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On Navier–Stokes Equations with Slip Boundary Conditions in an Infinite Pipe

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Abstract. The paper examines steady Navier–Stokes equations in a two-dimensional infinite pipe with slip boundary conditions. At both inlet and outlet, the velocity of flow is assumed to be constant. The main results show the existence of weak and regular solutions with no restrictions of smallness of the flux vector, also simply connectedness of the domain is not required.

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1. Introduction

In this paper, we study a steady flow of incompressible viscous Newtonian fluids in an infinite two-dimensional pipe-like domain with an obstacle inside. We assume that there is no friction on the boundary of the considered model. At infinity, at the inlet and outlet of the pipe, the velocity of the fluid is assumed to be constant.

We describe such a flow by applying the steady Navier–Stokes equations with slip boundary conditions which read

$$\begin{aligned}
&(v\nabla)v - \nu\Delta v + \nabla p = 0, \quad \text{in } \Omega,\\
&\text{div } v = 0, \quad \text{in } \Omega,\\
&v \cdot n = 0, \quad \text{on } \partial\Omega,\\
&n \cdot \mathbf{T}(v, p) \cdot \tau = 0, \quad \text{on } \partial\Omega,\\
&v \to (v_\infty, 0), \quad \text{as } x_1 \to \pm\infty,
\end{aligned} \tag{1.1}$$

where v is the velocity of the fluid, p – the pressure, ν – constant positive viscous coefficient, n and τ are normal and tangent vectors to the boundary $\partial\Omega$, the dot $\cdot$ denotes the scalar product in $\mathbf{R}^2$ and

$$\mathbf{T}(v, p) = \nu\mathbf{D}(v) - p\text{Id} \tag{1.2}$$

is the stress tensor, Id is the identity matrix and

$$\mathbf{D}(v) = \{v^i_{,j} + v^j_{,i}\}_{i,j=1,2}. \tag{1.3}$$

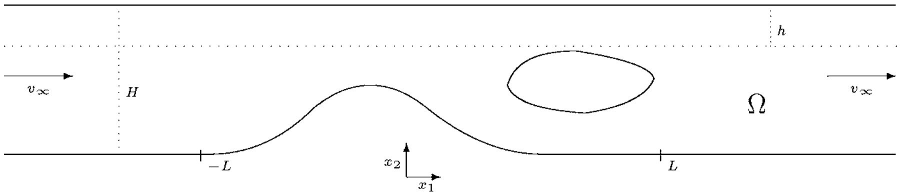

Figure 1.

Our aim is to prove the existence of weak solutions to problem (1.1). Next, applying the extra properties of the slip boundary conditions, we will be able to regularize them under the assumption of smoothness of the boundary.

Problem (1.1) can be treated as a type of Leray problem [1, 2, Chap. XI]. The classical Leray problem consists of Equation $(1.1)_{1,2}$ with no slip boundary conditions $v|_{\partial\Omega} = 0$ and with the Poiseuille flow at infinity. The slip boundary condition $(1.1)_{2,3}$ describes a perfect boundary which does not admit friction between the surface of the pipe and the fluid. This model compared to the zero Dirichlet condition, has not only different physical interpretation but also gives different mathematical properties. This kind of relation enables us to describe the vorticity of the velocity field on the boundary as a function of the velocity and a curvature of the boundary.

The existence of solutions to the two-dimensional Leray problem 'in the large' is still an open question. The difficulty is hidden in the Dirichlet integral $\int_\Omega \nabla v : \nabla v \, dx$. For Leray's problem, this quantity is infinite [1, 2, Chap. XI]. By using the known techniques, we are able to prove only the existence of solutions for small data [6]. For large fluxes, there is the result of Ladyzhenskaya and Solonnikov [5], but only for a modification of Leray's problem which does not involve the conditions at infinity. As we will see in the problem considered in this paper, the Dirichlet integral is finite for any v_∞.

The slip boundary conditions are not so popular in considerations dealing with Navier–Stokes equations, but in general the results are the same [3, 9]. Constrains $(1.1)_{2,3}$ arise naturally from the Neumann boundary conditions for the free boundary problem in steady cases [7, 8]. But they are also a special case of the friction (slip) conditions $n \cdot \mathbf{T}(v, p) \cdot \tau + f v \cdot \tau = 0$ if friction f is neglectable (if $f \to +\infty$, we get the Dirichlet no-slip data and Leray's problem). The properties of conditions of this type, in particular the relation with the problem on the vorticity of the velocity, have been used also in a three-dimensional evolutional case for a problem in a bounded domain in [11].

Solutions to problem (1.1) can be treated as a perturbation of the flow for a case when there is no obstacle inside the pipe. Such an unperturbed solution is equal to a constant flow $\bar{v} = (v_\infty, 0)$ which follows from the properties of slip conditions $(1.1)_{2,3}$. It follows that condition $(1.1)_5$ is quite natural from the physical point of view for our model. Since we do not require that domain Ω be a simply connected model, (1.1) can be treated also as an approximation of a flow around an obstacle in the whole space.

