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Abstract

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with a perfect residue field k. For a semi-stable scheme over the ring of integers O_K of K or, more generally, for a log smooth scheme of semi-stable type over k, we define nearby cycles as a single $\mathcal{D}$-module endowed with a monodromy $\partial_t^{\log}$, whose cohomology should give the log crystalline cohomology. We also explicitly describe the monodromy filtration of the $\mathcal{D}$-module with respect to the endomorphism $\partial_t^{\log}$, and construct a weight spectral sequence for the cohomology of the nearby cycles.

1. Introduction

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with a perfect residue field k and let X_K be a proper smooth scheme over K. If X_K has semi-stable reduction, i.e. there exists a proper regular model X flat over the ring of integers O_K of K such that the special fiber X_0 is a reduced divisor with normal crossings on X, then the p-adic nearby cycles, which compute the p-adic étale cohomology of the generic fiber, are known to be described in terms of the syntomic complexes originated from certain de Rham complexes 'with log poles along X_0'. For a prime l different from p, we often understand the l-adic étale cohomology through a local analysis of the l-adic nearby cycles even when X_K has worse reduction. Therefore, it is natural to ask whether we have a corresponding p-adic theory which still works for X with reduction worse than semi-stable. A natural framework in which we work on this problem would be the category of $\mathcal{D}$-modules (without log poles). However, even in the semi-stable reduction case, p-adic theory has been studied by adding log poles and eliminating singularities, and is not yet well understood from the viewpoint of $\mathcal{D}$-modules except for the local results by Gros and Narváez-Macarro in [GN00] and [Gro04]. Note that we have a complete theory of nearby cycles for $\mathcal{D}$-modules on complex analytic varieties (cf. [Kas83, Mal83, MS89, Sab87, Sai88]).

When X is semi-stable, we define, in this paper, nearby cycles as a single $\mathcal{D}$-module which should compute the log crystalline cohomology (Hyodo–Kato cohomology, cf. [Hyo91, HK94]) of X_0; more explicitly, we assume that $p > 2$ and there exists a closed immersion of X_0 into a smooth scheme Y over $\mathrm{Spec}(W_N)$, where $W_N = W_N(k)$, and define nearby cycles as a $\mathcal{D}_{Y/W_N}$-module endowed with a monodromy $\partial_t^{\log}$. We also explicitly describe the monodromy filtration of the nearby cycles and its graded quotients, and construct a weight spectral sequence for the cohomology of the nearby cycles (cf. [Sai88] for the case of complex analytic varieties). The main

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differences from the results of Gros and Narváez-Macarro mentioned above are that we work with torsion coefficients, i.e. over W_N, and that the nearby cycles are defined globally. The weight spectral sequence was already constructed by Mokrane for the log crystalline cohomology [Mok93] (see also [Nak05]), and recently by Nakkajima for the relative log crystalline cohomology. However, our method may have an advantage, for example, in studying functoriality of the weight spectral sequences because we use filtrations of sheaves: $\mathcal{D}$-modules, whereas their methods use filtrations of complexes: de Rham–Witt complexes and de Rham complexes, respectively. The author plans to study the functoriality in a subsequent paper (cf. [Sai03] for the l-adic case). The coincidence of the cohomology of our nearby cycles with the log crystalline cohomology is left as a question (Question 4.5.3) in this paper. After the first version of this paper was written, Berthelot proved the coincidence by using his new theory interpreting direct images of $\mathcal{D}$-modules in terms of crystalline topos.

This paper is organized as follows. In § 2, we summarize basic facts on $\mathcal{D}$-modules for log schemes which will be used in the following sections. Since we need to consider $\mathcal{D}$-modules also for a certain kind of non-smooth morphisms of log schemes, we try to give the details although most of the arguments are found in the literature: [Ber90, Ber96, Ber00, Ber02, Mon02]. In § 3, we define and study nearby cycles when a lifting of the log scheme X_0 over the scheme $\mathrm{Spec}(W_N)$ endowed with the log structure associated to $\mathbb{N} \to W_N; 1 \mapsto 0$ is given. In § 4, we glue the nearby cycles constructed in § 3 and study their properties under the assumption that $p > 2$.

Notation. A log scheme is denoted by a single letter such as X, $Y, \ldots$ and the log structures (respectively the structure sheaves of the underlying schemes, respectively the underlying schemes) of log schemes X, $Y, \ldots$ are denoted by M_X, $M_Y, \ldots$ (respectively $\mathcal{O}_X$, $\mathcal{O}_Y, \ldots$, respectively $\mathring{X}$, $\mathring{Y}, \ldots$). Fiber products of fine log schemes are always considered in the category of fine log schemes (cf. [Kat89, (2.8)]). Unless we consider log structures, sheaves are always considered in the Zariski topology. Let k be a perfect field of characteristic $p > 0$, and let S_0 (respectively T_0) be $\mathrm{Spec}(k)$ with the trivial log structure (respectively the log structure associated to $\mathbb{N} \to k; 1 \mapsto 0$). Let N be a positive integer and let S (respectively T) be $\mathrm{Spec}(W_N(k))$ with the trivial log structure (respectively the log structure associated to $\mathbb{N} \to W_N(k); 1 \mapsto 0$). Let t denote the image of $1 \in \mathbb{N}$ in $\Gamma(T, M_T)$.

2. Preliminaries on the rings of differential operators for log schemes

Let X be a fine log scheme smooth over T. In this section, we define the rings of differential operators $\mathcal{D}_{X/T}$ and $\mathcal{D}_{X/S}$, give interpretations of left $\mathcal{D}_{X/B}$-modules and right $\mathcal{D}_{X/B}$-modules ($B = S, T$) in terms of stratifications and costratifications, and discuss a natural right action of $\mathcal{D}_{X/B}$ on $\wedge^d \Omega_{X/T}$ when X is purely of dimension d and satisfies certain conditions. We also define and study tensor products, inverse images, and direct images. We have a natural homomorphism of rings $\mathcal{D}_{X/T} \to \mathcal{D}_{X/S}$ (2.1.8) and we also discuss the compatibility of the restriction of scalars by this homomorphism with stratifications, costratifications, right actions on $\wedge^d \Omega_{X/T}$, tensor products, inverse images, and direct images (cf. Lemma 2.2.9, the second paragraph of § 2.3, Lemma 2.4.7, (2.5.1), and Corollary 2.6.10). See Montagnon's thesis [Mon02] for the case $B = T$, where the rings of differential operators for log smooth morphisms of fine log schemes were studied. Note that $X \to S$ is not log smooth so that we need to slightly generalize the construction in [Mon02] to define $\mathcal{D}_{X/S}$. See also [Ber90, Ber96, Ber00, Ber02], where the case without log structures is discussed. If we follow the notation in the above references, we should write $\mathcal{D}^{(0)}_{X/B}$ for $\mathcal{D}_{X/B}$. However, since we consider only $\mathcal{D}^{(0)}_{X/B}$ throughout this paper, we omit the superscript (0).

2.1 Rings of differential operators

For an integer r, let $X^{r+1}_{/B}$ be the fiber product over B of $r+1$ copies of X and let $P_{X/B}(r)$ be the PD envelope of the diagonal immersion $X \hookrightarrow X^{r+1}_{/B}$ compatible with the canonical PD structure of pW_N. Let $\mathcal{P}_{X/B}(r)$ be the structure sheaf of the underlying scheme of $P_{X/B}(r)$ and let $\mathcal{J}_{X/B}(r)$ be its PD ideal. We define $P^n_{X/B}(r) \hookrightarrow P_{X/B}(r)$ to be the exact closed immersion defined by the $(n+1)$th divided power $\mathcal{J}_{X/B}(r)^{[n+1]}$. We have a sequence of exact closed immersions:

$$X = P^0_{X/B}(r) \hookrightarrow P^1_{X/B}(r) \hookrightarrow P^2_{X/B}(r) \hookrightarrow \cdots \hookrightarrow P^n_{X/B}(r) \hookrightarrow P^{n+1}_{X/B}(r) \hookrightarrow \cdots \hookrightarrow P_{X/B}(r).$$

Let $\mathcal{P}^n_{X/B}(r)$ denote the structure sheaf $\mathcal{P}_{X/B}(r)/\mathcal{J}_{X/B}(r)^{[n+1]}$ of $P^n_{X/B}(r)$ and let $\mathcal{J}^n_{X/B}(r)$ denote its ideal $\mathcal{J}_{X/B}(r)/\mathcal{J}_{X/B}(r)^{[n+1]}$, which is endowed with the PD structure induced by that of $\mathcal{J}_{X/B}(r)$. We omit (r) from the notation when $r=1$. We have natural PD morphisms $P_{X/T}(r) \to P_{X/S}(r)$ and $P^n_{X/T}(r) \to P^n_{X/S}(r)$ compatible with the above sequence of exact closed immersions.

First we give a local explicit description of $\mathcal{P}_{X/B}$. Let $p_{B,i}$ denote the projection to the ith component $P_{X/B} \to X$ for $i = 1, 2$.

Proposition 2.1.1. *Assume that we are given $t_1, \ldots, t_d \in \Gamma(X, M_X)$ such that $\{d \log t_\nu\}$ is a basis of $\Omega^1_{X/T}$. Then, since $X \hookrightarrow P_{X/B}$ is an exact nilimmersion, there exists a unique $u_{B,\nu} \in 1 + \mathcal{J}_{X/B}$ such that $u_{B,\nu} \cdot p^*_{B,1}(t_\nu) = p^*_{B,2}(t_\nu)$ in $\Gamma(P_{X/B}, M_{P_{X/B}})$ for $1 \leqslant \nu \leqslant d$. The image of $u_{S,\nu}$ in $\mathcal{P}_{X/T}$ is $u_{T,\nu}$, so that we abbreviate $u_{B,\nu}$ to u_ν if there is no risk of confusion. In the case $B = S$, there exists a unique $u \in 1 + \mathcal{J}_{X/S}$ such that $u \cdot p^*_{S,1}(t) = p^*_{S,2}(t)$ in $\Gamma(P_{X/S}, M_{P_{X/S}})$. Let i be 1 or 2, and regard $\mathcal{P}_{X/B}$ as a sheaf of $\mathcal{O}_X$-algebras via $p^*_{B,i}$. Then:*

(1) *the PD homomorphism of sheaves of $\mathcal{O}_X$-algebras*

$$\mathcal{O}_X\langle V_1, \ldots, V_d\rangle \longrightarrow \mathcal{P}_{X/T}$$

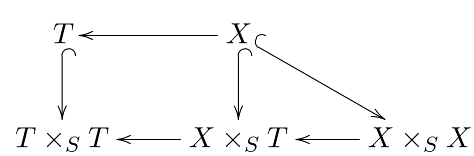
defined by $V_\nu \mapsto u_\nu - 1$ is an isomorphism;

(2) *the PD homomorphism of sheaves of $\mathcal{O}_X$-algebras*

$$\mathcal{O}_X\langle V, V_1, \ldots, V_d\rangle \longrightarrow \mathcal{P}_{X/S}$$

defined by $V \mapsto u - 1$ and $V_\nu \mapsto u_\nu - 1$ is an isomorphism.

Proof. (1) is a special case of [Kat89, Proposition 6.5]. We prove (2) for $i = 1$. Consider the following commutative diagram, whose left square is cartesian.



