > 0$ and hence S_0 is a worse strategy.

Next suppose that L does build edges. The edges can be of six different types: L builds an edge to (a) a line $L_j^{x_i}$ for some $3 \le i \le q$ and $1 \le j \le q$; (b) a point x_j^i, for some $1 \le i \le r$ and $1 \le j \le q$; (c) an edge $L_j^{x_1}$ or $L_j^{x_2}$, for some $1 \le j \le q$; (d) a point x_j^i, for some $r+1 \le i \le q-1$ and $1 \le j \le q$; (e) a line L_i, for some $r+1 \le i \le q-1$; (f) a point x_1 or x_2. In the following we investigate all of these cases, which are also depicted in Figure 4.

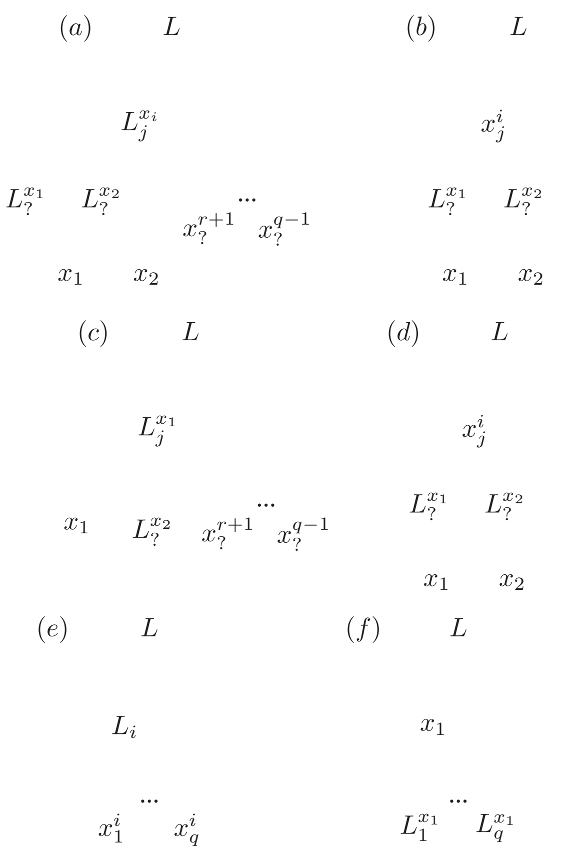

Figure 4: The effect of edges of types (a – f).

Case (a): The line $L_j^{x_i}$ is connected to one line from $L_1^{x_1}, \ldots, L_q^{x_1}$, which is linked to x_1, and to one line from $L_1^{x_2}, \ldots, L_q^{x_2}$, which is linked to x_2. Additionally $L_j^{x_i}$ is connected to one point from $x_1^k, \ldots, x_q^k$, for any $r+1 \le k \le q-1$. Thus, setting a link to $L_j^{x_i}$, line L can save a cost of at most $s+5$ relative to S_0. Hence L can save a cost of at most $s+5$ no matter how other links are laid out by L. In other words, removing the edge to $L_j^{x_i}$ results in an increase in the shortest path distance cost of at most $s+5$.

Case (b): Point x_j^i is connected to one line from $L_1^{x_1}, \ldots, L_q^{x_1}$ and to one line from $L_1^{x_2}, \ldots, L_q^{x_2}$. From there x_1 and x_2 can be reached. By laying out an edge to x_j^i, line L saves a shortest path distance cost of 5 relative to S_0 and hence a value of at most 5 relative to any other strategy. Again, removing this link can increase the shortest path distance cost by at most 5.

Case (c): Assume w.l.o.g. that an edge to $L_j^{x_1}$ is built. The analysis of a link to $L_j^{x_2}$ is similar. Line $L_j^{x_1}$ is linked to x_1 and to one line from $L_1^{x_2}, \ldots, L_q^{x_2}$. Furthermore $L_j^{x_1}$ is linked to one point from $x_1^i, \ldots, x_q^i$, for any $r+1 \le i \le q-1$. Relative to S_0 the shortest path distances decrease by $s+5$. Removing the edge results in an increase of at most $s+5$.