About a domain $\Omega \subset \mathbf{R}^2$ – see Figure 1 – we require that Ω is connected, $\partial\Omega$ is sufficiently smooth – at least C^2 and, moreover,

$$\Omega \subset (-\infty, +\infty) \times (0, H), \tag{1.4}$$

and there exists a number $h > 0$ such that

$$\begin{aligned} &\Omega \setminus [-L, L] \times [0, H-h] \\ &\quad = ((-\infty, +\infty) \times (0, H)) \setminus ([-L, L] \times [0, H-h]). \end{aligned} \tag{1.5}$$

Condition (1.5) says that the obstacle is bounded – the perturbation is local – and the following inclusion holds:

$$(-\infty, +\infty) \times (H-h, H) \subset \Omega. \tag{1.6}$$

Regularity of $\partial\Omega$ implies that the measure of $\partial(\Omega \cap ((-L, L) \times (0, H)))$ is finite. We do not assume that Ω is simply connected.

Since the flux of the flow by $(1.1)_2$ is equal to

$$\int_{I(x_1)} v^1(x_1, x_2)\,\mathrm{d}x_2 = H v_\infty \neq 0, \tag{1.7}$$

where $I(x_1) = \{x_2 : (x_1, x_2) \in \Omega\}$, we have difficulties with the integration of velocity v over Ω. The solution is the same as in the classical Leray problem. We have to find a vector field a: $\Omega \to \mathbf{R}^2$ with features similar to the sought-after solution, i.e. we need to construct a field which satisfies

$$\begin{aligned} &\operatorname{div} a = 0 \quad \text{in } \Omega, \\ &a \cdot n = 0, \quad n \cdot \mathbf{D}(a) \cdot \tau = 0 \quad \text{on } \partial\Omega, \\ &a \to (v_\infty, 0) \quad \text{as } x_1 \to \pm\infty. \end{aligned} \tag{1.8}$$

By $(1.8)_1$, we see that

$$\int_{I(x_1)} a^1(x_1, x_2)\,\mathrm{d}x_2 = H v_\infty. \tag{1.9}$$

Thus introducing a new variable

$$u = v - a, \tag{1.10}$$

we obtain a new sought-after function with zero flux

$$\int_{I(x_1)} u^1(x_1, x_2)\,\mathrm{d}x_2 = 0. \tag{1.11}$$

Using (1.10) from problem (1.1), we obtain the equations for function u:

$$\begin{aligned} &(u\nabla)u + (a\nabla)u - \nu\Delta u + \nabla p = -(u\nabla)a - (a\nabla)a + \nu\Delta a, \quad \text{in } \Omega, \\ &\operatorname{div} u = 0, \quad \text{in } \Omega, \\ &u \cdot n = 0, \quad n \cdot \mathbf{T}(u, p) \cdot \tau = 0, \quad \text{on } \partial\Omega, \\ &u \to 0 \quad \text{as } x_1 \to \pm\infty. \end{aligned} \tag{1.12}$$

We define a weak solution for problem (1.12), but first we have to obtain the compability of Equations $(1.12)_1$ with boundary condition $(1.12)_3$. By $(1.12)_2$ and (1.2), we note

$$-\nu\Delta u + \nabla p = -\mathrm{div}\,\mathbf{T}(u, p) = -\nu \mathrm{div}\,\mathbf{D}(u) + \nabla p. \tag{1.13}$$

This remark leads to the definition: We say that u is a *weak solution* to problem (1.12) if $u \in V$ and the identity

$$\begin{aligned} &\nu \int_\Omega \mathbf{D}(u) : \nabla\phi \,\mathrm{d}x + \int_\Omega ((u\nabla)u + (a\nabla)u + (u\nabla)a) \cdot \phi \,\mathrm{d}x \\ &\quad = -\int_\Omega (a\nabla)a \cdot \phi \,\mathrm{d}x - \nu \int_\Omega \mathbf{D}(a) : \nabla\phi \,\mathrm{d}x \end{aligned} \tag{1.14}$$

holds for all $\phi \in V$, where

$$V = \left\{ f \in H^1(\Omega; \mathbf{R}^2) : \ f \cdot n|_{\partial\Omega} = 0, \ \mathrm{div}\, f = 0 \right\}. \tag{1.15}$$

The main result of this paper is the following:

THEOREM A. *At least one weak solution of* (1.12) *exists in the sense of* (1.14) *such that $u \in V$ and*

$$\|\nabla u\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)} \leqslant M(v_\infty). \tag{1.16}$$

In the formulation of the weak solution, a vector field a is used satisfying conditions (1.8). Moreover, to show Theorem A, we need some extra properties of this field. Thus, the next result is the following:

THEOREM B. *Let $\delta > 0$ be fixed. There exists a vector field a: $\Omega \to \mathbf{R}^2$ satisfying* (1.8) *such that $a \in C^\infty(\bar{\Omega})$ and*

$$\|\nabla a\|_{L_2(\Omega)} + \|(a\nabla)a\|_{L_2(\Omega)} \leqslant c(v_\infty, \delta). \tag{1.17}$$

Moreover, for any $u \in V$,

$$\left| \int_\Omega (u\nabla)a \cdot u \,\mathrm{d}x \right| \leqslant \delta \|u\|^2_{H^1(\Omega)}. \tag{1.18}$$

We use standard techniques in the proof of Theorem A. The difference with Leray's problem lies in the existence of a suitable field a. A key element of the method is estimate (1.18) which, for the classical Leray problem, is able to obtain only for small data. In our case, Theorem B gives the estimate for any $\delta > 0$ which guarantees the existence of general solutions to problem (1.12). The construction of field a is based on an old idea of Ladyzhenskaya [4] giving estimate (1.18). By estimates (1.16) and (1.17), we conclude that the velocity, v being the solution to problem (1.1) given by (1.10), defines the finite Dirichlet integral $\int_\Omega \nabla v : \nabla v \,\mathrm{d}x$ for any v_∞.