Put $T(1) = T \times_S T$. The chart $\mathbb{N}_T \to M_T$ induces a chart $(\mathbb{N} \oplus \mathbb{N})_{T(1)} \to M_{T(1)}$. We define $\widetilde{T}(1)$ to be $T(1) \times_{\mathrm{Spec}(\mathbb{Z})[\mathbb{N}\oplus\mathbb{N}]} \mathrm{Spec}(\mathbb{Z})[\mathbb{N} \oplus \mathbb{Z}]$, where $\mathrm{Spec}(\mathbb{Z})[P]$ denotes $\mathrm{Spec}(\mathbb{Z}[P])$ with the log structure associated to $P \hookrightarrow \mathbb{Z}[P]$. The morphism $\mathrm{Spec}(\mathbb{Z})[\mathbb{N} \oplus \mathbb{Z}] \to \mathrm{Spec}(\mathbb{Z})[\mathbb{N} \oplus \mathbb{N}]$ is defined by $\mathbb{N} \oplus \mathbb{N} \to \mathbb{N} \oplus \mathbb{Z}$; $(n, m) \mapsto (n+m, m)$. Then the morphism $\widetilde{T}(1) \to T(1)$ is étale and the closed immersion $T \to T(1)$ factors through the exact closed immersion $T \to \widetilde{T}(1)$ induced by $\mathbb{N} \oplus \mathbb{Z} \to \mathbb{N}$; $(n, m) \mapsto n$. If we denote by v the image of $(0, 1) \in \mathbb{N} \oplus \mathbb{Z}$ in $\Gamma(\widetilde{T}(1), M_{\widetilde{T}(1)})$, then we have an isomorphism $W_N[V] \cong \Gamma(\widetilde{T}(1), \mathcal{O}_{\widetilde{T}(1)})$ sending V to $v - 1$. By the construction of PD envelopes in [Kat89, Proposition 5.3], we see that the PD envelope $P_{T/S}$ of T in $T\times_S T$ is $\mathrm{Spec}(W_N\langle v-1\rangle)$ endowed with the log structure defined by $\mathbb{N}\oplus\mathbb{Z}\to W_N\langle v-1\rangle$; $(n,m)\mapsto v^m$ (if $n=0$), 0 (if $n>0$). Since the morphism $X\to T$ is smooth and integral [Kat89, Corollary 4.4], the lower left horizontal morphism is smooth and integral; especially, it is flat in the underlying schemes [Kat89, Corollary 4.5]. Hence, the PD envelope of X in $X\times_S T$ is isomorphic to $X\times_{T,p_{T,1}}P_{T/S}$, where $p_{T,1}$ denotes the projection to the first component $P_{T/S}\to T$. Especially, the structure sheaf of rings of the PD envelope is $\mathcal{O}_X\langle v-1\rangle$. Since the morphism $X\times_S X\to X\times_S T$ is smooth, the claim follows from [Tsu00, Proposition 1.8]. (Use the inverse images of t_ν by the second projection $X\times_S X\to X$ and the projection $X\times_S T\to X$.) Note that the image of v in $\mathcal{P}_{X/S}$ is u. □

COROLLARY 2.1.2 (cf. [Ber96, Proposition 2.1.3]).

(1) *There exists a unique PD structure on the ideal of the structure sheaf of* $P^n_{X/B}\times_X P^{n'}_{X/B}$ *defining the exact closed immersion* $X\hookrightarrow P^n_{X/B}\times_X P^{n'}_{X/B}$ *such that the two projections* $P^n_{X/B}\times_X P^{n'}_{X/B}\to P^n_{X/B}, P^{n'}_{X/B}$ *are PD morphisms. Furthermore, the* $(n+n'+1)$*th divided power of the PD ideal is* 0. *Here we regard the left* $P^n_{X/B}$ *(respectively the right* $P^{n'}_{X/B}$*) as an* X*-scheme by the second (respectively the first) projection to* X.

(2) *There exists a unique PD structure on the ideal of the structure sheaf of* $P^n_{X/B}\times_X P^{n'}_{X/B}\times_X P^{n''}_{X/B}$ *defining the exact closed immersion* $X\hookrightarrow P^n_{X/B}\times_X P^{n'}_{X/B}\times_X P^{n''}_{X/B}$ *such that the three projections*

$$P^n_{X/B}\times_X P^{n'}_{X/B}\times_X P^{n''}_{X/B}\to P^n_{X/B}, P^{n'}_{X/B}, P^{n''}_{X/B}$$

are PD morphisms. Furthermore, the $(n+n'+n''+1)$*th divided power of the PD ideal is* 0.

Proof. We write $\mathcal{P}^m$ and $\mathcal{J}^m$ for $\mathcal{P}^m_{X/B}$ and $\mathcal{J}^m_{X/B}$ to simplify the notation.

(1) By Proposition 2.1.1, we see that $p_i^*\colon\mathcal{O}_X\to\mathcal{P}^n$ is flat for $i=1,2$. Hence, the PD structure on $\mathcal{J}^n$ (respectively $\mathcal{J}^{n'}$) induces a PD structure of $\mathcal{J}^n\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}$ (respectively $\mathcal{P}^n\otimes_{\mathcal{O}_X}\mathcal{J}^{n'}$). Since $\mathcal{P}^m=p_i^{-1}(\mathcal{O}_X)\oplus\mathcal{J}^m$, the intersection of the above PD ideal is $\mathcal{J}^n\otimes_{\mathcal{O}_X}\mathcal{J}^{n'}$, on which the two PD structures coincide:

$$\gamma_m(a)\otimes b^m=m!\gamma_m(a)\otimes\gamma_m(b)=a^m\otimes\gamma_m(b)\quad\text{for } a\in\mathcal{J}^n \text{ and } b\in\mathcal{J}^{n'}.$$

The second claim follows from the formula

$$\gamma_m(c+d)=\sum_{m=m_1+m_2}\gamma_{m_1}(c)\gamma_{m_2}(d)$$

for $c\in\mathcal{J}^n\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}$ and $d\in\mathcal{P}^n\otimes_{\mathcal{O}_X}\mathcal{J}^{n'}$.

(2) By (1) and the flatness of $p_i^*\colon\mathcal{O}_X\to\mathcal{P}^n$, we can define PD structures on

$$(\mathcal{J}^n\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}+\mathcal{P}^n\otimes_{\mathcal{O}_X}\mathcal{J}^{n'})\otimes_{\mathcal{O}_X}\mathcal{P}^{n''}\quad\text{and}\quad(\mathcal{P}^n\otimes_{\mathcal{O}_X}\mathcal{P}^{n'})\otimes_{\mathcal{O}_X}\mathcal{J}^{n''}.$$

We can verify that they coincide on the intersection

$$(\mathcal{J}^n\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}+\mathcal{P}^n\otimes_{\mathcal{O}_X}\mathcal{J}^{n'})\otimes_{\mathcal{O}_X}\mathcal{J}^{n''}$$

in the same way as (1) and obtain the desired PD structure. □

By Corollary 2.1.2, the morphism

$$P^n_{X/B}\times_X P^{n'}_{X/B}\hookrightarrow(X\times_B X)\times_X(X\times_B X)=X\times_B X\times_B X\xrightarrow{p_{13}}X\times_B X$$

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induces a PD homomorphism

$$\delta^{n,n'}\colon \mathcal{P}^{n+n'}_{X/B} \to \mathcal{P}^{n}_{X/B} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/B}. \tag{2.1.3}$$

Here p_{13} denotes the projection to the first and third components. The homomorphism (2.1.3) is compatible with n and n' in the obvious sense.

Corollary 2.1.4. *The following diagram is commutative.*

$$\begin{array}{ccc}
\mathcal{P}^{n+n'+n''}_{X/B} & \xrightarrow{\delta^{n+n',n}} & \mathcal{P}^{n+n'}_{X/B} \otimes_{\mathcal{O}_X} \mathcal{P}^{n''}_{X/B} \\
\downarrow{\scriptstyle \delta^{n,n'+n''}} & & \downarrow{\scriptstyle \delta^{n,n'}\otimes \mathrm{id}} \\
\mathcal{P}^{n}_{X/B} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'+n''}_{X/B} & \xrightarrow{\mathrm{id}\otimes\delta^{n',n''}} & \mathcal{P}^{n}_{X/B} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/B} \otimes_{\mathcal{O}_X} \mathcal{P}^{n''}_{X/B}
\end{array}$$

Proof. We see that every homomorphism in the diagram is a PD homomorphism for the PD structures defined in Corollary 2.1.2. Furthermore, by the definition of $\delta^{n,n'}$, we see that the two PD homomorphisms $\mathcal{P}^{n+n'+n''}_{X/B} \rightrightarrows \mathcal{P}^{n}_{X/B} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/B} \otimes_{\mathcal{O}_X} \mathcal{P}^{n''}_{X/B}$ give commutative diagrams

$$\begin{array}{ccccc}
P^{n}_{X/B} \times_X P^{n'}_{X/B} \times_X P^{n''}_{X/B} & \hookrightarrow & X^2_{/B} \times_X X^2_{/B} \times_X X^2_{/B} & = & X^4_{/B} \\
\downarrow\downarrow & & & & \downarrow{\scriptstyle p_{14}} \\
P^{n+n'+n''}_{X/B} & & \hookrightarrow & & X^2_{/B}
\end{array}$$

where p_{14} denotes the projection to the first and fourth components. By the universality of the PD envelope of X in $X^2_{/B}$, we see that the two left vertical morphisms coincide. $\square$

Following [Ber96, Mon02], we define $\mathcal{D}_{X/B,n}$ by

$$\mathcal{D}_{X/B,n} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{n}_{X/S}, \mathcal{O}_X),$$

where we regard $\mathcal{P}^{n}_{X/S}$ as an $\mathcal{O}_X$-module via p_1^*. By Proposition 2.1.1, $\mathcal{D}_{X/B,n}$ is a locally free sheaf of $\mathcal{O}_X$-modules locally of finite type and the surjection $\mathcal{P}^{n'}_{X/S} \to \mathcal{P}^{n}_{X/S}$ induces an injection $\mathcal{D}_{X/S,n} \to \mathcal{D}_{X/B,n'}$ for $n' \geqslant n$. We define $\mathcal{D}_{X/B}$ by

$$\mathcal{D}_{X/B} = \varinjlim_{n} \mathcal{D}_{X/B,n}.$$

For $P \in \mathcal{D}_{X/B}$ and $P' \in \mathcal{D}_{X/B}$, if $P \in \mathcal{D}_{X/B,n}$ and $P' \in \mathcal{D}_{X/B,n'}$, then we define the product $P \cdot P'$ to be the composite:

$$\mathcal{P}^{n+n'}_{X/B} \xrightarrow{\delta^{n,n'}} \mathcal{P}^{n}_{X/B} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/B} \xrightarrow{\mathrm{id}\otimes P'} \mathcal{P}^{n}_{X/B} \xrightarrow{P} \mathcal{O}_X.$$

By the compatibility of $\delta^{n,n'}$ for n and n', this is well defined. By Corollary 2.1.4, the multiplication defined above is associative, and we see easily that $\mathcal{D}_{X/B}$ with this product structure becomes a sheaf of rings. The homomorphism

$$\mathcal{O}_X \xrightarrow{\cong} \mathcal{D}_{X/B,0} \subset \mathcal{D}_{X/B}$$

is a ring homomorphism.

We define the action of $P \in \mathcal{D}_{X/B,n}$ on $\mathcal{O}_X$ to be the composite of

$$\mathcal{O}_X \xrightarrow{p_2^*} \mathcal{P}^{n}_{X/B} \xrightarrow{P} \mathcal{O}_X.$$

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We can verify that this defines an action of the ring $\mathcal{D}_{X/B}$ by using the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{P}^{n+n'}_{X/B} & \xrightarrow{\delta^{n,n'}} & \mathcal{P}^{n}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X/B} \\
{\scriptstyle p_2^*}\uparrow & & \uparrow \\
\mathcal{O}_X & \xrightarrow{\quad p_2^* \quad} & \mathcal{P}^{n'}_{X/B}
\end{array}$$

Proposition 2.1.5 (cf. [Mon02, § 2.3.2, C]). *Let t_ν and u_ν $(1\leqslant \nu\leqslant d)$ be as in Proposition 2.1.1. We put $t_0=t$ and $u_0=u$. Let ν_0 be 0 (respectively 1) if $B=S$ (respectively $B=T$). For $n\geqslant 0$, the elements*

$$\left\{\prod_{\nu_0\leqslant\nu\leqslant d}(u_\nu-1)^{[n_\nu]}\mid \underline{n}=(n_{\nu_0},\dots,n_d)\in\mathbb{N}^{d+1-\nu_0},\ |\underline{n}|:=n_{\nu_0}+\cdots+n_d\leqslant n\right\}$$

are a basis of $\mathcal{P}^n_{X/B}$ as a left $\mathcal{O}_X$-module (Proposition 2.1.1) and we define $\partial^{\langle\underline{n}\rangle}\in\mathcal{D}_{X/B,n}$ to be its dual basis. We denote by ∂_ν the dual of $u_\nu-1$. This definition is consistent with respect to the inclusion $\mathcal{D}_{X/B,n}\hookrightarrow\mathcal{D}_{X/B,n+1}$. We have the following formulae in $\mathcal{D}_{X/B}$:

(1) $\partial_\nu\partial_\mu=\partial_\mu\partial_\nu$ *for* $\nu_0\leqslant\nu,\mu\leqslant d$;
(2) $\partial^{\langle\underline{n}\rangle}=\prod_{\nu=\nu_0}^{d}\prod_{j=0}^{n_\nu-1}(\partial_\nu-j)$ *for* $\underline{n}\in\mathbb{N}^{d+1-\nu_0}$;
(3) $\partial_\nu\cdot f=\partial_\nu(f)+f\cdot\partial_\nu$ *for* $f\in\mathcal{O}_X$.