Case (d): Point x_j^i is connected to one line from $L_1^{x_1}, \ldots, L_q^{x_1}$ and to one line from $L_1^{x_2}, \ldots, L_q^{x_2}$. From there x_1 and x_2 can be reached. Building an edge to x_j^i saves a shortest path distance cost of 6 relative to S_0. Not building this edge results in an increase of at most 6.

The last two cases are studied under the condition that the other edges built by L are also of type (e) or (f).

Case (e): If L builds only edges of type (e) and (f), then points $x_1^i, \ldots, x_q^i$ are still at distance 3 and by setting a link to L_i the shortest path distance cost reduces by $q+1$.

Case (f): Again, assume that L builds only edges of type (e) and (f). Without an edge to x_1, lines $L_1^{x_1}, \ldots, L_q^{x_1}$ are a distance of 3 away from L and x_1 is a distance of 4 away. Building an edge to x_1 reduces the shortest path distance cost by $q+3$.

With the above case distinction (a–f) we are able to finish the proof. Recall that L builds at most $s+2$ edges. If S contains edges of types (a–d), then we simultaneously replace all of these edges by edges of type (e) or (f). Any such edge replacement increases the shortest path distance cost by at most 6 or $s+5$ while the decrease is at least $q+1$. Since, for $q > 10$, we have $q+1 > q/2+6 \ge s+5 \ge 6$, strategy S is worse than L's strategy defined by graph G. So suppose that S only builds edges of types (e) or (f). If S builds less than $s+2$ edges, then we introduce additional edges of types (e) or (f) until a total of $s+2$ edges are laid out. For any additional edge, there is an edge building cost of α while the shortest path distance cost decreases by at least $q+1$. If $\alpha < q+1$, there is a net cost saving and S is worse than L's original strategy given by G. If $\alpha = q+1$, then L's original strategy is at least as good. □

LEMMA 2.2. *For α in the range $1 < \alpha \le q+1$, no player associated with a point x has a different strategy that achieves a cost equal to or smaller than that of x's original strategy. For $\alpha = 1$, no player associated with a point has a strategy that achieves a smaller cost.*

The above two lemmata yield the main result of this section.

THEOREM 2.1. *Let $q > 10$. The graph G is a strong Nash* *equilibrium, for $1 < \alpha < q+1$, and a Nash equilibrium, for $1 \le \alpha \le q+1$.*

3 Improved bounds for the price of anarchy

We first consider the case that $\alpha \ge 12n\lceil \log n \rceil$, proving a constant price of anarchy. Then we address the remaining range of α. In both cases, for a given equilibrium graph $G(\vec{S})$, we need the concept of a shortest path tree rooted at a certain vertex u. The root of $T(u)$ is vertex u and this vertex represents *layer* 0 of the tree. Given vertex layers 0 to $i-1$, layer i is constructed as follows. A node w belongs to layer i if it is not yet contained in layers 0 to $i-1$ and there is a vertex v in layer $i-1$ such that there is an edge connecting v and w, i.e. $\{v, w\} \in E$. We add this edge to the shortest path tree. We emphasize that if w is linked to several vertices of layer $i-1$, only one such edge is added to the tree at this point. Suppose that all vertices of V have been added to $T(u)$ in this fashion. The edges inserted so far are referred to as *tree edges*. We now add all remaining edges of E to $T(u)$ and refer to these edges as *non-tree edges*. Essentially, $T(u)$ is just a layered version of G with distinguished tree edges.

3.1 Constant price of anarchy for $\alpha \ge 12n\lceil \log n \rceil$

In order to establish a constant price of anarchy, we prove that if $\alpha \ge 12n\lceil \log n \rceil$, then every Nash equilibrium graph is a tree. This implies an upper bound of 5 on the price of anarchy [10]. However, we here give an improved upper bound of 1.5 for the considered range of α.

Our proof has the following structure. Given an equilibrium graph whose girth (i.e., the length of the minimal cycle in the graph) is at least $12\lceil \log n \rceil$, we prove that the graph diameter is bounded by $6\lceil \log n \rceil$. The proof is by contradiction. We assume that there exists a vertex u with eccentricity at least $6\lceil \log n \rceil$ and examine its shortest path tree $T(u)$. We show that the maximal depth of $T(u)$ is less than $6\lceil \log n \rceil$. This immediately implies that the equilibrium graph is a tree, given the bound on the girth. Also, since we have chosen an arbitrary vertex this implies that the diameter is at most $6\lceil \log n \rceil$. We complete the proof by showing that for high edge costs the graph has a high girth.