Finally, we want to regularize the weak solutions obtained from Theorem A. To do this, we apply an important feature of the slip boundary condition. By simple calculations, we can completely determine the Dirichlet boundary data to the equation on the vorticity of the velocity field. Two-dimensional properties simplify the problem to the following one:

$$\begin{aligned} &(v\nabla)\alpha - \nu\Delta\alpha = \nu\Delta\operatorname{rot} a - (v\nabla)\operatorname{rot} a, \quad \text{in } \Omega, \\ &\alpha = 2u\cdot\tau\chi, \quad \text{on } \partial\Omega, \\ &\alpha \to 0 \quad \text{as } x_1 \to \pm\infty, \end{aligned} \tag{1.19}$$

where χ is the curvature of $\partial\Omega$ and $\alpha = \operatorname{rot} u = u^2_{,1} - u^1_{,2}$.

Applying this information, we easily show:

THEOREM C. *If boundary $\partial\Omega$ is C^∞-smooth, then solutions u and ∇p to problem* (1.12) *given by Theorem* A*, belong to $C^\infty(\bar{\Omega})$, in particular*

$$\|u\|_{H^2(\Omega)} + \|\nabla p\|_{L_2(\Omega)} \leqslant c(v_\infty). \tag{1.20}$$

An application to prove Theorem C of problem (1.19) enables us to increase the regularity of the velocity without any knowledge of the pressure. By the standard theory, we are able to get only local information about a norm of the pressure [2, 10]. Using this way it would be hard for us to obtain estimate (1.20) which is global (for whole Ω). Let us note that Theorem C does not give information about the limits of the pressure at infinity. In particular, the function p can tend to infinity.

2. Notation

In our consideration, we try to restrict ourselves to the use of standard notations.

By $H^m(\Omega)$, for $m \in \mathbf{N}$, we mean the closure of $C^\infty(\Omega)$ in the norm

$$\|f\|^2_{H^m(\Omega)} = \sum_{0\leqslant|\alpha|\leqslant m} \int_\Omega |\partial^\alpha f|^2 \,\mathrm{d}x, \tag{2.1}$$

where

$$\partial^\alpha = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}, \qquad \alpha = (\alpha_1, \alpha_2) \in \mathbf{N}\times\mathbf{N} \quad \text{and} \quad |\alpha| = \alpha_1 + \alpha_2.$$

LEMMA 2.1 (The Korn inequality). *Let Ω satisfy conditions defined above, then there exists $\tilde{\nu} > 0$ such that*

$$\nu \int_\Omega (\mathbf{D}(u))^2 \,\mathrm{d}x \geqslant \tilde{\nu}\|u\|^2_{H^1(\Omega)} \quad \textit{for all } u \in V, \tag{2.2}$$

where V is defined by (1.15) *and $\tilde{\nu}/\nu$ is dependent of every quantities of domain Ω.*

Proof of Lemma 2.1 is in the Appendix – or in [9].

In the proof of Theorem A, where we apply the Galerkin method, we need in particular we need to solve the approximated problem in finite-dimensional spaces. We use the well-known result presented in [10, Chap. II].

LEMMA 2.2. *Let H be a finite-dimensional Hilbert space with a scalar product $(\cdot,\cdot)$. If $P\colon H \to H$ is a continuous map which satisfies the condition*

$$(P(\xi),\xi)_H > 0, \quad \text{for all } \xi : \|\xi\| = M, \tag{2.3}$$

for some $M > 0$, then there exists $\xi_* \in H$ such that

$$P(\xi_*) = 0 \quad \text{and} \quad \|\xi_*\| \leqslant M. \tag{2.4}$$

We denote all constants by a letter c. Constants which are fixed in each proof are denoted by $A_1, A_2, \ldots$.

3. Proof of Theorem A

We prove the existence of weak solutions by applying the Galerkin method. From the elementary theory, there exists a base of space V – see definition (1.15) – such that

$$V = \overline{\operatorname{span}\{w_1, w_2, \ldots, w_n, \ldots\}}^{\|\cdot\|_{H^1(\Omega)}}, \tag{3.1}$$

where $(w_k, w_l)_{H^1(\Omega)} = \delta_{kl}$.

Space V can be approximated by a finite-dimensional Hilbert space

$$V^N = \operatorname{span}\{w_1, w_2, \ldots, w_N\}. \tag{3.2}$$

To construct an approximation of solution u, which is searched for in the form of

$$u^N = \sum_{k=1}^{N} c_k^N w_k, \tag{3.3}$$

we solve the following system on coefficients c_k^N:

$$\begin{aligned} \nu \int_\Omega \mathbf{D}(u^N)\colon \nabla w_k \,\mathrm{d}x + \int_\Omega \big((u^N\nabla)u^N + (a\nabla)u^N + (u^N\nabla)a\big)\cdot w_k \,\mathrm{d}x \\ = -\int_\Omega (a\nabla)a \cdot w_k \,\mathrm{d}x - \nu \int_\Omega \mathbf{D}(a) : \nabla w_k \,\mathrm{d}x \end{aligned} \tag{3.4}$$

for $k = 1, \ldots, N$. Problem (3.4) follows from the weak formulation (1.14).