We also have the following equality for $d\colon\mathcal{O}_X\to\Omega^1_{X/B}$:

(4) $d(f)=\sum_{\nu=\nu_0}^{d}\partial_\nu(f)\,d\log(t_\nu)$.

Proof. Put $\tau_\nu=u_\nu-1$. Then (4) follows from the fact that d is given by

$$p_2^*-p_1^*\colon\mathcal{O}_X\to\operatorname{Ker}(\mathcal{P}^1_{X/B}\to\mathcal{O}_X)=\Omega^1_{X/B}$$

and $d\log(t_\nu)=\tau_\nu$ in $\mathcal{P}^1_{X/B}$.

(1) and (2) For $\underline{n}\in\mathbb{N}^{d+1-\nu_0}$ and $n=|\underline{n}|$, the image of

$$\prod_{\nu_0\leqslant\nu\leqslant d}\tau_\nu^{[m_\nu]}\quad(\underline{m}\in\mathbb{N}^{d+1-\nu_0},\ |\underline{m}|\leqslant n+1)$$

by the homomorphism $\delta^{n,1}\colon\mathcal{P}^{n+1}_{X/B}\longrightarrow\mathcal{P}^n_{X/B}\otimes_{\mathcal{O}_X}\mathcal{P}^1_{X/B}$ is

$$\begin{aligned}\prod_{\nu_0\leqslant\nu\leqslant d}(u_\nu\otimes u_\nu-1)^{[m_\nu]}&=\prod_{\nu_0\leqslant\nu\leqslant d}((\tau_\nu+1)\otimes\tau_\nu+\tau_\nu\otimes 1)^{[m_\nu]}\\&=\prod_{\nu_0\leqslant\nu\leqslant d}(\tau_\nu^{[m_\nu]}\otimes 1+m_\nu\tau_\nu^{[m_\nu]}\otimes\tau_\nu+\tau_\nu^{[m_\nu-1]}\otimes\tau_\nu).\end{aligned}$$

For $\nu_0\leqslant\nu\leqslant d$, the image of this element by

$$\mathcal{P}^n_{X/B}\otimes_{\mathcal{O}_X}\mathcal{P}^1_{X/B}\xrightarrow{\partial_\nu}\mathcal{P}^n_{X/B}\xrightarrow{\partial^{\langle\underline{n}\rangle}}\mathcal{O}_X$$

is n_ν if $\underline{m}=\underline{n}$, 1 if $\underline{m}=\underline{n}+\epsilon_\nu$, and 0 otherwise. Here ϵ_ν denotes the element of $\mathbb{N}^{d+1-\nu_0}$ whose νth component is 1 and other components are 0. Hence, we have

$$\partial^{\langle\underline{n}\rangle}\partial_\nu=n_\nu\partial^{\langle\underline{n}\rangle}+\partial^{\langle\underline{n}+\epsilon_\nu\rangle}\Longleftrightarrow\partial^{\langle\underline{n}+\epsilon_\nu\rangle}=\partial^{\langle\underline{n}\rangle}(\partial_\nu-n_\nu).$$

This implies that $\partial_\nu\partial_\mu=\partial^{\langle\epsilon_\nu+\epsilon_\mu\rangle}=\partial_\mu\partial_\nu$ if $\nu\neq\mu$. We obtain (2) by induction on $|\underline{n}|$ using the above formula.

(3) In $\mathcal{P}^1_{X/B}$, we have

$$p_2^*(f) = p_1^*(f) + \sum_{\nu_0 \leqslant \nu \leqslant d} p_1^*(\partial_\nu(f))\tau_\nu$$

by (4). Hence, the images of 1, τ_ν, and τ_μ ($\mu \neq \nu$) by

$$\mathcal{P}^1_{X/B} \xrightarrow{p_2^*(f)} \mathcal{P}^1_{X/B} \xrightarrow{\partial_\nu} \mathcal{O}_X$$

are $\partial_\nu(f)$, f, and 0, respectively. Hence, $\partial_\nu \cdot f = \partial_\nu(f) + f \cdot \partial_\nu$. □

Corollary 2.1.6. *Let the notation and assumptions be the same as in Proposition 2.1.5. For either of the left $\mathcal{O}_X$-action or the right $\mathcal{O}_X$-action on $\mathcal{D}_{X/B,n}$, $\mathcal{D}_{X/B,n}$ is a free $\mathcal{O}_X$-module with a basis $\{\prod_{\nu_0 \leqslant \nu \leqslant d} \partial_\nu^{n_\nu} \mid n_\nu \in \mathbb{N}, \sum_{\nu_0 \leqslant \nu \leqslant d} n_\nu \leqslant n\}$.*

Proof. For the left action, the claim immediately follows from Proposition 2.1.5. The proposition also implies that the two actions of $\mathcal{O}_X$ on $\mathcal{D}_{X/B,n}/\mathcal{D}_{X/B,n-1}$ coincide. Hence, the claim also holds for the right action. □

We define the increasing filtration $F_n\mathcal{D}_{X/B}$ ($n \in \mathbb{Z}$) of $\mathcal{D}_{X/B}$ by $F_n\mathcal{D}_{X/B} = \mathcal{D}_{X/B,n}$ ($n \geqslant 0$) and $F_n\mathcal{D}_{X/B} = 0$ ($n < 0$). Since $F_n\mathcal{D}_{X/B} \cdot F_m\mathcal{D}_{X/B} \subset F_{n+m}\mathcal{D}_{X/B}$ for n, $m \in \mathbb{Z}$, $\mathrm{gr}^F \mathcal{D}_{X/B}$ becomes a sheaf of rings. Put $T_{X/B} = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega^1_{X/B}, \mathcal{O}_X)$.

Corollary 2.1.7. *The sheaf of rings $\mathrm{gr}^F \mathcal{D}_{X/B}$ is commutative and the isomorphism of $\mathcal{O}_X$-modules $\mathrm{gr}^F_1 \mathcal{D}_{X/B} \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathrm{Ker}(p_{1*}\mathcal{P}^1_{X/B} \to \mathcal{O}_X), \mathcal{O}_X) \cong T_{X/B}$ induces an isomorphism of graded $\mathcal{O}_X$-algebras: $\mathrm{Sym}_{\mathcal{O}_X} T_{X/B} \xrightarrow{\cong} \mathrm{gr}^F \mathcal{D}_{X/B}$.*

Proof. The map

$$F_n\mathcal{D}_{X/B} \times F_m\mathcal{D}_{X/B} \to \mathrm{gr}^F_{n+m} \mathcal{D}_{X/B}; (P, Q) \mapsto PQ - QP$$

is bilinear for the left actions of $\mathcal{O}_X$ since the two actions of $\mathcal{O}_X$ on $\mathrm{gr}^F \mathcal{D}_{X/B}$ coincide, as is mentioned in the proof of Corollary 2.1.6. Hence, by Proposition 2.1.5(1) and (2), we see that the above map is 0 and $\mathrm{gr}^F \mathcal{D}_{X/B}$ is commutative. The latter claim follows from Corollary 2.1.6. □

The morphism $X \times_T X \to X \times_S X$ induces PD morphisms $P_{X/T} \to P_{X/S}$ and $P^n_{X/T} \to P^n_{X/S}$. By taking the dual of the homomorphisms between the structure sheaves, we obtain a homomorphism of sheaves of $\mathcal{O}_X$-modules:

$$\mathcal{D}_{X/T} \to \mathcal{D}_{X/S}. \tag{2.1.8}$$

Proposition 2.1.9. *The homomorphism (2.1.8) is a ring homomorphism.*

Proof. It suffices to prove the compatibility of $\delta^{n,n'}$ with the homomorphisms $\mathcal{P}^m_{X/S} \to \mathcal{P}^m_{X/T}$ ($m = n, n', n+n'$). The morphism $P^n_{X/T} \times_X P^{n'}_{X/T} \to P^n_{X/S} \times_X P^{n'}_{X/S}$ is a PD morphism. Hence, the commutative diagram

$$\begin{array}{ccc} X \times_T X \times_T X & \longrightarrow & X \times_S X \times_S X \\ \downarrow{\scriptstyle p_{13}} & & \downarrow{\scriptstyle p_{13}} \\ X \times_T X & \longrightarrow & X \times_S X \end{array}$$

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induces the following commutative diagram.

$$\begin{array}{ccc} P^n_{X/T}\times_X P^{n'}_{X/T} & \longrightarrow & P^n_{X/S}\times_X P^{n'}_{X/S} \\ \downarrow & & \downarrow \\ P^{n+n'}_{X/T} & \longrightarrow & P^{n+n'}_{X/S} \end{array}$$

□

Assume that the underlying scheme $\mathring{X}$ of X is smooth over S and $X=\mathring{X}\times_S T$. Then the diagonal immersion $X\hookrightarrow X\times_T X$ is an exact closed immersion. Hence, the underlying scheme of $P_{X/T}$ is the PD envelope of $\mathring{X}\hookrightarrow \mathring{X}\times_S\mathring{X}$ endowed with the inverse image of M_T. This implies that we have a canonical isomorphism of sheaves of $\mathcal{O}_X$-algebras:

$$\mathcal{D}_{X/T}\xrightarrow{\cong}\mathcal{D}_{\mathring{X}/S}. \tag{2.1.10}$$

On the other hand, we have

$$\Omega^1_{X/S}\cong\Omega^1_{\mathring{X}/S}\oplus(\mathcal{O}_X\otimes_{\mathcal{O}_T}\Omega^1_{T/S})\quad\text{and}\quad\Omega^1_{T/S}=\mathcal{O}_T\,d\log(t).$$

By taking the dual of $d\log(t)$, we obtain a canonical element $\partial_t^{\log}$ of

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X\otimes_{\mathcal{O}_T}\Omega^1_{T/S},\mathcal{O}_X)\subset\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S},\mathcal{O}_X)\subset\mathcal{D}_{X/S,1}.$$

Proposition 2.1.11. *Under the notation and assumptions as above, the differential operator $\partial_t^{\log}$ is contained in the center of $\mathcal{D}_{X/S}$ and the homomorphism (2.1.8) induces an isomorphism of sheaves of rings:*

$$\mathcal{D}_{X/T}[V]\xrightarrow{\cong}\mathcal{D}_{X/S};V\mapsto\partial_t^{\log}.$$

Proof. Since the question is étale local, we may assume that there exist $t_1,\dots,t_d\in\Gamma(X,\mathcal{O}_X^\times)$ such that $\{dt_\nu\}$ is a basis of $\Omega^1_{\mathring{X}/S}$. Since the image of $d\colon\mathcal{O}_X\to\Omega^1_{X/S}$ is contained in $\Omega^1_{\mathring{X}/S}$, Proposition 2.1.5(4) implies that $\partial_t^{\log}(f)=0$ for $f\in\mathcal{O}_X$. By Proposition 2.1.5(1), (2), and (3), we see that $\partial_t^{\log}$ is contained in the center of $\mathcal{D}_{X/S}$. The second claim follows from Corollary 2.1.6. □

2.2 $\mathcal{D}_{X/B}$-modules and stratifications

For $n\in\mathbb{N}$, let p_i^n $(i=1,2)$ denote the morphism $P^n_{X/B}\to X$ induced by the projection to the ith component $X^2_{/B}\to X$ and let q_j^n $(j=1,2,3)$ denote the morphism $P^n_{X/B}(2)\to X$ induced by the projection to the jth component $X^3_{/B}\to X$. Let q^n_{ij} $((i,j)=(1,2),(2,3),(1,3))$ be the morphism $P^n_{X/B}(2)\to P^n_{X/B}$ induced by the morphism $X^3_{/B}\to X^2_{/B}$ whose composite with the first (respectively second) projection $X^2_{/B}\to X$ is the ith (respectively jth) projection $X^3_{/B}\to X$. Let ι^n denote the exact closed immersion $P^n_{X/B}\to P^{n+1}_{X/B}$.

Theorem 2.2.1 (cf. [Ber96, Proposition 2.3.2] and [Mon02, Proposition 2.6.1]). *The category of left $\mathcal{D}_{X/B}$-modules is canonically equivalent to the following category.*

Object: an $\mathcal{O}_X$-module $\mathcal{E}$ endowed with a family of isomorphisms $\{\varepsilon_n\colon p_2^{n*}\mathcal{E}\xrightarrow{\cong}p_1^{n*}\mathcal{E}\}_{n\in\mathbb{N}}$ of $\mathcal{P}^n_{X/B}$-modules satisfying the following properties.

(i) $\varepsilon_0=\mathrm{id}_{\mathcal{E}}$.