We classify the vertices of the equilibrium graph according to their location in the tree $T(u)$. We refer to the vertices at depth exactly $6\lceil \log n \rceil$ as vertices in the *Boundary* level. We classify the vertices in the levels before the Boundary level according to the number of descendent their children have in the Boundary level. We have three types of vertices. The first are *Expanding vertices* which lead to an exponential growth, the second, and the most problematic, are *Neutral vertices* that do not lead to a growth but have descendants in the Boundary level, and the third are *Degenerate vertices* that have no descendants in the Boundary level. The vertices of the Boundary level, and at levels of larger depth, are unclassified. We now give the formal definition.

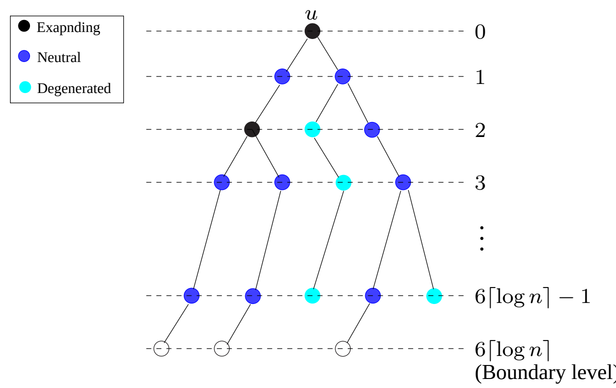

Figure 5: A classification of the vertices of T(u).

DEFINITION 2. *Let $G(\vec{S})$ be an equilibrium graph and let $u \in V$. Let $T(u)$ be a shortest path tree rooted at u. We say that a vertex $v \in V$, at a depth smaller than $6\lceil \log n \rceil$ in $T(u)$, is:*

- **Expanding** *- If v has at least two children with at least one descendent in the Boundary level.*
- **Neutral** *- If v has exactly one child with at least one descendent in the Boundary level.*
- **Degenerate** *- If v does not have any descendent in the Boundary level.*

An example to this classification is given in Figure 5. Note that vertices at level $6\lceil \log n \rceil$ (the Boundary level) and higher levels are not classified. Our target is to show that there are n vertices in the Boundary level. This implies that there are no vertices in levels higher than $6\lceil \log n \rceil$. It is important to note that since the graph has girth at least $12\lceil \log n \rceil$, there is a unique tree $T(u)$ up to level $6\lceil \log n \rceil$ (the Boundary level).

In the next Lemma we show that Degenerate children of a Neutral vertex v and their descendants are connected only through v to vertices out of the subtree of v in $T(u)$.

LEMMA 3.1. *Let $G(\vec{S})$ be an equilibrium graph whose girth is at least $12\lceil \log n \rceil$. Let v be a Neutral vertex in $T(u)$ and let $D_u(v)$ be the set of its Degenerate children and their descendants at $T(u)$. Every path from $x \in D_u(v)$ to $y \in V \setminus D_u(v)$ in $G(\vec{S})$ must go through v.*

Proof. Suppose that there is a path that does not go through v then either it goes through a vertex z from the Boundary level or the entire path does not cross the Boundary level. However, x is Degenerate and wlog z is its descendant and can not be in the Boundary level since it violates the definition Degenerate vertex. Thus, it must be that $\delta(u, z) <$

$6\lceil \log n \rceil$. Now if every vertex z on the path from x to y satisfies that $\delta(u,z) < 6\lceil \log n \rceil$ then there is a cycle of length less than $12\lceil \log n \rceil$. We conclude that any path from x to y must go through v. □

Lemma 3.1 shows that Neutral vertices have a crucial role in connecting Degenerate vertices. The next Lemma will use this property to show that although many Neutral vertices can be found in the tree, the number of times that two Neutral vertices can appear consecutively on a path from u is limited.