As we see, it is not so easy to solve (3.4) and to show the existence of solutions to this problem, we apply Lemma 2.2. Let us define operator $P(\cdot)$ in our case. For $u^N \in V^N$ defined as in (3.4)

$$\begin{aligned} P(u^N) \;=\; \sum_{k=1}^{N} \Bigg(\nu \int_\Omega (\mathbf{D}(u^N) + \mathbf{D}(a)) : \nabla w_k \,\mathrm{d}x + \\ + \int_\Omega \big((u^N\nabla)u^N + (a\nabla)u^N + (a\nabla)a + (u^N\nabla)a\big) w_k \,\mathrm{d}x \Bigg) \cdot w_k. \end{aligned} \tag{3.5}$$

It is obviously that $P\colon V^N \to V^N$ and it is a continuous map. To show condition (2.3), we put $\|u^N\|_{V^N} = k$. Note that, by the properties of the base vectors,

$$\|u^N\|^2_{V^N} = \|u^N\|^2_{H^1(\Omega)}, \qquad (u^N, \tilde{u}^N)_{V^N} = (u^N, \tilde{u}^N)_{H^1(\Omega)}. \tag{3.6}$$

We examine

$$\begin{aligned} &(P(u^N), u^N)_{V^N} \\ &\quad = \nu \int_\Omega \big(\mathbf{D}(u^N) + \mathbf{D}(a)\big) : \nabla u^N \,\mathrm{d}x + \\ &\qquad + \int_\Omega \big((u^N\nabla)u^N + (a\nabla)u^N + (a\nabla)a + (u^N\nabla)a\big)u^N \,\mathrm{d}x. \end{aligned} \tag{3.7}$$

Estimate the right-hand side of (3.7). By Lemma 2.1,

$$\nu \int_\Omega \mathbf{D}(u^N) : \nabla u^N \,\mathrm{d}x = \nu \int_\Omega \big(\mathbf{D}(u^N)\big)^2 \,\mathrm{d}x \geqslant \tilde{\nu}\|u^N\|^2_{H^1(\Omega)}. \tag{3.8}$$

The Schwarz inequality gives

$$\left|\nu \int_\Omega \mathbf{D}(a) : \nabla u^N \,\mathrm{d}x\right| \leqslant A_1 \|\nabla a\|_{L_2(\Omega)} \|u^N\|_{H^1(\Omega)},$$
$$\left|\int_\Omega (a\nabla)au^N \,\mathrm{d}x\right| \leqslant A_2 \|(a\nabla)a\|_{L_2(\Omega)} \|u^N\|_{H^1(\Omega)}.$$

Since $u^N \in V$, in particular $\operatorname{div} u^N = 0$ and by $(1.8)_1$ we get

$$\int_\Omega (u^N\nabla)u^N u^N \,\mathrm{d}x = 0 \quad \text{and} \quad \int_\Omega (a\nabla)u^N u^N \,\mathrm{d}x = 0.$$

The last term of the right-hand side of (3.7) can be bounded by Theorem B – estimate (1.18) with $\delta = 1/2\tilde{\nu}$, where $\tilde{\nu}$ is taken to be, as in the Korn inequality (2.2),

$$\left|\int_\Omega (u^N\nabla)a \cdot u^N \,\mathrm{d}x\right| \leqslant \tfrac{1}{2}\tilde{\nu}\|u^N\|^2_{H^1(\Omega)}. \tag{3.9}$$

Summing up, by (3.8) and (3.9) from (3.7), we obtain

$$\begin{aligned} &(P(u^N), u^N)_{V^N} \\ &\quad \geqslant \|u^N\|_{H^1(\Omega)} \bigg(\frac{\tilde{\nu}}{2}\|u^N\|_{H^1(\Omega)} - A_3(\|\nabla a\|_{L_2(\Omega)} + \|(a\nabla)a\|_{L_2(\Omega)})\bigg). \end{aligned} \tag{3.10}$$

If $M = \|u^N\|_{V^N}$ is so large that

$$\frac{\tilde{\nu}}{2}M - A_3(\|\nabla a\|_{L_2(\Omega)} + \|(a\nabla)a\|_{L_2(\Omega)}) > 0, \tag{3.11}$$

then inequality (3.10) implies

$$(P(u^N), u^N)_{V^N} > 0, \quad \text{for all } \|u^N\|_{V^N} = M. \tag{3.12}$$

By Lemma 2.2 and remark (3.6), we conclude that there exists $u_*^N \in V^N$ such that

$$P(u_*^N) = 0. \tag{3.13}$$

Hence, we have found the coefficients c_k^n ($u_*^N = \sum_{k=1}^N c_k^N w_k$) and, moreover, by (3.12) and (2.4), we have

$$\|u_*^N\|_{H^1(\Omega)} \leqslant M, \tag{3.14}$$

where k is independent of N – see (3.11).

By the properties of Hilbert spaces, there exists a subsequence $\{u_*^{N_k}\}_{k=0}^\infty$ such that

$$u_*^{N_k} \rightharpoonup u_* \text{ as } k \to \infty \quad \text{weakly in } V. \tag{3.15}$$

Without loss of generality, we can take a subsequence $\{u_*^{N_k}\}_{k=0}^\infty$ in such a way that

$$u_*^{N_k} \to u_* \quad \text{as } k \to \infty \quad \text{strongly in } L_4(\Omega \cap ([-k,k] \times (0,H)), \tag{3.16}$$

for all $k \in \mathbf{N}$, which is possible by the Rellich theorem. Then we can prove that

$$\int_\Omega (u_*^{N_k}\nabla)u_*^{N_k} \cdot \phi \, \mathrm{d}x \to \int_\Omega (u_*\nabla)u_* \cdot \phi \, \mathrm{d}x, \quad \text{for all } \phi \in V. \tag{3.17}$$

And by the Galerkin method, function u_* is a weak solution in the sense of (1.14). Estimate (1.16) is concluded from (3.14) and by (3.11) we find a bound

$$M > \frac{2A_3(\|\nabla a\|_{L_2(\Omega)} + \|(a\nabla)a\|_{L_2(\Omega)})}{\tilde{\nu}}. \tag{3.18}$$

4. Proof of Theorem B

In this part of the paper, we construct a vector field $a: \Omega \to \mathbf{R}^2$. A method from [4] is adapted here to obtain a field near the obstacle. To simplify, we assume that $v_\infty \geqslant 0$ which does not change our considerations.