(ii) *For every $n\in\mathbb{N}$, $\varepsilon_n=\iota_n^*(\varepsilon_{n+1})$.*

(iii) *For every $n \in \mathbb{N}$, the following diagram is commutative.*

$$\begin{array}{ccc} q_3^{n*}\mathcal{E} & \xrightarrow{q_{23}^{n*}(\varepsilon_n)} & q_2^{n*}\mathcal{E} \\ & \searrow^{q_{13}^{n*}(\varepsilon_n)} & \downarrow{\scriptstyle q_{12}^{n*}(\varepsilon_n)} \\ & & q_1^{n*}\mathcal{E} \end{array}$$

Morphism: a morphism $(\mathcal{E}, \{\varepsilon_n\}) \to (\mathcal{E}', \{\varepsilon'_n\})$ is a $\mathcal{O}_X$-linear homomorphism $\alpha\colon \mathcal{E} \to \mathcal{E}'$ such that $\varepsilon'_n \circ p_2^{n*}(\alpha) = p_1^{n*}(\alpha) \circ \varepsilon_n$ for every $n \in \mathbb{N}$.

Definition 2.2.2 (cf. [Ber96, Définition 2.3.1] and [Mon02, Définition 2.6.1]). Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. *A PD stratification on $\mathcal{E}$ relative to B* is a family of isomorphisms $\{\varepsilon_n \colon p_2^{n*}\mathcal{E} \xrightarrow{\cong} p_1^{n*}\mathcal{E}\}_{n\in\mathbb{N}}$ of $\mathcal{P}^n_{X/B}$-modules satisfying the conditions (i), (ii), and (iii) in Theorem 2.2.1.

Proposition 2.2.3 (cf. [Ber96, Proposition 2.3.2] and [Mon02, Proposition 2.6.1]). *The category of left $\mathcal{D}_{X/B}$-modules is canonically equivalent to the following category.*

Object: an $\mathcal{O}_X$-module $\mathcal{E}$ endowed with a family of homomorphisms $\{\theta_n \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/B}\}_{n\in\mathbb{N}}$ $\mathcal{O}_X$-linear for the right $\mathcal{O}_X$-module structures of the targets satisfying the following properties.

(i) $\theta_0 = \mathrm{id}_{\mathcal{E}}$.

(ii) *For every $n \in \mathbb{N}$, the composite of θ_{n+1} with the homomorphism*

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{n+1}_{X/B} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/B}$$

coincides with θ_n.

(iii) *For every $n, n' \in \mathbb{N}$, the following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\theta_{n'}} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/B} \\ \downarrow{\scriptstyle \theta_{n+n'}} & & \downarrow{\scriptstyle \theta_n \otimes \mathrm{id}_{\mathcal{P}^{n'}_{X/B}}} \\ \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{n+n'}_{X/B} & \xrightarrow{\mathrm{id}_{\mathcal{E}} \otimes \delta^{n,n'}} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/B} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/B} \end{array}$$

Morphism: a morphism $(\mathcal{E}, \{\theta_n\}_{n\in\mathbb{N}}) \to (\mathcal{E}', \{\theta'_n\}_{n\in\mathbb{N}})$ is an $\mathcal{O}_X$-linear homomorphism $\alpha\colon \mathcal{E} \to \mathcal{E}'$ such that $\theta'_n \circ \alpha = (\alpha \otimes \mathrm{id}_{\mathcal{P}^n_{X/B}}) \circ \theta_n$ for every $n \in \mathbb{N}$.

Lemma 2.2.4. *We regard an $\mathcal{O}_X$-module as a left $\mathcal{O}_X$-module. For $\mathcal{O}_X$-modules or $\mathcal{O}_X$-bimodules, let $\mathcal{H}\mathrm{om}_{lr}(-,-)$ denote the sheaf of $\mathcal{O}_X$-linear homomorphisms for the left $\mathcal{O}_X$-action on the source and the right $\mathcal{O}_X$-action on the target. We define $\mathcal{H}\mathrm{om}_{ll}$, $\mathcal{H}\mathrm{om}_{rl}$, and $\mathcal{H}\mathrm{om}_{rr}$ in the same way.*

Let $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ be $\mathcal{O}_X$-modules, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-bimodules locally free of finite type as left $\mathcal{O}_X$-modules. Note that $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is locally free of finite type as a left $\mathcal{O}_X$-module. For an $\mathcal{O}_X$-bimodule $\mathcal{L}$ locally free of finite type as a left $\mathcal{O}_X$-module, let $\mathcal{L}^\vee$ denote the $\mathcal{O}_X$-bimodules $\mathcal{H}\mathrm{om}_{ll}(\mathcal{L}, \mathcal{O}_X)$. We have a natural isomorphism of $\mathcal{O}_X$-bimodules $(\mathcal{L}^\vee)^\vee \cong \mathcal{L}$.

(1) *There exists a canonical $\mathcal{O}_X$-bilinear isomorphism:*

$$\mathcal{H}\mathrm{om}_{ll}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}_{lr}(\mathcal{E}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}^\vee)$$

functorial on $\mathcal{E}$ and $\mathcal{F}$. Here we regard the left-hand side (LHS) (respectively the right-hand side (RHS)) as an $\mathcal{O}_X$-bimodule through the actions of $\mathcal{O}_X$ on $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}$ (respectively $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}^\vee$).

(2) *The canonical isomorphism:*

$$(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N})^\vee\cong\mathcal{M}^\vee\otimes_{\mathcal{O}_X}\mathcal{N}^\vee$$

given by (1) is an isomorphism of $\mathcal{O}_X$-bimodules. For $f\in\mathcal{M}^\vee$ and $g\in\mathcal{N}^\vee$, $f\otimes g$ of the RHS corresponds to $h=f\circ(\mathrm{id}_\mathcal{M}\otimes g)$ of the LHS. Furthermore,

$$\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}\cong((\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N})^\vee)^\vee\cong(\mathcal{M}^\vee\otimes_{\mathcal{O}_X}\mathcal{N}^\vee)^\vee\cong(\mathcal{M}^\vee)^\vee\otimes_{\mathcal{O}_X}(\mathcal{N}^\vee)^\vee\cong\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}$$

is the identity map.

(3) *For $f\in\mathcal{H}om_{ll}(\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{E},\mathcal{F})$ and $g\in\mathcal{H}om_{ll}(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{F},\mathcal{G})$, let $f^\vee\in\mathcal{H}om_{lr}(\mathcal{E},\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{N}^\vee)$ and $g^\vee\in\mathcal{H}om_{lr}(\mathcal{F},\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{M}^\vee)$ be the homomorphisms corresponding to f and g by (1). Then the composite of*

$$\mathcal{E}\xrightarrow{f^\vee}\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{N}^\vee\xrightarrow{g^\vee\otimes\mathrm{id}_{\mathcal{N}^\vee}}\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{M}^\vee\otimes_{\mathcal{O}_X}\mathcal{N}^\vee\overset{(2)}{\cong}\mathcal{G}\otimes_{\mathcal{O}_X}(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N})^\vee$$

corresponds to the composite of

$$\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{E}\xrightarrow{\mathrm{id}_\mathcal{M}\otimes f}\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{F}\xrightarrow{g}\mathcal{G}$$

by (1).

(3)′ *For $f\in\mathcal{H}om_{ll}(\mathcal{N}^\vee\otimes_{\mathcal{O}_X}\mathcal{E},\mathcal{F})$ and $g\in\mathcal{H}om_{ll}(\mathcal{M}^\vee\otimes_{\mathcal{O}_X}\mathcal{F},\mathcal{G})$, let $f^\vee\in\mathcal{H}om_{lr}(\mathcal{E},\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{N})$ and $g^\vee\in\mathcal{H}om_{lr}(\mathcal{F},\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{M})$ be the homomorphisms corresponding to f and g by (1). Then the composite of*

$$\mathcal{E}\xrightarrow{f^\vee}\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{N}\xrightarrow{g^\vee\otimes\mathrm{id}_\mathcal{N}}\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}$$

corresponds to the composite of

$$(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N})^\vee\otimes_{\mathcal{O}_X}\mathcal{E}\overset{(2)}{\cong}\mathcal{M}^\vee\otimes_{\mathcal{O}_X}\mathcal{N}^\vee\otimes_{\mathcal{O}_X}\mathcal{E}\xrightarrow{\mathrm{id}_{\mathcal{M}^\vee}\otimes f}\mathcal{M}^\vee\otimes_{\mathcal{O}_X}\mathcal{F}\xrightarrow{g}\mathcal{G}$$

by (1).

(4) *Let $f\colon\mathcal{M}\to\mathcal{N}$ be a homomorphism of $\mathcal{O}_X$-bimodules and let $f^\vee\colon\mathcal{N}^\vee\to\mathcal{M}^\vee$ be its dual. Then the following diagram is commutative.*

$$\begin{array}{ccc}
\mathcal{H}om_{ll}(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{E},\mathcal{F}) & \underset{(1)}{\xrightarrow{\cong}} & \mathcal{H}om_{lr}(\mathcal{E},\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{M}^\vee)\\
\uparrow{\scriptstyle -\circ(f\otimes\mathrm{id}_\mathcal{E})} & & \uparrow{\scriptstyle (\mathrm{id}_\mathcal{F}\otimes f^\vee)\circ-}\\
\mathcal{H}om_{ll}(\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{E},\mathcal{F}) & \underset{(1)}{\xrightarrow{\cong}} & \mathcal{H}om_{lr}(\mathcal{E},\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{N}^\vee)
\end{array}$$

Proof. (1) The map from the LHS to the RHS is defined as follows. We identify $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{M}^\vee$ with $\mathcal{H}om_{ll}(\mathcal{M},\mathcal{F})$. Note that $\mathcal{M}$ is locally free of finite type as a left $\mathcal{O}_X$-module by assumption. For a local section of the LHS $f\colon\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{E}\to\mathcal{F}$, we define the corresponding map $g\colon\mathcal{E}\to\mathcal{H}om_{ll}(\mathcal{M},\mathcal{F})$ by $g(e)=f_e$, $f_e(m)=f(m\otimes e)$. One can verify that $f_e\in\mathcal{H}om_{ll}(\mathcal{M},\mathcal{F})$, $g\in\mathcal{H}om_{lr}(\mathcal{E},\mathcal{H}om_{ll}(\mathcal{M},\mathcal{F}))$ and the map thus obtained is a homomorphism of $\mathcal{O}_X$-bimodules. The map from the RHS to the LHS is defined as follows. For a local section of the RHS $g\colon\mathcal{E}\to\mathcal{H}om_{ll}(\mathcal{M},\mathcal{F})$, we define $f\colon\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{E}\to\mathcal{F}$ by $f(m\otimes e)=\{g(e)\}(m)$. One can verify that f is well defined and contained in the LHS. It is straightforward to see that these two maps are the inverse of each other.

(2) The natural isomorphism

$$\mathcal{M}^\vee\otimes_{\mathcal{O}_X}\mathcal{N}^\vee\cong\mathcal{H}om_{lr}(\mathcal{N},\mathcal{H}om_{ll}(\mathcal{M},\mathcal{O}_X))$$

sends $f \otimes g$ to φ defined by $\varphi(n) = f \cdot g(n)$. Hence, the image h of $f \otimes g$ in $\mathcal{H}om_{ll}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{O}_X)$ is given by

$$h(m \otimes n) = \{\varphi(n)\}(m) = \{f \cdot g(n)\}(m) = f(m \cdot g(n)) = f \circ (\mathrm{id}_{\mathcal{M}} \otimes g)(m \otimes n).$$

This implies that the isomorphism is compatible with the left and right actions of $\mathcal{O}_X$. For the last claim, using the above description, one can verify that the images of $m \otimes n \in \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ in

$$(\mathcal{M}^\vee \otimes_{\mathcal{O}_X} \mathcal{N}^\vee)^\vee = \mathrm{Hom}_{ll}(\mathcal{M}^\vee \otimes_{\mathcal{O}_X} \mathcal{N}^\vee, \mathcal{O}_X)$$

by the two maps (the composite of the left two isomorphisms and that of the right two) are the same and given by $f \otimes g \mapsto f(m \cdot g(n))$.