LEMMA 3.2. *Let $G(\vec{S})$ be an equilibrium graph whose girth is at least $12\lceil \log n \rceil$. Let $u = w_0, w_1, \ldots, w_l = v$ be a shortest path from u to v. An edge on the path is said to be a Neutral edge if both of its endpoints are Neutral vertices. The total number of Neutral edges is at most $2\lceil \log n \rceil$.*

Proof. Let (w_{i-1}, w_i) be a Neutral edge on the path from u to v. There are two possible types of Neutral edges. Edges which are bought by their tail (i.e. w_{i-1}) or edges which are bought by their head (i.e. w_i). We assume w.l.o.g that the number of edges which are bought by their tail is larger than the number of edges which are bought by their head. We bound the total number of such Neutral edges by $\log n$. This gives the desired bound of $2\log n$.

Let $(w_{i_1-1}, w_{i_1}), (w_{i_2-1}, w_{i_2}), \ldots, (w_{i_m-1}, w_{i_m})$ be the Neutral edges on the path which are bought by their tail. We show that $m \leq \log n$. Let $D_u(w_{i_j})$ be the set of all the Degenerate children of w_{i_j} and their descendants. By Lemma 3.1 every path from a vertex in $V \setminus D_u(w_{i_j})$ to a vertex in $D_u(w_{i_j})$ goes through w_{i_j}. Let n_j denote the size of $D_u(w_{i_j})$. Now since we are in equilibrium the benefit of w_{i_j-1} from buying the edge (w_{i_j-1}, w_{i_j}) is larger than the benefit from buying the edge (w_{i_j-1}, w_{i_j+1}). Thus, $n_j \geq \sum_{k=j+1}^{m} n_k$. As a result $n_j \geq 2^{m-j-1}$ and m is bounded by $\lceil \log n \rceil$. □

Based on Lemma 3.2 we prove the main result of this section. We show that every equilibrium graph whose girth is at least $12\log n$ must be a tree whose maximal depth is $6\log n$.

PROPOSITION 1. *If $G(\vec{S})$ is an equilibrium graph whose girth is at least $12\lceil \log n \rceil$ then the diameter of $G(\vec{S})$ is less than $6\lceil \log n \rceil$ and $G(\vec{S})$ is a tree.*

Proof. For the sake of contradiction, we start by assuming that the diameter is at least $6\lceil \log n \rceil$. Let $u \in V$ be a vertex on one of the endpoints of the diameter. We look on a shortest path tree rooted at u. Since u is one of the diameter endpoints our assumption implies that u is either Neutral or Expanding vertex. We show that the number of descendants at the Boundary level (i.e. vertices at a depth of **exactly** $6\lceil \log n \rceil$) is at least n. As it is not possible to have n vertices in the Boundary level we reach to a contradiction. This obviously implies that the maximal depth is at most $6\lceil \log n \rceil$ and that there are no cycles. Let $v \in V$, we denote with d the depth of v in $T(u)$ and with b the number of Neutral edges on the path from u to v. We label a vertex by (d,b). For example, the label for the root u is $(0,0)$ because $d = 0$ and $b = 0$. Let v be a non-Degenerate vertex whose label is (d,b), and let $N(d,b)$ be a lower bound on the number of its descendants at the Boundary level. (Note that two vertices might have the same label, but have different number of descendants at the boundary level.) We claim for every even d that $N(d,b) \geq 2^{\frac{6\lceil \log n \rceil - d}{2} - (2\lceil \log n \rceil - b)}$. This implies for the root that $N(0,0) \geq 2^{\frac{6\lceil \log n \rceil - 0}{2} - (2\lceil \log n \rceil - 0)} = n$, thus proving the claim will lead to the desired contradiction.