To define a field a, we need two quantities. The first one is a function η: $[0, H] \to [0, 1]$ such that

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant \varrho \mathrm{e}^{-1/\varepsilon} = R, \\ -\varepsilon \ln \dfrac{t}{\varrho} & \text{for } R < t \leqslant \varrho, \\ 0 & \text{for } \varrho < t \leqslant H, \end{cases} \tag{4.1}$$

where $\varrho < h$. By definition (4.1) we see that

$$\eta'(t) = \begin{cases} 0 & \text{for } 0 \leqslant t \leqslant \varrho \mathrm{e}^{-1/\varepsilon} = R, \\ -\dfrac{\varepsilon}{t} & \text{for } R < t \leqslant \varrho, \\ 0 & \text{for } \varrho < t \leqslant H. \end{cases} \tag{4.2}$$

The function η defines

$$\xi(x_2) = m * \partial_{x_2}\eta(H - x_2), \tag{4.3}$$

where m is a standard mollifier such that

$$\|(m * \partial_{x_2}\eta(H - x_2))^2 - (\partial_{x_2}\eta(H - x_2))^2\|_{L_1(0,H)} \leqslant \varepsilon^2/H. \tag{4.4}$$

Thus definition (4.3) implies the smoothness of the function $\xi(\cdot)$.

The second quantity is a vector field b: $[0, D] \times [0, H]$,

$$b^1(x_1, x_2) = \xi(x_2) + (1/H - \xi(x_2))\pi(x_1), \tag{4.5}$$

$$b^2(x_1, x_2) = \pi'(x_1)\int_0^{x_2} (\xi(x_2') - 1/H)\,\mathrm{d}x_2', \tag{4.6}$$

where D will be specified later and $\pi(\cdot)$ is a smooth increasing function such that

$$\pi(0) = \pi'(0) = \pi'(D) = 0, \qquad \pi(D) = 1 \quad \text{and}$$
$$\|\pi'\|_{L_\infty(0,D)} \leqslant 2/D. \tag{4.7}$$

We see that a field b given by (4.5)–(4.6) satisfies

$$\operatorname{div} b = 0, \quad b \in C^1([0, D] \times [0, H]),$$
$$b|_{x_1=0} = (\xi(x_2), 0), \qquad b|_{x_1=D} = (1/H, 0).$$

Having a function ξ and a vector field b, we define a field a in the following way:

$$a(x_1, x_2) = Hv_\infty \cdot \begin{cases} (\xi(x_2), 0), & \text{for } x_1 \in [-D, D], \\ b(x_1 - D, x_2), & \text{for } x_1 \in (D, 2D], \\ b(-x_1 - D, x_2), & \text{for } x_1 \in [-2D, -D), \\ (1/H, 0), & \text{for } x_1 \in (-\infty, -2D) \cup (2D, +\infty). \end{cases} \tag{4.8}$$

This construction guarantees that conditions (1.8) are satisfied (by (4.3) and definition (4.1), we get $\int_0^H (\xi(x_2) - 1/H)\,\mathrm{d}x_2 = 0$).

Now we consider the main difficulty of Theorem B – estimation for $\int_\Omega u \cdot \nabla a u\,\mathrm{d}x$. We have

$$\int_\Omega u \cdot \nabla a u\,\mathrm{d}x = \int_\Omega u \cdot \nabla a^1 u^1\,\mathrm{d}x + \int_\Omega u \cdot \nabla a^2 u^2\,\mathrm{d}x = I_1 + I_2$$

and

$$I_1 = \int_\Omega u^1 a^1_{,1} u^1\,\mathrm{d}x + \int_\Omega u^2 a^1_{,2} u^1\,\mathrm{d}x = I_{11} + I_{12}.$$

As we see, I_{12} is a more complex term. By definition (4.8), the first component of a reads

$$a^1(x_1, x_2) = Hv_\infty\{\xi(x_2)(1 - \tilde{\pi}(x_1)) + 1/H\,\tilde{\pi}(x_1)\}, \tag{4.9}$$

where $\tilde{\pi}(\cdot)$ is a smooth function such that

$$\begin{aligned}&\tilde{\pi}(x_1) = 0 \quad \text{for } x_1 \in [-D, D],\\ &\tilde{\pi}(x_1) = \pi(x_1 - D) \quad \text{for } x_1 \in (D, 2D],\\ &\tilde{\pi}(x_1) = \pi(-x_1 - D) \quad \text{for } x_1 = [-2D, D)\end{aligned}$$

and

$$\tilde{\pi}(x_1) = 1 \quad \text{for } x_1 \in (-\infty, 2D) \cup (2D, +\infty).$$

It then follows that

$$\begin{aligned}I_{12} &= -Hv_\infty \int_\Omega \xi(x_2)(1-\tilde{\pi})u^2 u^1_{,2}\,\mathrm{d}x - Hv_\infty \int_\Omega \xi(x_2)(1-\tilde{\pi})u^2_{,2}u^1\,\mathrm{d}x\\ &= I_{121} + I_{122}.\end{aligned}$$