(3) The homomorphisms $f^\vee : \mathcal{E} \to \mathcal{H}om_{ll}(\mathcal{N}, \mathcal{F})$ and $g^\vee : \mathcal{F} \to \mathcal{H}om_{ll}(\mathcal{M}, \mathcal{G})$ are given by $\{f^\vee(x)\}(n) = f(n \otimes x)$ and $\{g^\vee(y)\}(m) = g(m \otimes y)$. This implies that the homomorphism

$$(g^\vee \otimes \mathrm{id}_{\mathcal{N}^\vee}) \circ f^\vee : \mathcal{E} \mapsto \mathcal{H}om_{lr}(\mathcal{N}, \mathcal{H}om_{ll}(\mathcal{M}, \mathcal{G}))$$

is given by

$$\varphi : x \mapsto \{n \mapsto [m \mapsto g(m \otimes f(n \otimes x))]\}.$$

On the other hand, the homomorphism $\mathcal{E} \to \mathcal{H}om_{ll}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{G})$ corresponding to $g \circ (\mathrm{id}_{\mathcal{M}} \otimes f)$ is given by $\psi : x \mapsto \{m \otimes n \mapsto g(m \otimes f(n \otimes x))\}$. Hence, the claim follows from the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{H}om_{ll}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{G}) & \xrightarrow[(1)]{\cong} & \mathcal{H}om_{lr}(\mathcal{N}, \mathcal{H}om_{ll}(\mathcal{M}, \mathcal{G})) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H}om_{ll}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{O}_X) & \xrightarrow[\mathrm{id}_{\mathcal{G}} \otimes (1)]{\cong} & \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H}om_{lr}(\mathcal{N}, \mathcal{H}om_{ll}(\mathcal{M}, \mathcal{O}_X))
\end{array}$$

Note that the upper horizontal map sends $\psi(x)$ to $\varphi(x)$ for $x \in \mathcal{E}$.

(3)′ By (3), we are reduced to the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{H}om_{ll}(\mathcal{M}^\vee \otimes_{\mathcal{O}_X} \mathcal{N}^\vee \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{G}) & \xrightarrow[(1)]{\cong} & \mathcal{H}om_{lr}(\mathcal{E}, \mathcal{G} \otimes_{\mathcal{O}_X} (\mathcal{M}^\vee \otimes_{\mathcal{O}_X} \mathcal{N}^\vee)^\vee) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
\mathcal{H}om_{ll}((\mathcal{M} \otimes \mathcal{N})^\vee \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{G}) & \xrightarrow[(1)]{\cong} & \mathcal{H}om_{lr}(\mathcal{E}, \mathcal{G} \otimes_{\mathcal{O}_X} ((\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})^\vee)^\vee)
\end{array}$$

where the left (respectively right) vertical map is induced by $\alpha : (\mathcal{M} \otimes \mathcal{N})^\vee \overset{(2)}{\cong} \mathcal{M}^\vee \otimes \mathcal{M}^\vee$ (respectively $\beta : (\mathcal{M}^\vee \otimes \mathcal{N}^\vee)^\vee \overset{(2)}{\cong} (\mathcal{M}^\vee)^\vee \otimes (\mathcal{N}^\vee)^\vee \cong \mathcal{M} \otimes \mathcal{N} \cong ((\mathcal{M} \otimes \mathcal{N})^\vee)^\vee$). By (2), we see that β is the dual of α, and the commutativity is reduced to (4).

(4) Straightforward computation. □

Proof of Proposition 2.2.3. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. By Lemma 2.2.4(1) and (4), giving a system $\{\theta_n\}$ satisfying (i) and (ii) is equivalent to giving a system of homomorphisms $\{\eta_n : \mathcal{D}_{X/B,n} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E}\}_{n \in \mathbb{N}}$ linear for the left $\mathcal{O}_X$-module structures of the sources such that $\eta_0 = \mathrm{id}_{\mathcal{E}}$ and the composite of η_{n+1} with $\mathcal{D}_{X/B,n} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{D}_{X/B,n+1} \otimes_{\mathcal{O}_X} \mathcal{E}$ coincides with η_n for every $n \in \mathbb{N}$. By Lemma 2.2.4(3)′ and (4), the condition (iii) is equivalent to the commutativity of the following diagram.

$$\begin{array}{ccccc}
(\mathcal{D}_{X/B,n}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n'})\otimes_{\mathcal{O}_X}\mathcal{E} & \xrightarrow[\text{Lemma 2.2.4(2)}]{\cong} & (\mathcal{P}^n_{X/B}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X/B})^\vee\otimes_{\mathcal{O}_X}\mathcal{E} & \xrightarrow{(\delta^{n,n'})^\vee\otimes\mathrm{id}_{\mathcal{E}}} & \mathcal{D}_{X/B,n+n'}\otimes_{\mathcal{O}_X}\mathcal{E} \\
\downarrow{\scriptstyle \mathrm{id}_{\mathcal{D}_{X/B,n}}\otimes\eta_{n'}} & & & & \downarrow{\scriptstyle \eta_{n+n'}} \\
\mathcal{D}_{X/B,n}\otimes_{\mathcal{O}_X}\mathcal{E} & & \xrightarrow{\qquad\eta_n\qquad} & & \mathcal{E}
\end{array}$$

Since the composite of the isomorphism $\mathcal{D}_{X/B,n}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n'}\cong(\mathcal{P}^n_{X/B}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X/B})^\vee$ with $(\delta^{n,n'})^\vee$ is the ring product by the description of the isomorphism in Lemma 2.2.4(2), this is equivalent to saying that the homomorphism $\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}\to\mathcal{E}$ induced by $\{\eta_n\}$ makes $\mathcal{E}$ a left $\mathcal{D}_{X/B}$-module. By the functoriality of the isomorphism in Lemma 2.2.4(1), we see that the above correspondence gives the desired equivalence of categories. □

Proof of Theorem 2.2.1. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module, let $\{\theta_n\colon\mathcal{E}\to\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{P}^n_{X/B}\}_{n\in\mathbb{N}}$ be a family of homomorphisms $\mathcal{O}_X$-linear for the right $\mathcal{O}_X$-module structures of the targets, and let

$$\{\varepsilon_n\colon p_2^{n*}\mathcal{E}=\mathcal{P}^n_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}\to\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{P}^n_{X/B}=p_1^{n*}\mathcal{E}\}_{n\in\mathbb{N}}$$

be the family of homomorphisms of $\mathcal{P}_{X/B,n}$-modules associated to $\{\theta_n\}$. By Proposition 2.2.3, it suffices to prove that the conditions (i), (ii), and (iii) in the theorem are equivalent to those in Proposition 2.2.3 and that ε_n is an isomorphism. The equivalence for (i) and (ii) is trivial. Assume that the conditions (i) and (ii) are satisfied. Let $p_{12}^{n,n'}$ and $p_{23}^{n,n'}$ denote the projections from $P^n_{X/B}\times_X P^{n'}_{X/B}$ to $P^n_{X/B}$ and $P^{n'}_{X/B}$, respectively, and let $p_{13}^{n,n'}$ denote the PD morphism $P^n_{X/B}\times_X P^{n'}_{X/B}\to P^{n+n'}_{X/B}$ considered after Corollary 2.1.2. For $j=1,2,3$, let $r_j^{n,n'}$ denote the composite of $P^n_{X/B}\times_X P^{n'}_{X/B}\to X^3_{/B}$ with the jth projection $X^3_{/B}\to X$. Then we have PD morphisms

$$P^{\min\{n,n'\}}_{X/B}(2)\to P^n_{X/B}\times_X P^{n'}_{X/B}\to P^{n+n'+1}_{X/B}(2)$$

compatible with $q_{ij}^\bullet$, $q_j^\bullet$ and $p_{ij}^{n,n'}$, $r_j^{n,n'}$ in the obvious sense. Hence, the condition (iii) in the theorem is equivalent to

$$p_{12}^{n,n'*}(\varepsilon_n)\circ p_{23}^{n,n'*}(\varepsilon_{n'})=p_{13}^{n,n'*}(\varepsilon_{n+n'})$$

for every $n,n'\in\mathbb{N}$. Noting that $\delta^{n,n'}$ is the homomorphism induced by $p_{13}^{n,n'}$, we see that the above equality is equivalent to

$$(\theta_n\otimes\mathrm{id}_{\mathcal{P}^{n'}_{X/B}})\circ\theta_{n'}=(\mathrm{id}_{\mathcal{E}}\otimes\delta^{n,n'})\circ\theta_{n+n'}.$$

It remains to prove that ε_n is an isomorphism. Let τ be the PD morphism $P^n_{X/B}\to P^n_{X/B}(2)$ induced by the morphism $(p_1,p_2,p_1)\colon X^2_{/B}\to X^3_{/B}$ and let ς be the PD isomorphism $P^n_{X/B}\to P^n_{X/B}$ induced by the isomorphism $(p_2,p_1)\colon X^2_{/B}\to X^2_{/B}$ exchanging the two components. Then by pulling back the commutative diagram of the condition (iii) in the theorem by τ and $\tau\circ\varsigma$, we obtain $\varepsilon_n\circ\varsigma^*(\varepsilon_n)=\mathrm{id}_{p_1^{n*}(\mathcal{E})}$ and $\varsigma^*(\varepsilon_n)\circ\varepsilon_n=\mathrm{id}_{p_2^{n*}(\mathcal{E})}$. □

Let $\alpha\colon\mathcal{A}\to\mathcal{B}$ be a homomorphism of sheaves of commutative rings on a topological space. We define functors α^* and $\alpha^\natural$ from the category of $\mathcal{A}$-modules to that of $\mathcal{B}$-modules by $\alpha^*(\mathcal{M})=\mathcal{B}\otimes_{\mathcal{A}}\mathcal{M}$ and $\alpha^\natural(\mathcal{M})=\mathcal{H}om_{\mathcal{A}}(\mathcal{B},\mathcal{M})$. Let α_* denote the functor from the category of $\mathcal{B}$-modules to that of $\mathcal{A}$-modules obtained by regarding $\mathcal{B}$-modules as $\mathcal{A}$-modules via α. The functor α^* is a left adjoint of α_*. The functor $\alpha^\natural$ is a right adjoint of α_*; for an $\mathcal{A}$-module $\mathcal{M}$

and a $\mathcal{B}$-module $\mathcal{N}$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{N}, \mathcal{M}) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{M})).$$

An element φ of the LHS and an element ψ of the RHS correspond to each other by the following formulae: $\{\psi(n)\}(b) = \varphi(bn)$ and $\varphi(n) = \{\psi(n)\}(1)$. By the above property, we have a canonical isomorphism $\beta^\natural \circ \alpha^\natural \cong (\beta \circ \alpha)^\natural$ for another homomorphism of sheaves of commutative rings $\beta\colon \mathcal{B} \to \mathcal{C}$; for an $\mathcal{A}$-module $\mathcal{M}$, we have a canonical isomorphism

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{M})) \cong \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{M}).$$

A local section φ of the LHS and a local section ψ of the RHS correspond to each other by the following formulae: $\psi(c) = \{\varphi(c)\}(1)$ and $\{\varphi(c)\}(b) = \psi(cb)$. For an $\mathcal{A}$-module $\mathcal{M}$ and a $\mathcal{B}$-module $\mathcal{N}$, the adjunction maps $\alpha_*\alpha^\natural\mathcal{M} \to \mathcal{M}$ and $\mathcal{N} \to \alpha^\natural\alpha_*\mathcal{N}$ are explicitly given by

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{M}) \to \mathcal{M};\ \varphi \mapsto \varphi(1) \quad \text{and} \quad \mathcal{N} \to \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{N});\ n \mapsto \{b \mapsto b \cdot n\}.$$

For a morphism of schemes $f\colon U' \to U$ such that the underlying morphism of topological spaces is a homeomorphism, we define a functor $f^\natural$ from the category of $\mathcal{O}_U$-modules to that of $\mathcal{O}_{U'}$-modules by

$$f^\natural(\mathcal{M}) = \mathcal{H}om_{f^{-1}(\mathcal{O}_U)}(\mathcal{O}_{U'}, f^{-1}(\mathcal{M})).$$

Then $f^\natural$ is a right adjoint of the direct image functor f_*. We have a canonical isomorphism $g^\natural \circ f^\natural \cong (f \circ g)^\natural$ for another morphism of schemes $g\colon U'' \to U'$ satisfying the same condition as f.

For $n \in \mathbb{N}$, let

$$p_i^n : P^n_{X/B} \to X \quad (i = 1, 2), \qquad q_j : P^n_{X/B}(1) \to X \quad (j = 1, 2, 3),$$

and $\iota^n\colon P^n_{X/B} \to P^{n+1}_{X/B}$ be the same as before Theorem 2.2.1.