The proof will be by a backwards induction on d and b. As for the induction basis we show that $N(6\lceil \log n \rceil, b) \geq 2^{-(2\lceil \log n \rceil - b)}$ and $N(d, 2\lceil \log n \rceil) \geq 2^{\frac{6\lceil \log n \rceil - d}{2}}$. We first show that $N(6\lceil \log n \rceil, b) \geq 2^{-(2\lceil \log n \rceil - b)}$. The only descendent at the Boundary level is the vertex itself and $N(6\lceil \log n \rceil, b) = 1$. Thus, we need to show that $2^{-(2\lceil \log n \rceil - b)} \leq 1$. This follows directly from Lemma 3.2 since $b \leq 2\lceil \log n \rceil$. Next, we prove that $N(d, 2\lceil \log n \rceil) \geq 2^{\frac{6\lceil \log n \rceil - d}{2}}$. The proof here is a bit more subtle and a secondary induction on d is needed. The basis for the secondary induction, $N(6\lceil \log n \rceil, 2\lceil \log n \rceil) \geq 1$, trivially holds. We assume that $N(d', 2\lceil \log n \rceil) \geq 2^{\frac{6\lceil \log n \rceil - d'}{2}}$ for every even $d' > d$ and prove it for an even d. Let v be a vertex at an even depth d with $b = 2\lceil \log n \rceil$ which may be either Expanding or Neutral. We show that in either case v has at least two descendants at depth $d+2$ which are either Expanding or Neutral. For the case that v is Expanding it follows from the definition of Expanding vertex that v has at least two descendants at depth $d+2$ which are either Expanding or Neutral. For the case that v is Neutral it follows that v cannot have a Neutral child since $b = 2\lceil \log n \rceil$ and there are at most $2\lceil \log n \rceil$ Neutral edges by Lemma 3.2. Thus, v must have an Expanding child which again has by definition at least two children which are either Expanding or Neutral. We conclude that in both cases, i.e. v is Expanding or Neutral, it has at least two descendants at depth $d+2$ which are either Expanding or Neutral. The induction hypothesis holds for these descendants of v (recall that we assume it only for even values of d) and we get that:

$$\begin{aligned} N(d, 2\lceil \log n \rceil) &\geq 2N(d+2, 2\lceil \log n \rceil) \\ &\geq 2 \cdot 2^{\frac{6\lceil \log n \rceil - d - 2}{2}} \\ &= 2^{\frac{6\lceil \log n \rceil - d}{2}} \end{aligned}$$

This completes the proof of the basis of the primary induction. We assume the induction hypothesis holds for every even $d' \geq d$ and every $b' \geq b$ (note that one inequality must be sharp). Let v be a vertex at an even depth d with b Neutral edges on the path from v. Let w be a child of v. There are four possibilities: both v and w are Expanding, v is Expanding and w is Neutral, v is Neutral and w is Expanding and both v and w are Neutral. In the first three possibilities, as we already discussed above, v has at least two descendants at depth $d+2$ which are either Expanding or Neutral and thus the induction hypothesis holds for them and we have:

$$\begin{aligned} N(d,b) &\geq 2N(d+2,b) \\ &\geq 2 \cdot 2^{\frac{6\lceil \log n \rceil - d - 2}{2} - (2\lceil \log n \rceil - b)} \\ &= 2^{\frac{6\lceil \log n \rceil - d}{2} - (2\lceil \log n \rceil - b)} \end{aligned}$$

In the fourth case in which both v and w are Neutral there is one more Neutral edge and we have

$$\begin{aligned} N(d,b) &= N(d+2,b+1) \\ &= 2^{\frac{6\lceil \log n \rceil - d - 2}{2} - (2\lceil \log n \rceil - b - 1)} \\ &= 2^{\frac{6\lceil \log n \rceil - d}{2} - (2\lceil \log n \rceil - b)} \quad \square \end{aligned}$$

So far the only assumption that we used in our proofs on the equilibrium graph is that its girth is of length at least $12\lceil \log n \rceil$. The next lemma connects between the girth of an equilibrium graph and the edge cost α.

LEMMA 3.3. *Let $G(\vec{S})$ be an equilibrium graph and c be any positive constant. If $\alpha > cn\lceil \log n \rceil$ then the length of the girth of $G(\vec{S})$ is more than $c\lceil \log n \rceil$.*

Proof. Suppose for the sake of contradiction that the size of the minimal cycle is $c\lceil \log n \rceil$, and look on a vertex u on the cycle that buys a cycle edge. The benefit of u from this edge is at most $(c\lceil \log n \rceil - 1)n$, which is strictly less than $cn\lceil \log n \rceil = \alpha$ the cost of an edge. Therefore, this is not an equilibrium graph and we reach to a contradiction. $\square$

We are ready to state our main results, which is a characterization of every Nash equilibrium and a constant price of anarchy whenever $\alpha \geq 12n\lceil \log n \rceil$.