For I_{121}, one can see that

$$|I_{121}| \leqslant \left(H^2 v_\infty^2 \int_\Omega (\xi(x)_2)(1-\tilde{\pi})^2 (u^2)^2\,\mathrm{d}x\right)^{1/2} \|u^1_{,2}\|_{L_2(\Omega)}. \tag{4.10}$$

We have to find an estimate for the first integral in the right-hand side of (4.10). By the Sobolev imbedding theorem, (4.4) and (4.2), we have

$$\begin{aligned}&H^2 v_\infty^2 \int_\Omega (\xi(x_2)(1-\tilde{\pi}))^2 (u^2)^2\,\mathrm{d}x\\ &\quad \leqslant H^2 v_\infty^2 \varepsilon^2 \left(\|u^2_{,2}\|^2_{L_2(I(x_1))} + \int_{-\infty}^{+\infty}\int_{H-\varrho}^{H} \frac{(u^2)^2\,\mathrm{d}x_2}{(H-x_2)^2}\,\mathrm{d}x_1\right).\end{aligned} \tag{4.11}$$

To find a bound for the last term of the right-hand side of (4.11) we note that

$$\begin{aligned}\int_{H-\varrho}^{H} \frac{(u^2)^2\,\mathrm{d}x_2}{(H-x_2)^2} &= \int_{H-\varrho}^{H} \left(\frac{1}{H-x_2}\right)_{,2} (u^2)^2\,\mathrm{d}x_2\\ &= \left(\frac{(u^2)^2(x_1,x_2)}{H-x_2}\Big|_{H-\varrho}^{H}\right) - \int_{H-\varrho}^{H} \frac{2u^2u^2_{,2}\,\mathrm{d}x_2}{H-x_2}\\ &\leqslant -\frac{(u^2)^2(x_1, H-\varrho)}{\varrho} - \int_{H-\varrho}^{H} \frac{2u^2u^2_{,2}\,\mathrm{d}x_2}{H-x_2} \leqslant \left|\int_{H-\varrho}^{H} \frac{2u^2u^2_{,2}}{H-x_2}\right|\\ &\leqslant 2\left(\int_{H-\varrho}^{H} \frac{(u^2)^2\,\mathrm{d}x_2}{(H-x_2)^2}\right)^{1/2} \|u^2_{,2}\|_{L_2(I(x_1))}.\end{aligned} \tag{4.12}$$

Therefore we conclude that

$$\int_{H-\varrho}^{H} \frac{(u^2)^2\,\mathrm{d}x_2}{(H-x_2)^2} \leqslant 4\|u^2_{,2}\|^2_{L_2(I(x_1))}. \tag{4.13}$$

Estimation (4.13) can be found in [4]. Inserting (4.13) to (4.11), we obtain

$$|I_{121}| \leqslant \sqrt{5}Hv_\infty \varepsilon \|\nabla u\|^2_{L_2(\Omega)}. \tag{4.14}$$

We take ε to be so small that

$$\sqrt{5} H v_\infty \varepsilon \leqslant \tfrac{1}{4}\tilde{\nu}. \tag{4.15}$$

If ϱ and ε are chosen, we can define parameter D. To estimate I_{122} we note that, since $\operatorname{div} u = 0$,

$$I_{122} = -\int_\Omega a^1 u^1_{,1} u^1 \, \mathrm{d}x = \frac{1}{2}\int_\Omega a^1_{,1} (u^1)^2 \, \mathrm{d}x, \tag{4.16}$$

where we used the fact that field a vanishes near the obstacle, hence the boundary terms in the integration by parts are zero. By the definition of a field a – see (4.8) $|a^1_{,1}| \leqslant c\|\xi\|_{L_\infty}/D$. (Note that $\|\xi\|_{L_\infty} \to \infty$ as $\varepsilon \to 0$.) By definition (4.2), we see that $\|\xi\|_{L_\infty} \sim \mathrm{e}^{v_\infty}$. But a parameter ε has already been fixed, hence we can choose D so large that

$$|a^1_{,1}| \leqslant \frac{1}{16}\tilde{\nu} \quad \text{and} \quad |I_{122}| \leqslant \frac{\tilde{\nu}}{16}\|u\|^2_{H^1(\Omega)}.$$

For I_{11} we have the same estimate, $|I_{11}| \leqslant \tilde{\nu}/16\|u\|^2_{H^1(\Omega)}$. Finally to estimate I_2, we note that for the same reason as for (4.16), $|\nabla a^2| \leqslant c\|\xi\|_{L_\infty}/D$. Thus, $|I_2| \leqslant \tilde{\nu}/16\|u\|^2_{H^1(\Omega)}$.

Summing up,

$$\left| \int_\Omega u \cdot \nabla a u \, \mathrm{d}x \right| \leqslant \frac{\tilde{\nu}}{2}\|u\|^2_{H^1(\Omega)}. \tag{4.17}$$

Since $\tilde{\nu}$ can be chosen as small, we get (1.18) for any δ, but in the proof of Theorem A, we need (4.17).

5. Proof of Theorem C

To obtain regularity of the solutions obtained by Theorem A, we use extra properties of problem (1.1). We want to apply the equations on the vorticity of the velocity of the fluid. Since, on this level of our considerations, we have only weak solutions, we use formulation (1.14).