Theorem 2.2.5 (cf. [Ber00, Proposition 1.1.4]). *The category of right $\mathcal{D}_{X/B}$-modules is canonically equivalent to the following category.*

Object: an $\mathcal{O}_X$-module $\mathcal{M}$ endowed with a family of isomorphisms $\{\varepsilon_n\colon p_1^{n\natural}\mathcal{M} \xrightarrow{\cong} p_2^{n\natural}\mathcal{M}\}_{n\in\mathbb{N}}$ of $\mathcal{P}^n_{X/B}$-modules satisfying the following properties.

(i) $\varepsilon_0 = \mathrm{id}_{\mathcal{M}}$.

(ii) *For $n \in \mathbb{N}$, $\varepsilon_n = \iota^{n\natural}(\varepsilon_{n+1})$.*

(iii) *For $n \in \mathbb{N}$, the following diagram is commutative.*

$$\begin{array}{ccc} q_1^{n\natural}\mathcal{M} & \xrightarrow{q_{12}^{n\natural}(\varepsilon_n)} & q_2^{n\natural}\mathcal{M} \\ & \searrow^{q_{13}^{n\natural}(\varepsilon_n)} & \downarrow{\scriptstyle q_{23}^{n\natural}(\varepsilon_n)} \\ & & q_3^{n\natural}\mathcal{M} \end{array}$$

Morphism: a morphism $(\mathcal{M}, \{\varepsilon_n\}) \to (\mathcal{M}', \{\varepsilon'_n\})$ is an $\mathcal{O}_X$-linear homomorphism $\alpha\colon \mathcal{M} \to \mathcal{M}'$ such that $\varepsilon'_n \circ p_1^{n\natural}(\alpha) = p_2^{n\natural}(\alpha) \circ \varepsilon_n$.

Definition 2.2.6 (cf. [Ber00, Définition 1.1.3]). Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. *A PD costratification on $\mathcal{M}$ relative to B* is a family of isomorphisms $\{\varepsilon_n\colon p_1^{n\natural}\mathcal{M} \xrightarrow{\cong} p_2^{n\natural}\mathcal{M}\}_{n\in\mathbb{N}}$ of $\mathcal{P}^n_{X/B}$-modules satisfying the conditions (i), (ii), and (iii) in Theorem 2.2.5.

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For $n,\, n'\in\mathbb{N}$, let $p_{ij}^{n,n'}$ $((i,j)=(1,2),\,(2,3),\,(1,3))$, and $r_j^{n,n'}$ $(j=1,2,3)$ be the same as in the proof of Theorem 2.2.1. We have

$$r_1^{n,n'}=p_1^n\circ p_{12}^{n,n'}=p_1^{n+n'}\circ p_{13}^{n,n'},\quad r_2^{n,n'}=p_2^n\circ p_{12}^{n,n'}=p_1^{n'}\circ p_{23}^{n,n'},$$

and

$$r_3^{n,n'}=p_2^{n+n'}\circ p_{13}^{n,n'}=p_2^{n'}\circ p_{23}^{n,n'}.$$

PROPOSITION 2.2.7 (cf. [Ber00, Proposition 1.1.4]). *The category of right $\mathcal{D}_{X/B}$-modules is canonically equivalent to the following category.*

Object: an $\mathcal{O}_X$-module $\mathcal{M}$ endowed with a family of homomorphisms $\{\mu_n\colon p_{2*}^n p_1^{n\natural}\mathcal{M}\to\mathcal{M}\}_{n\in\mathbb{N}}$ of $\mathcal{O}_X$-modules satisfying the following properties.

(i) $\mu_0=\mathrm{id}_{\mathcal{M}}$.

(ii) *The composite of μ_{n+1} with*

$$p_{2*}^n p_1^{n\natural}\mathcal{M}\cong p_{2*}^{n+1}\iota_*^n\iota^{n\natural}p_1^{n+1\natural}\mathcal{M}\xrightarrow{p_{2*}^{n+1}(\mathrm{adj})}p_{2*}^{n+1}p_1^{n+1\natural}\mathcal{M}$$

coincides with μ_n for every $n\in\mathbb{N}$.

(iii) *For $n,n'\in\mathbb{N}$, the following diagram is commutative.*

$$\begin{array}{ccc} r_{3*}^{n,n'}r_1^{n,n'\natural}\mathcal{M} & \longrightarrow & p_{2*}^{n'}p_1^{n'\natural}\mathcal{M}\\ \downarrow & & \downarrow{\scriptstyle \mu_{n'}}\\ p_{2*}^{n+n'}p_1^{n+n'\natural}\mathcal{M} & \xrightarrow{\mu_{n+n'}} & \mathcal{M}\end{array}$$

Here the upper horizontal homomorphism is induced by μ_n as follows. The composite of μ_n with the homomorphism

$$p_{1*}^{n'}p_{23*}^{n,n'}p_{12}^{n,n'\natural}p_1^{n\natural}\mathcal{M}=p_{2*}^n p_{12*}^{n,n'}p_{12}^{n,n'\natural}p_1^{n\natural}\mathcal{M}\xrightarrow{p_{2*}^n(\mathrm{adj})}p_{2*}^n p_1^{n\natural}\mathcal{M}$$

induces

$$p_{23*}^{n,n'}p_{12}^{n,n'\natural}p_1^{n\natural}\mathcal{M}\longrightarrow p_1^{n'\natural}\mathcal{M}.$$

Taking $p_{2*}^{n'}$, we obtain the desired homomorphism. The left vertical homomorphism is induced by the adjunction $p_{13*}^{n,n'}p_{13}^{n,n'\natural}\to\mathrm{id}$.

Proof. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. The homomorphism

$$p_{2*}p_1^{n\natural}\mathcal{M}\to p_{2*}^{n+1}p_1^{n+1\natural}\mathcal{M}$$

is the homomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(p_{1*}^n\mathcal{P}_{X/B}^n,\mathcal{M})\to\mathcal{H}om_{\mathcal{O}_X}(p_{1*}^{n+1}\mathcal{P}_{X/B}^{n+1}\mathcal{M})$$

defined by the composition with $\iota^{n*}\colon\mathcal{P}_{X/B}^{n+1}\to\mathcal{P}_{X/B}^n$. Hence, giving a system $\{\mu_n\}$ satisfying (i) and (ii) is equivalent to giving a system of homomorphisms $\{\kappa_n\colon\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n}\to\mathcal{M}\}_{n\in\mathbb{N}}$ linear for the right $\mathcal{O}_X$-module structures of the sources such that $\kappa_0=\mathrm{id}$ and compatible with respect to the injection

$$\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n}\hookrightarrow\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n+1}\quad\text{for } n\in\mathbb{N}.$$

Hence, it suffices to prove that the condition (iii) is equivalent to the commutativity of the following diagram.

$$\begin{array}{ccc} \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n'} & \xrightarrow{\kappa_n\otimes\mathrm{id}_{\mathcal{D}_{X/B,n'}}} & \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n'} \\ \downarrow{\scriptstyle \mathrm{id}_{\mathcal{M}}\otimes\mathrm{prod}} & & \downarrow{\scriptstyle \kappa_{n'}} \\ \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n+n'} & \xrightarrow{\kappa_{n+n'}} & \mathcal{M} \end{array}$$

The upper horizontal homomorphism of the diagram in (iii) is the homomorphism

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(r_{1*}(\mathcal{P}^n_{X/B}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X/B}),\mathcal{M})\longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(p^{n'}_{1*}\mathcal{P}^{n'}_{X/B},\mathcal{M})$$

sending φ to ψ defined by

$$\psi(b)=\mu_n(\{p^n_{1*}\mathcal{P}^n_{X/B}\to\mathcal{M};a\mapsto\varphi(a\otimes b)\}).$$

The left vertical homomorphism is

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(r_{1*}(\mathcal{P}^n_{X/B}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X/B}),\mathcal{M})\longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(p^{n+n'}_{1*}\mathcal{P}^{n+n'}_{X/B},\mathcal{M})$$

defined by the composition with $\delta^{n,n'}$. Hence, via the isomorphism

$$\begin{aligned}\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B,n'} &\xrightarrow[\text{Lemma 2.2.4(2)}]{\cong}\mathcal{M}\otimes_{\mathcal{O}_X}(\mathcal{P}^n_{X/B}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X/B})^\vee\\ &\xrightarrow{\cong}\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(r_{1*}(\mathcal{P}^n_{X/B}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X/B}),\mathcal{M})\end{aligned}$$

sending $m\otimes P\otimes Q$ to $(P\circ\mathrm{id}_{\mathcal{P}^n_{X/B}}\otimes Q)\cdot m$, the two diagrams in question coincide. □

Proof of Theorem 2.2.5. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Let $\{\mu_n\colon p^n_{2*}p^{n\natural}_1(\mathcal{M})\to\mathcal{M}\}_{n\in\mathbb{N}}$ be a family of homomorphisms of $\mathcal{O}_X$-modules, and let $\{\varepsilon_n\colon p^{n\natural}_1\mathcal{M}\to p^{n\natural}_2\mathcal{M}\}_{n\in\mathbb{N}}$ be the family of homomorphisms of $\mathcal{P}^n_{X/B}$-modules associated to μ_n. By Proposition 2.2.7, it suffices to prove that the conditions (i), (ii), and (iii) in the theorem are equivalent to those in Proposition 2.2.7 and that ε_n is an isomorphism. The equivalence for (i) is trivial. By taking the adjoints of the condition (ii) in Proposition 2.2.7 with respect to p^{n+1}_2 and then to ι^n, we obtain the condition (ii) in the theorem. Now we assume that the conditions (i) and (ii) are satisfied. To simplify the notation, we abbreviate $p^{n,n'}_{i,j}$ and $r^{n,n'}_j$ to p_{ij} and r_j in the following. By the same argument as in the proof of Theorem 2.2.1, we see that the condition (iii) in the theorem is equivalent to saying that we have $p^\natural_{23}(\varepsilon_{n'})\circ p^\natural_{12}(\varepsilon_n)=p^\natural_{13}(\varepsilon_{n+n'})$ on $P^n_{X/B}\times_X P^{n'}_{X/B'}$ for every $n,n'\in\mathbb{N}$. On the other hand, the condition (iii) in Proposition 2.2.7 is equivalent to saying that the following diagram obtained by taking $r^\natural_3$ of the diagram in Proposition 2.2.7(iii) and composing with $r^\natural_1\mathcal{M}\to r^\natural_3r_{3*}r^\natural_1\mathcal{M}$ is commutative.

$$\begin{array}{ccc} r^\natural_1\mathcal{M} & \longrightarrow & r^\natural_3p^{n'}_{2*}p^{n'\natural}_1\mathcal{M} \\ \downarrow & & \downarrow \\ r^\natural_3p^{n+n'}_{2*}p^{n+n'\natural}_1\mathcal{M} & \longrightarrow & r^\natural_3\mathcal{M} \end{array}$$

The composite of the right vertical homomorphism with

$$\varphi\colon p^\natural_{23}p^{n'\natural}_1\mathcal{M}\to p^\natural_{23}p^{n'\natural}_2p^{n'}_{2*}p^{n'\natural}_1\mathcal{M}=r^\natural_3p^{n'}_{2*}p^{n'\natural}_1\mathcal{M}$$

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is $p_{23}^{\natural}(\varepsilon_{n'})$. The upper horizontal homomorphism of the diagram in Proposition 2.2.7(iii) is defined to be $p_{2*}^{n'}(\psi)$ for a morphism $\psi\colon p_{23*}r_1^{\natural}\mathcal{M}\to p_1^{n'\natural}\mathcal{M}$. This implies that the upper horizontal homomorphism of the above diagram is the composite of

$$r_1^{\natural}\mathcal{M}\xrightarrow{\text{adj}} p_{23}^{\natural}p_{23*}r_1^{\natural}\mathcal{M}\xrightarrow{p_{23}^{\natural}(\psi)} p_{23}^{\natural}p_1^{n'\natural}\mathcal{M}=r_2^{\natural}\mathcal{M}$$

with the homomorphism φ. The former homomorphism is obtained by taking the adjoint of

$$p_{2*}^{n}p_{12*}p_{12}^{\natural}p_1^{n\natural}\mathcal{M}\xrightarrow{p_{2*}^{n}(\text{adj})} p_{2*}^{n}p_1^{n\natural}\mathcal{M}\xrightarrow{\mu_n}\mathcal{M}$$

with respect to $r_2=p_2^n\circ p_{12}=p_1^{n'}\circ p_{23}$, and it coincides with $p_{12}^{\natural}(\varepsilon_n)$ by Lemma 2.2.8 below. Similarly, the composite of the left vertical homomorphism with the bottom horizontal one is the adjoint of

$$p_{2*}^{n+n'}p_{13*}p_{13}^{\natural}p_1^{n+n'\natural}\mathcal{M}\xrightarrow{p_{2*}^{n+n'}(\text{adj})} p_{2*}^{n+n'}p_1^{n+n'\natural}\mathcal{M}\xrightarrow{\mu_{n+n'}}\mathcal{M}$$

with respect to $r_3=p_2^{n+n'}\circ p_{13}$, and it coincides with $p_{13}^{\natural}(\varepsilon_{n+n'})$ by Lemma 2.2.8 below. Thus, we obtain the desired equivalence for (iii). By the same argument as in the proof of Theorem 2.2.1, we see that the conditions (i), (ii), and (iii) of the theorem imply that ε_n is an isomorphism. □