THEOREM 3.1. *For $\alpha \geq 12n\lceil \log n \rceil$ the price of anarchy is bounded by $1 + \frac{6n\lceil \log n \rceil}{\alpha} \leq 1.5$ and any equilibrium graph is a tree.*

Proof. The fact that the graph is a tree follows form Lemma 3.3 and Proposition 1. The *social cost* of the optimum, a star graph, is $\alpha(n-1)+2(n-1)^2$. By Proposition 1 we know that every Nash equilibrium graph is a tree whose maximal depth is $6 \log n$. Therefore, the cost of every equilibrium graph is bounded by $\alpha(n-1) + 6n^2\lceil \log n \rceil$ and the price of anarchy is bounded by

$$\begin{aligned} \frac{\alpha(n-1) + 6n^2\lceil \log n \rceil}{\alpha(n-1) + 2(n-1)^2} &\leq 1 + \frac{6n^2\lceil \log n \rceil}{\alpha n + 2(n-1)^2 - \alpha} \\ &\leq 1 + \frac{6n\lceil \log n \rceil}{\alpha} \quad \square \end{aligned}$$

3.2 Improved upper bound for $\alpha < 12n\lceil \log n \rceil$

We give a new upper bound for $\alpha < 12n\lceil \log n \rceil$. In fact, the following theorem holds for any α and is stated in this general form so that it can be generalized to a weighted game in Section 5. Furthermore, it implies a constant upper bound for $\alpha = O(\sqrt{n})$.

THEOREM 3.2. *Let $\alpha > 0$. For any Nash equilibrium S, the price of anarchy is bounded by* $15(1 + (\min\{\frac{\alpha^2}{n}, \frac{n^2}{\alpha}\})^{1/3})$.

Sketch of proof: Consider an arbitrary Nash equilibrium $\vec{S}$ and let $G(\vec{S}) = (V, E)$ be the corresponding equilibrium graph. Given a shortest path tree $T(u)$ and a vertex v, let $\ell(v)$ be the index of the layer v belongs to in $T(u)$. We show

LEMMA 3.4. *For any $T(u)$ and any $v, w \in V$, the shortest path between v and w in G consists of at least $|\ell(v) - \ell(w)|$ edges.* $\square$

Let $Cost(S)$ be the cost of S and let $Cost(v)$ be the cost paid by player $v \in V$ in S, i.e. $Cost(S) = \sum_{v \in V} Cost(v)$. The cost incurred by v consists of the cost for building edges and $Dist(v)$, the sum of the shortest path distances from v to all the other vertices in the equilibrium graph. Fix an arbitrary $v_0 \in V$. We prove,

$$Cost(S) \leq 2\alpha(n-1) + nDist(v_0) + (n-1)^2.$$

The main part of the proof is to analyze $Dist(v_0)$ for the case $1 \leq \alpha \leq n^2$. Let d be the depth of $T(v_0)$, i.e. d is the maximum layer number $\max_{v \in V} \ell(v)$. If $d \leq 9$, we are easily done so we restrict ourselves to the case $d \geq 10$.

Determine c, $1/3 \leq c \leq 1$, such that $\alpha = n^{3c-1}$. Let $V' = \{v \in V \mid \ell(v) \leq \lfloor \frac{2}{5}d \rfloor \text{ in } T(v_0)\}$ be the set of vertices of depth at most $\lfloor \frac{2}{5}d \rfloor$ in $T(v_0)$. If $|V'| \geq \frac{2}{3}n^c$, then consider a vertex w_0 at depth d in $T(v_0)$. By Lemma 3.4, the shortest path distance between w_0 and any vertex $v \in V'$ is at least $\lceil \frac{3}{5}d \rceil$. If there was an edge between w_0 and v_0, then the distance between w_0 and v would be at most $\lfloor \frac{2}{5}d \rfloor + 1$. Since w_0 did not build an edge to v_0 we have $\alpha > |V'| \left(\lceil \frac{3}{5}d \rceil - \lfloor \frac{2}{5}d \rfloor - 1 \right) \geq \frac{2}{3}n^c \left(\frac{1}{5}d - 1 \right) \geq \frac{2}{3}n^c \frac{1}{10}d$ and hence $d \leq \frac{15\alpha}{n^c}$. On the other hand, if $|V'| < \frac{2}{3}n^c$, more involved calculations show that $d \leq 15\sqrt{\frac{\alpha}{n^{1-c}}}$. By the choice of c, both bounds on d are identical.