Introduce a subclass of test functions defined in the following way:

$$\phi = (\partial_{x_2}\varphi, -\partial_{x_1}\varphi), \qquad n \cdot \phi|_{\partial\Omega} = 0, \qquad \varphi|_{\partial\Omega} = 0, \tag{5.1}$$

where $\varphi \in H^2(\Omega)$.

Inserting such functions into the weak formulation (1.14), we get

$$\nu \int_\Omega \alpha \Delta\varphi \, \mathrm{d}x - 2\nu \int_{\partial\Omega} \chi u \cdot \tau \frac{\partial \varphi}{\partial n} \, \mathrm{d}\sigma = \int_\Omega (v\nabla)\varphi\alpha \, \mathrm{d}x - \int_\Omega \operatorname{rot} F \varphi \, \mathrm{d}x, \tag{5.2}$$

where $F = (u\nabla)a + (a\nabla)a - \nu\Delta a$. As we see, problem (5.2) is a weak formulation of (1.19) to obtain the vorticity in $L_2(\Omega)$.

Introduce a function $d\colon \Omega \to \mathbf{R}$ which is an extension of the boundary datum such that

$$d|_{\partial\Omega} = 2(u \cdot \tau)\chi \quad \text{and} \quad \|d\|_{H^1(\Omega)} \leqslant c\|u\|_{H^1(\Omega)}. \tag{5.3}$$

Moreover we require that supp $d \subset [-L, L] \times [0, H]$.

Thus, the vorticity can be searched for in the form of

$$\alpha = \beta + d, \tag{5.4}$$

where β satisfies the following problem:

$$\begin{aligned} \nu \int_\Omega \nabla\beta \cdot \nabla\varphi \, \mathrm{d}x \; &= \; -\nu \int_\Omega \nabla d \cdot \nabla\varphi \, \mathrm{d}x - \int_\Omega (v\nabla)\varphi\beta \, \mathrm{d}x - \\ &- \int_\Omega (v\nabla)\varphi d \, \mathrm{d}x + \int_\Omega \operatorname{rot} F\varphi \, \mathrm{d}x \end{aligned} \tag{5.5}$$

for all $\varphi \in H^1_0(\Omega)$. But Theorems A and B guarantee $F \in L_2(\Omega)$, thus by the Lax–Milgram theorem, having (5.3), we can note that $\beta \in H^1_0(\Omega)$ with the following estimate

$$\|\beta\|_{H^1(\Omega)} \leqslant c(v_\infty). \tag{5.6}$$

Hence, we conclude that the vorticity given by (5.4) also satisfies (5.2), thus $\alpha \in H^1(\Omega)$ with a suitable estimate which follows from (5.3) and (5.6).

To obtain the information about the regularity of the velocity, we use the following elliptic problem:

$$\begin{aligned} &\operatorname{rot} u = \alpha, \quad \text{in } \Omega, \\ &\operatorname{div} u = 0, \quad \text{in } \Omega, \\ &n \cdot u = 0, \quad \text{on } \partial\Omega, \\ &u \to 0 \quad \text{as } x_1 \to \pm\infty. \end{aligned} \tag{5.7}$$

Since Ω cannot be a simply connected domain, the vector field can be represented as a gradient of a scalar function ($\operatorname{div} u = 0$) only locally, but the L_2-norm is known, hence we can localize problem (5.7) and get $u \in H^2(\Omega)$ with the following estimate

$$\|u\|_{H^2(\Omega)} \leqslant c(v_\infty). \tag{5.8}$$

Moreover, from $(1.12)_1$ we also get the information about the gradient of the pressure

$$\|\nabla p\|_{L_2(\Omega)} \leqslant c(v_\infty). \tag{5.9}$$

Thus we get (1.20).

To get the full smoothness, it is enough to note that, since $u \in H^2(\Omega)$, then

$$(u\nabla)u \in H^1(\Omega). \tag{5.10}$$

Using the standard technique, we can easily increase regularity up to $H^m(\Omega)$ for any $m \in \mathbf{N}$ and, by the Sobolev imbedding theorem, we get the finiteness of $C^m(\bar{\Omega})$-norms of u and ∇p for all $m \in \mathbf{N}$. Theorem C has been proved. □

6. Appendix

Proof of Lemma 2.1. The proof is based on the results from [9, Lemma 4]. First we prove that if $u \in V$ – see definition (1.15), then

$$\|\nabla u\|_{L_2(\Omega)} \geqslant c\|u\|_{H^1(\Omega)}. \tag{6.1}$$

To show it, first we note that, since

$$u \cdot n|_{\partial\Omega} = 0, \quad u \to 0 \quad \text{as } x_1 \to \pm\infty \quad \text{and} \quad \operatorname{div} u = 0, \tag{6.2}$$

we have

$$\int_{I(x_1)} u^1 \,\mathrm{d}x_2 = 0, \quad \text{for all } x_1 \quad \text{and} \quad u^2|_{x_2=H} = 0. \tag{6.3}$$

This implies the Poincaré inequality for a domain

$$Q = \Omega \setminus ([-L, L] \times (0, H)), \quad \text{i.e. } \|\nabla u\|_{L_2(Q)} \geqslant c\|u\|_{H^1(Q)}. \tag{6.4}$$

Since $\Omega \setminus Q$ is connected and bounded by the trace theorem, we conclude (6.1).