Lemma 2.2.8. *Let $\mathcal{A}\overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}}\mathcal{B}\xrightarrow{\beta}\mathcal{C}$ be morphisms of sheaves of commutative rings on a topological space. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{A}$-modules and let $\varphi\colon \alpha_{2*}\alpha_1^{\natural}\mathcal{F}\to\mathcal{G}$ be an $\mathcal{A}$-linear homomorphism, which induces a $\mathcal{B}$-linear homomorphism $\psi\colon \alpha_1^{\natural}\mathcal{F}\to\alpha_2^{\natural}\mathcal{G}$. Then the homomorphism $\beta^{\natural}(\psi)\colon \beta^{\natural}\alpha_1^{\natural}\mathcal{F}\to\beta^{\natural}\alpha_2^{\natural}\mathcal{G}$ coincides with that induced by the composite of φ with*

$$\alpha_{2*}(\text{adj}):\alpha_{2*}\beta_*\beta^{\natural}\alpha_1^{\natural}\mathcal{F}\to\alpha_{2*}\alpha_1^{\natural}\mathcal{G}.$$

Proof. Exercise. □

The equivalence of categories in Theorems 2.2.1 and 2.2.5 for the two bases S and T is compatible with the change of base $T\to S$ as follows.

Lemma 2.2.9. *Let r^n denote the natural morphism $P_{X/T}^n\to P_{X/S}^n$.*

(1) *Let $\mathcal{E}$ be a left $\mathcal{D}_{X/S}$-module and let $\mathcal{E}'$ be the $\mathcal{O}_X$-module $\mathcal{E}$ regarded as a left $\mathcal{D}_{X/T}$-module via the ring homomorphism $\mathcal{D}_{X/T}\to\mathcal{D}_{X/S}$ (2.1.8). Then the PD stratification of $\mathcal{E}'$ relative to T is obtained by taking r^{n*} of the PD stratification of $\mathcal{E}$ relative to S.*

(2) *Let $\mathcal{M}$ be a right $\mathcal{D}_{X/S}$-module and let $\mathcal{M}'$ be the $\mathcal{O}_X$-module $\mathcal{M}$ regarded as a right $\mathcal{D}_{X/T}$-module via the ring homomorphism $\mathcal{D}_{X/T}\to\mathcal{D}_{X/S}$ (2.1.8). Then the PD costratification of $\mathcal{M}'$ relative to T is obtained by taking $r^{n\natural}$ of the PD costratification of $\mathcal{M}$ relative to S.*

Proof. Straightforward. □

2.3 The right $\mathcal{D}_{X/B}$-module $\Omega_{X/T}^{d}$

For a noetherian scheme Y, let $D(Y)$ denote the derived category of the category of $\mathcal{O}_Y$-modules and let $D_c^+(Y)$ denote the full subcategory of $D(Y)$ consisting of complexes bounded below with coherent cohomology. Then, for a morphism of noetherian schemes $g\colon Y\to Z$ which is of finite type, we have a functor $g^!\colon D_c^+(Z)\to D_c^+(Y)$ with the trace map $\mathrm{Tr}_g\colon \mathbb{R}g_*g^!\to 1$ [Har66]. If g is finite, then the functor $g^!$ is canonically isomorphic to the functor $\overline{g}^*\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{O}_Z}(g_*\mathcal{O}_Y,-)$, where $\overline{g}$ denotes the morphism of ringed spaces $(Y,\mathcal{O}_Y)\to(Z,g_*\mathcal{O}_Y)$. Especially, if the underlying morphism of topological spaces of g is a homeomorphism and g is flat, then, for a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_Z$-modules $\mathcal{M}$, we have a canonical isomorphism $g^{\natural}\mathcal{M} \cong g^!\mathcal{M}$ in $D_c^+(Y)$. If g is smooth and $\Omega^1_{Y/Z}$ has constant rank d, then the functor $g^!$ is canonically isomorphic to the functor $\mathbb{L}g^*(-) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \Omega^d_{Y/Z}[d]$.

Assume that M_X is saturated, the morphism $f\colon X \to T$ is universally saturated (see [Tsu99, Definition 2.17] for example), $\Omega^1_{X/T}$ has constant rank d, and $\mathring{f}$ is of finite type. We define the ideal I_f of M_X as in [Tsu99, § 2]. Then we have a canonical isomorphism $f^!(\mathcal{O}_T) \cong I_f\Omega^d_{X/T}[d]$ by [Tsu99, Theorem 2.21(ii)]. The underlying morphisms of schemes of the composite of f with the two projections $p_i^n\colon P^n_{X/B} \to X$ $(i = 1, 2)$ coincide. Hence, we have a canonical isomorphism $p_1^{n\natural}(I_f\Omega^d_{X/T}) \cong p_2^{n\natural}(I_f\Omega^d_{X/T})$ satisfying the conditions (i), (ii), and (iii) in Theorem 2.2.5. Hence, $I_f\Omega^d_{X/T}$ is canonically regarded as a right $\mathcal{D}_{X/B}$-module. By Lemma 2.2.9(2), we see that the action of $\mathcal{D}_{X/T}$ on $I_f\Omega^d_{X/T}$ coincides with that induced by the action of $\mathcal{D}_{X/S}$ on $I_f\Omega^d_{X/S}$ through the homomorphism (2.1.8).

First assume that the underlying scheme $\mathring{X}$ of X is smooth over S and $X = \mathring{X} \times_S T$. Then the morphism $X \times_S X \to \mathring{X} \times_S \mathring{X}$ induces PD morphisms $P^n_{X/S} \to P^n_{\mathring{X}/S}$. Here we define $P^n_{\mathring{X}/S}$ in the same way as $P^n_{X/S}$ using the PD envelope of the diagonal immersion $\mathring{X} \hookrightarrow \mathring{X} \times_S \mathring{X}$. By taking the dual of the homomorphisms between the structure sheaves, we obtain a homomorphism of sheaves of $\mathcal{O}_X$-modules:

$$\mathcal{D}_{X/S} \to \mathcal{D}_{\mathring{X}/S}. \tag{2.3.1}$$

By a similar argument as in the proof of Proposition 2.1.9, we obtain the following proposition.

Proposition 2.3.2. *The homomorphism (2.3.1) is a ring homomorphism.*

The composite of (2.3.1) with (2.1.8) coincides with the isomorphism (2.1.10).

Lemma 2.3.3. *Let $\partial_t^{\log}$ be as before Proposition 2.1.11. Then the image of $\partial_t^{\log}$ under the homomorphism (2.3.1) is* 0.

Proof. This immediately follows from the definition of $\partial_t^{\log}$ and the isomorphisms $\mathcal{P}^1_{X/S} \cong \mathcal{O}_X \oplus \Omega^1_{X/S}$ and $\mathcal{P}^1_{\mathring{X}/S} \cong \mathcal{O}_X \oplus \Omega^1_{\mathring{X}/S}$, where the LHS's are regarded as $\mathcal{O}_X$-modules via p_1^*. □

Since $X = \mathring{X} \times_S T$, we have a canonical isomorphism $\Omega^d_{\mathring{X}/S} \xrightarrow{\cong} \Omega^d_{X/T}$ and $I_f = M_X$.

Proposition 2.3.4. *If we identify $\Omega^d_{\mathring{X}/S}$ with $\Omega^d_{X/T}$ by the above isomorphism, then the natural action of $\mathcal{D}_{X/S}$ on $\Omega^d_{X/T}$ coincides with the action of $\mathcal{D}_{X/S}$ on $\Omega^d_{\mathring{X}/S}$ through the homomorphism (2.3.1).*

Proof. We have a commutative diagram of fine log schemes.

$$\begin{array}{ccc} P^n_{X/S} & \xrightarrow{\pi^n} & P^n_{\mathring{X}/S} \\ {\scriptstyle p_1^n}\downarrow\downarrow{\scriptstyle p_2^n} & & {\scriptstyle p_1^n}\downarrow\downarrow{\scriptstyle p_2^n} \\ X & \longrightarrow & \mathring{X} \end{array}$$

Hence, the isomorphism $p_1^{n\flat}\Omega^d_{X/T} \cong p_2^{n\flat}\Omega^d_{X/T}$ on $P^n_{X/S}$ is obtained by applying the functor $\pi_n^{\flat}$ to

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the isomorphism $p_1^{n\flat}\Omega^d_{\mathring{X}/S} \cong p_2^{n\flat}\Omega^d_{\mathring{X}/S}$ on $P^n_{\mathring{X}/S}$. Hence, by Lemma 2.2.8,

$$\Omega^d_{X/T} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \cong p^n_{2*}p_1^{n\flat}\Omega^d_{X/T} \to \Omega^d_{X/T}$$

coincides with the composite

$$p^n_{2*}\pi^n_*\pi^{n\natural}p_1^{n\natural}\Omega^d_{\mathring{X}/S} \xrightarrow{p^n_{2*}(\mathrm{adj})} p^n_{2*}p^{n\natural}_{1*}\Omega^d_{\mathring{X}/S} \to \Omega^d_{\mathring{X}/S}.$$

The first homomorphism coincides with the homomorphism

$$\Omega^d_{\mathring{X}/S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} \to \Omega^d_{\mathring{X}/S} \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathring{X}/S}$$

induced by (2.3.1). □

We again consider a smooth and universally saturated morphism $f\colon X \to T$ of fine and saturated log schemes such that $\Omega^1_{X/T}$ has constant rank d and $\mathring{f}$ is of finite type. Assume that $I_f = M_X$. We give an explicit local description of the action of $\mathcal{D}_{X/B}$. Since the action of $\mathcal{D}_{X/T}$ coincides with the action of $\mathcal{D}_{X/T}$ via the homomorphism (2.1.8), it suffices to consider the case $B = S$. Assume that we are given $t_1, \dots, t_d \in \Gamma(X, M_X)$ as in the beginning of Proposition 2.1.1. We define $\partial_\nu \in \mathcal{D}_{X/S}$ $(0 \leqslant \nu \leqslant d)$ as in Proposition 2.1.5.

PROPOSITION 2.3.5. *Let the notation and assumptions be as above. Then, for $0 \leqslant \nu \leqslant d$ and $x \in \mathcal{O}_X$, we have*

$$(x\, d\log t_1 \wedge d\log t_2 \wedge \cdots \wedge d\log t_d)\partial_\nu = -\partial_\nu(x) d\log t_1 \wedge d\log t_2 \wedge \cdots \wedge d\log t_d.$$

Especially, the action of ∂_0 on $\Omega^d_{X/T}$ is 0.