We finally determine the price of anarchy. We have $Dist(v_0) \leq (n-1)15\alpha/n^c \leq 15\alpha n^{1-c}$, which implies $Cost(S) \leq 2\alpha(n-1) + 15\alpha n^{2-c} + n^2$. Therefore, price of anarchy is bounded by

$$\frac{2\alpha(n-1) + 15\alpha n^{2-c} + n^2}{\alpha(n-1) + n^2} \leq 3 + \frac{15\alpha n^{2-c}}{\alpha(n-1) + n^2}.$$

The theorem then follows by inspecting cases $\alpha \leq n$ and $\alpha > n$, taking into account that $n^c = (\alpha n)^{1/3}$. $\square$

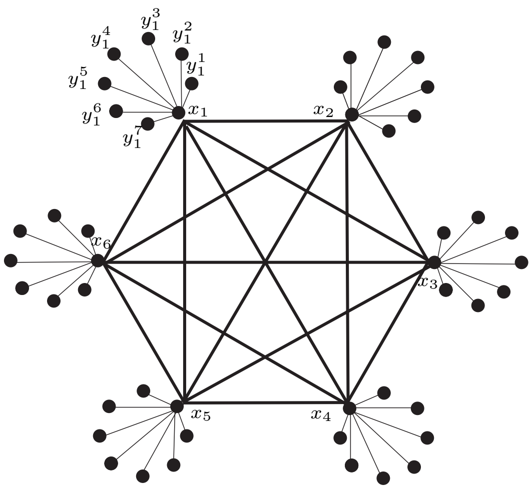

Figure 6: A $(6, 8)$ clique of stars graph, an equilibrium graph which is not a tree.

The next theorem implies that the only critical part in bounding the price of anarchy is the sum of the shortest path distances between players.

THEOREM 3.3. *In any Nash equilibrium S, the total cost incurred by the players in building edges is bounded by twice the cost of the social optimum. There exists a shortest path tree such that, for any player v, the number of non-tree edges built by v is bounded by $1 + \lfloor (n-1)/\alpha \rfloor$.*

4 Characterizations of Nash equilibria

We give further characterization of Nash equilibria. Our first contribution is to show that, for any n and any $\alpha < n/2$, there exist transient Nash equilibria which are not trees. We then show that every Nash equilibrium which is a chordal graph is a transient Nash equilibrium. An undirected graph is chordal if every cycle of length at least four has a chord, i.e. has an edge connecting two non-adjacent vertices on the cycle. Chordal graphs play a very important role in graph theory, see e.g. [9]. Finally, we show that for $\alpha < n/2$ every Nash equilibrium which is a tree must be star.

THEOREM 4.1. *For any integer n and for any integer cost $\alpha \le n/2$, there exists a Nash equilibrium forming a non-tree chordal graph on n vertices.*

Proof. We start by describing our non-tree chordal equilibrium graph. A (k, ℓ) *clique of stars* is a clique with k vertices, where each vertex of the clique is a root of a star with ℓ vertices. A $(6, 8)$ clique of stars is depicted in Figure 6.

We next prove that a (k, ℓ) clique of stars is a Nash equilibrium when $\alpha = \ell$. We show that the edges of each star are bought only by its root, and the clique edges are bought arbitrarily by one of their endpoints.

LEMMA 4.1. *Let $G(\vec{S})$ be a (k, ℓ) clique of stars. If the cost of an edge equals to ℓ and all the edges are bought by the clique vertices (and no edge is bought twice), then $G(\vec{S})$ is an equilibrium graph.*

Proof. We prove that a (k, ℓ) clique of stars is an equilibrium in this setting by showing that no player has an incentive to deviate from her strategy. We denote with $x_1, \ldots, x_k$ the vertices of the clique and with $y_i^1, \ldots, y_i^{\ell-1}$ the vertices of the star rooted at x_i.

We start by showing that the star vertices have no incentive to deviate from their strategy of not buying any edge. We look on an arbitrary star vertex y_i^j. The edge connecting it to the graph is bought by x_i. The benefit from buying the edge (y_i^j, x_p) for $p \neq i$ is ℓ, since y_i^j is getting closer by one only to the vertices of the star rooted at x_p. The cost of an edge is also ℓ therefore the player y_i^j is indifferent and will not deviate. The benefit from buying the edge $(y_i^j, y_{i'}^{j'})$ is only one and thus y_i^j will have no incentive to buy it. Since buying a set of edges is at most as beneficial as the sum of their benefits in a connected graph, y_i^j will not deviate.