Prove estimate (2.2).

$$\begin{aligned}
\int_\Omega (\mathbf{D}(u))^2 \,\mathrm{d}x &= \frac{1}{2}\int_\Omega \sum_{i,j=1}^{2} (u^i_{,j} + u^j_{,i})^2 \,\mathrm{d}x \\
&= \int_\Omega \sum_{i,j=1}^{2} \left((u^i_{,j})^2 + u^i_{,j}u^j_{,i}\right) \mathrm{d}x \\
&= \|\nabla u\|^2_{L_2(\Omega)} + \int_{\partial\Omega} \sum_{i,j=1}^{2} u^i_{,j}u^j_{,i} \,\mathrm{d}x.
\end{aligned} \tag{6.5}$$

Let us examine the last term of (6.5). Without loss of generality, we can assume that $u \in C^2(\Omega)$ and, by integration by parts, we get

$$\begin{aligned}
\int_\Omega \sum_{i,j=1}^{2} u^i_{,j}u^j_{,i} \,\mathrm{d}x &= -\int_\Omega u^i u^j_{,ij} \,\mathrm{d}x + \int_{\partial\Omega} \sum_{i,j=1}^{2} u^i u^j_{,i} n_j \,\mathrm{d}\sigma \\
&= \int_\Omega \sum_{i,j=1}^{2} u^i_{,i}u^j_{,j} \,\mathrm{d}x - \int_{\partial\Omega} \sum_{i,j=1}^{2} u^i n_i u^j_{,j} \,\mathrm{d}\sigma + \\
&\quad + \int_{\partial\Omega} \sum_{i,j=1}^{2} u^i u^j_{,i} n_j \,\mathrm{d}\sigma.
\end{aligned} \tag{6.6}$$

The first term of the right-hand side of (6.6) is equal to $\int (\operatorname{div} u)^2 \, dx = 0$ and the second one also vanishes, since $u \cdot n|_{\partial\Omega} = 0$. But condition $u \cdot n|_{\partial\Omega} = 0$ also implies that, at the boundary,

$$(u \cdot n)_{,i} = 0 \quad \text{or} \quad \sum_{j=1}^{2} u^j_{,i} n_j = -\sum_{j=1}^{2} u^j n_{j,i}. \tag{6.7}$$

Using (6.7) to (6.6), we get

$$\int_\Omega \sum_{i,j=1}^{2} u^i_{,j} u^j_{,i} \, dx = -\int_{\partial\Omega} \sum_{i,j=1}^{2} u^i u^j n_{i,j} \, d\sigma. \tag{6.8}$$

Of course $|n_{i,j}| \leqslant c\|\partial\Omega\|_{C^2}$, in particular by definition of Ω, we have $n_{i,j} \equiv 0$ for $|x_1| > L$, which guarantees that

$$\int_{\partial\Omega} u^i u^j n_{i,j} \, d\sigma = \int_{\partial O} u^i u^j n_{i,j} \, d\sigma, \tag{6.9}$$

where $\partial O = \partial\Omega \cap ([-L, L] \times [0, H])$, and it is obvious that ∂O is the finite measure giving the compactness of this set.

These considerations ((6.5), (6.8) and (6.9)) lead us to the conclusion that

$$\|u\|^2_{H^1(\Omega)} \leqslant A_1 \left(\int_\Omega (\mathbf{D}(u))^2 \, dx + \|u\|^2_{L_2(\partial O)} \right). \tag{6.10}$$

To finish the proof, we have to show that

$$\|u\|^2_{L_2(\partial O)} \leqslant \frac{1}{2A_1} \|u\|^2_{H^1(\Omega)} + A_2 \int_\Omega (\mathbf{D}(u))^2 \, dx. \tag{6.11}$$

We prove (6.11) by the contradiction. If a number A_2 does not exist, then there exists a bounded sequence $\{u^m\}_{m=0}^\infty \subset V$ such that

$$\|u^m\|^2_{L_2(\partial O)} \geqslant \frac{1}{2A_1} \|u^m\|^2_{H^1(\Omega)} + m \int_\Omega (\mathbf{D}(u^m))^2 \, dx, \tag{6.12}$$

and if we introduce $v^m = u^m / \|u^m\|_{L_2(\partial O)}$, then

$$\|v^m\|_{L_2(\partial O)} = 1 \quad \text{and} \quad m \int_\Omega (\mathbf{D}(v^m))^2 \, dx \leqslant 1. \tag{6.13}$$

Since by (6.12), the sequence $\{v^m\}$ is bounded in V, we can choose a subsequence $\{v^{m_k}\}_{k=0}^\infty$ which is weakly convergent in V and strongly in $L_2(\partial O)$ to a vector $v_* \in V$ and by (6.13)

$$\int_\Omega (\mathbf{D}(v^{m_k}))^2 \, dx \leqslant \frac{1}{m_k} \to 0. \tag{6.14}$$

These relations, together with (6.5) and (6.8), imply that $\|v^{m_k}\|_V \to \|v_*\|_V$ which causes $v^{m_k} \to v_*$ strongly in V which gives, in particular,

$$\int_\Omega (\mathbf{D}(v_*))^2 \,\mathrm{d}x = 0. \tag{6.15}$$

But functions satisfying (6.15) have the form: $v_*^1 = ax_2$ and $v_*^2 = -ax_1$ for $a \in \mathbf{R}$. Hence, since $v_* \in V$, we deduce that $v_* = 0$, which does not agree with (6.13). This shows the existence of a number A_2. To find the estimate for this constant it is necessary to have precise information about the boundary, but this construction is quite complex. Thus, we conclude estimate (2.2) from (6.11) and (6.1).

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