We now turn our attention to the clique vertices. We take an arbitrary vertex x_i. Its star vertices are connected with an edge of the form (x_i, y_i^j). If x_i does not buy one of these edges the graph get disconnected and the cost of x_i becomes infinity. Thus, these edges are necessary. Suppose that the edge (x_j, x_i) is bought by x_j, then x_j is indifference of buying or not buying the edge, since without the edge the distance to the star rooted at x_i is at least 2 while it is 1 with the edge. The benefit from buying the edge is ℓ which is also the cost of an edge. Clearly x_j can not benefit from buying an edge to a leaf of another star, say y_p^k, since $\alpha \ge 1$ and the benefit is exactly 1. Thus, x_j has no incentive to change its strategy and we conclude that $G(\vec{S})$ is an equilibrium graph. □

We now continue with the proof of Theorem 4.1. For every n we have a family of (k, ℓ) clique of stars with $k \cdot \ell = n$ and $\alpha = \ell$. This implies that we can build a non-tree equilibrium for $\alpha = n/3, n/4, \ldots, 1$. By a slightly more complicated construction it is possible to extend the (k, ℓ) clique of stars construction and to derive the desired theorem. The complete details are given in the full version of the paper. □

THEOREM 4.2. *Let $\alpha > 1$ and N be a Nash equilibrium that has a cycle in the associated graph $G = (V, E)$. If G is chordal, then N is transient.*

THEOREM 4.3. *For $\alpha < n/2$, the star is the only Tree which is an equilibrium graph.*

We note that for $\alpha = n/2$, the construction of Theorem 4.1 is an equilibrium graph which is also a tree with diameter 3, and as a result Theorem 4.3 is tight.

5 A weighted network creation game

So far, we have considered an unweighted network creation game in which all players incur the same traffic. We now study a weighted game in which player u sends a traffic amount of $w_{uv} > 0$ to player v, with $u \neq v$. In the cost of player u, the shortest path distance between u and v is multiplied by w_{uv}. Let $\mathcal{W} = (w_{uv})_{u,v}$ be the resulting $n \times n$ traffic matrix. We use $w_{\min} = \min_{u \neq v} w_{uv}$ to denote smallest traffic entry and $w_{\max} = \max_{u \neq v} w_{uv}$ to denote the largest one. Let $W = \sum_{u=1}^{n} \sum_{v=1}^{n} w_{uv}$ be the sum of the traffic values. We extend the upper bounds of Section 3 to the weighted case. Again we assume that there are at least $n \geq 2$ players. The following theorem is a generalization of Theorem 3.2. In the unweighted case we have $w_{\min} = 1$ and the bounds given in the next theorem are identical to that of Theorem 3.2, up to constant factors.

THEOREM 5.1. *a) Let $0 < \alpha \leq w_{\min} n^2$. For any Nash equilibrium S, the price of anarchy is bounded by $60(1 + \min\{(\alpha^2/(w_{\min}^2 n))^{1/3}, W/(w_{\min} n^4 \alpha)^{1/3}, n\})$.*

b) Let $w_{\min} n^2 < \alpha < w_{\max} n^2$. Then the price of anarchy is bounded by $12 + 3\min\{\sqrt{\alpha/w_{\min}}, W/(\sqrt{\alpha w_{\min}}(n-1)), n\}$.

c) Let $w_{\max} n^2 \leq \alpha$. Then the price of anarchy is bounded by 4.

6 Cost sharing

We study the effect of cost sharing where players can pay for a fraction of an edge. An edge exists if the total contribution is at least α. We first show that the bounds on the price of anarchy developed in Section 3 and 5 essentially carry over. We then prove that there exist strong Nash equilibria containing cycles in which the cost is split evenly among players.

THEOREM 6.1. *a) In the unweighted scenario the bounds of Theorem 3.2 hold. b) In the weighted scenario the bound of Theorem 5.1 hold.*

THEOREM 6.2. *For $n > 6$ and α in the range $\frac{1}{6}n^2 + n < \alpha < \frac{1}{2}n^2 - n$, there exist strong Nash equilibria with n players that contain cycle an in which the cost is split evenly among players.*

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