Proof. Put $\omega = d\log t_1 \wedge \cdots \wedge d\log t_d$. By Proposition 2.1.5(3), it suffices to prove that $\omega \cdot \partial_\nu = 0$. Let X' be the open subscheme $\{x \in \mathring{X} | M_{T,\bar{s}}/\mathcal{O}^\times_{T,\bar{s}} \xrightarrow{\cong} M_{X,\bar{x}}/\mathcal{O}^\times_{X,\bar{x}}\}$ of $\mathring{X}$ endowed with the inverse image of M_X. Then $\mathring{X}'$ is dense in $\mathring{X}$ [Tsu99, Lemma 2.18 and its proof] and the morphism $\mathring{X}' \to S$ is smooth [Kat89, Proposition 3.8]. Let j denote the morphism $X' \to X$, and let X_0, X'_0, and j_0 denote the reductions mod p of X, X', and j, respectively. Since $k[P]$ is Cohen–Macaulay for a finitely generated, saturated and integral monoid P [Hoc72], $\mathring{X}_0$ is Cohen–Macaulay by [Kat89, Theorem 3.5]. Hence, $\mathring{X}_0$ is reduced and the homomorphism $\Omega^d_{X_0/T_0} \to j_{0*}\Omega^d_{X'_0/T_0}$ is injective. Since $\mathring{X} \to S$ is flat [Kat89, Corollaries 4.4, 4.5], this implies that $\Omega^d_{X/T} \to j_*\Omega^d_{X'/T}$ is also injective. Hence, we may replace X with X' and assume that $M_X = f^*(M_T)$. We may also assume that $\mathring{X}$ is connected. Then we have $\Gamma(X, M_X) = t^{\mathbb{N}} \times \Gamma(X, \mathcal{O}^\times_X)$ and t_ν $(1 \leqslant \nu \leqslant d)$ is written in the form $t^{n_\nu}u_\nu$ for some $n_\nu \in \mathbb{N}$ and $u_\nu \in \Gamma(X, \mathcal{O}^\times_X)$. We have $d\log t_\nu = d\log u_\nu$ in $\Omega^1_{X/T}$. If we define $D_\nu \in \mathcal{D}_{\mathring{X}/S}$ $(1 \leqslant \nu \leqslant d)$ to be the differential operators corresponding to the dual basis of $du_1, \dots, du_d \in \Omega^1_{\mathring{X}/S}$, then the images of ∂_0 and ∂_ν $(1 \leqslant \nu \leqslant d)$ in $\mathcal{D}_{\mathring{X}/S}$ are 0 and $u_\nu D_\nu$. By Proposition 2.3.4 and [Ber00, Théorème 1.2.3], we obtain $\omega \cdot \partial_0 = 0$ and

$$\omega \cdot \partial_\nu = (d\log u_1 \wedge \cdots \wedge d\log u_d) \cdot u_\nu D_\nu = -D_\nu\bigg(\prod_{1 \leqslant \mu \leqslant d, \mu \neq \nu} u_\nu^{-1}\bigg)(du_1 \wedge \cdots \wedge du_d) = 0$$

for $1 \leqslant \nu \leqslant d$. □

2.4 Tensor products of $\mathcal{D}_{X/B}$-modules

PROPOSITION 2.4.1 (cf. [Ber96, Corollaire 2.3.3], [Ber00, Proposition 1.1.7] and [Mon02, Corollaire 2.6.1(i)]). *Let $\mathcal{E}$ and $\mathcal{F}$ be left $\mathcal{D}_{X/B}$-modules and let $\mathcal{M}$ and $\mathcal{N}$ be right $\mathcal{D}_{X/B}$-modules.*

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Then $\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ have natural left $\mathcal{D}_{X/B}$-module structures and $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{E}$ has a natural right $\mathcal{D}_{X/B}$-module structure.

Let $\alpha\colon\mathcal{A}\to\mathcal{B}$ be a homomorphism of sheaves of commutative rings on a topological space, and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{A}$-modules. The composite of

$$\alpha_*(\alpha^\natural\mathcal{F}\otimes_{\mathcal{B}}\alpha^*(\mathcal{G}))\xleftarrow{\cong}\alpha_*\alpha^\natural\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G}\xrightarrow{\text{adj}\otimes\text{id}}\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G}$$

induces a homomorphism

$$\alpha^\natural(\mathcal{F})\otimes_{\mathcal{B}}\alpha^*(\mathcal{G})\longrightarrow\alpha^\natural(\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G}),\tag{2.4.2}$$

and the composite of

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})\to\mathcal{H}om_{\mathcal{A}}(\alpha_*\alpha^\natural\mathcal{F},\mathcal{G})\xleftarrow{\cong}\alpha_*\mathcal{H}om_{\mathcal{B}}(\alpha^\natural\mathcal{F},\alpha^\natural\mathcal{G})$$

induces a homomorphism

$$\alpha^*(\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G}))\longrightarrow\mathcal{H}om_{\mathcal{B}}(\alpha^\natural\mathcal{F},\alpha^\natural\mathcal{G}).\tag{2.4.3}$$

Lemma 2.4.4. *Let α, $\mathcal{F}$, and $\mathcal{G}$ be as above.*

(1) *If $\mathcal{B}$ is locally free of finite type as a sheaf of $\mathcal{A}$-modules, then the homomorphisms (2.4.2) and (2.4.3) are isomorphisms.*

(2) *Let $\beta\colon\mathcal{B}\to\mathcal{C}$ be another homomorphism of sheaves of commutative rings and put $\gamma=\beta\circ\alpha$. Then the homomorphism (2.4.2) for γ: $\gamma^\natural(\mathcal{F})\otimes_{\mathcal{C}}\gamma^*(\mathcal{G})\to\gamma^\natural(\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G})$ coincides with the following composite of the homomorphisms (2.4.2) for α and*

$$\beta:\beta^\natural\alpha^\natural(\mathcal{F})\otimes_{\mathcal{C}}\beta^*\alpha^*(\mathcal{G})\longrightarrow\beta^\natural(\alpha^\natural(\mathcal{F})\otimes_{\mathcal{B}}\alpha^*(\mathcal{G}))\longrightarrow\beta^\natural\alpha^\natural(\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G}).$$

The same holds for (2.4.3).

Proof. (1) The homomorphism (2.4.2) is given by

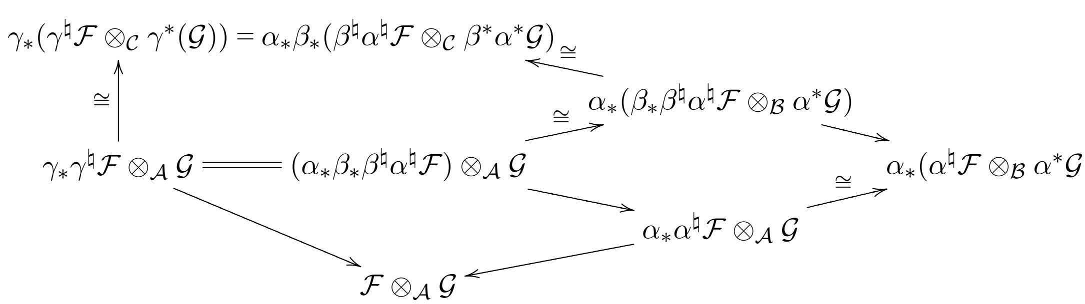
$$\mathcal{H}om_{\mathcal{A}}(\mathcal{B},\mathcal{F})\otimes_{\mathcal{B}}\alpha^*(\mathcal{G})\longrightarrow\mathcal{H}om_{\mathcal{A}}(\mathcal{B},\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G});\quad\varphi\otimes\alpha^*(x)\mapsto\psi,\psi(b)=\varphi(b)\otimes x.$$

The RHS of (2.4.3) is canonically isomorphic to $\mathcal{H}om_{\mathcal{A}}(\mathcal{H}om_{\mathcal{A}}(\mathcal{B},\mathcal{F}),\mathcal{G})$, and the homomorphism (2.4.3) is given by

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})\otimes_{\mathcal{A}}\mathcal{B}\longrightarrow\mathcal{H}om_{\mathcal{A}}(\mathcal{H}om_{\mathcal{A}}(\mathcal{B},\mathcal{F}),\mathcal{G});\quad\varphi\otimes b\mapsto\psi,\psi(\kappa)=\varphi(\kappa(b)).$$

By taking a basis of $\mathcal{B}$ over $\mathcal{A}$ locally, we see that these are isomorphisms.

(2) For (2.4.2), by using $\mathrm{Hom}_{\mathcal{C}}(-,\beta^\natural(\sim))=\mathrm{Hom}_{\mathcal{B}}(\beta_*(-),\sim)$ and then $\mathrm{Hom}_{\mathcal{B}}(-,\alpha^\natural(\sim))=\mathrm{Hom}_{\mathcal{A}}(\alpha_*(-),\sim)$, we are reduced to the commutativity of the following diagram.



Similarly, for (2.4.3), by using

$$\mathrm{Hom}_{\mathcal{C}}(\beta^*(-),\sim)=\mathrm{Hom}_{\mathcal{B}}(-,\beta_*(\sim))$$

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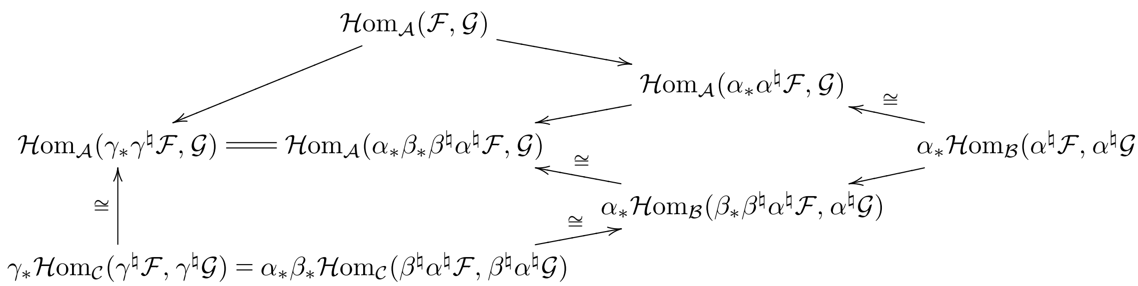
and then

$$\mathrm{Hom}_{\mathcal{B}}(\alpha^*(-), \sim) = \mathrm{Hom}_{\mathcal{A}}(-, \alpha_*(\sim)),$$

we are reduced to the commutativity of the following diagram.



Proof of Proposition 2.4.1. By Theorem 2.2.1 (respectively Theorem 2.2.5), there exist isomorphisms $p_2^{n*}\mathcal{E} \xrightarrow{\cong} p_1^{n*}\mathcal{E}$ and $p_2^{n*}\mathcal{F} \xrightarrow{\cong} p_1^{n*}\mathcal{F}$ (respectively $p_1^{n\natural}\mathcal{M} \xrightarrow{\cong} p_2^{n\natural}\mathcal{M}$ and $p_1^{n\natural}\mathcal{N} \xrightarrow{\cong} p_2^{n\natural}\mathcal{N}$) satisfying the conditions (i), (ii), and (iii) in Theorem 2.2.1 (respectively Theorem 2.2.5). By Lemma 2.4.4(1), these isomorphisms induce isomorphisms

$$p_2^{n*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \cong p_1^{n*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}), \quad p_2^{n*}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \cong p_1^{n*}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$$

and

$$p_1^{n\natural}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}) \cong p_2^{n\natural}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}).$$

Using Lemma 2.4.4, we can verify that the first and the second (respectively the third) isomorphisms satisfy the conditions (i), (ii), and (iii) in Theorem 2.2.1 (respectively Theorem 2.2.5). □

PROPOSITION 2.4.5 (cf. [Ber96, Corollaire 2.3.3], [Ber00, Proposition 1.1.7] and [Mon02, Corollaire 2.6.1(i)]). *Let $\mathcal{E}$ and $\mathcal{F}$ be left $\mathcal{D}_{X/B}$-modules, and let $\mathcal{M}$ and $\mathcal{N}$ be right $\mathcal{D}_{X/B}$-modules. Let $t_1, \ldots, t_d$ be as in Proposition 2.1.1 and let $\partial_\nu \in \mathcal{D}_{X/B}$ $(\nu_0 \leqslant \nu \leqslant d)$ be as in Proposition 2.1.5. Then the actions of ∂_ν on the left $\mathcal{D}_{X/B}$-modules $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ and the right $\mathcal{D}_{X/B}$-modules $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}$ are described as follows:*

$$\partial_\nu(e \otimes f) = (\partial_\nu e) \otimes f + e \otimes (\partial_\nu f), \quad e \in \mathcal{E},\ f \in \mathcal{F},$$
$$(\partial_\nu \varphi)(m) = \varphi(m\partial_\nu) - \varphi(m)\partial_\nu, \quad \varphi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}), m \in \mathcal{M},$$
$$(m \otimes e)\partial_\nu = (m\partial_\nu) \otimes e - m \otimes (\partial_\nu e), \quad m \in \mathcal{M}, e \in \mathcal{E}.$$

Proof. For a left $\mathcal{D}_{X/B}$-module $\mathcal{G}$, the associated $\mathcal{P}^1_{X/B}$-linear isomorphism

$$p_2^{1*}\mathcal{G} = \mathcal{P}^1_{X/B} \otimes_{\mathcal{O}_X} \mathcal{G} \xrightarrow{\cong} \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/B} = p_1^{1*}\mathcal{G}$$

is given by $1 \otimes x \mapsto x \otimes 1 + \sum_\nu \partial_\nu(x) \otimes (u_\nu - 1)$. Its inverse coincides with the pull-back by the isomorphism $\mathrm{inv}\colon P^1_{X/B} \xrightarrow{\cong} P^1_{X/B}$ induced by the isomorphism $X \times_B X \xrightarrow{\cong} X \times_B X$ exchanging the two components (cf. the proof of Theorem 2.2.1). Since the pull-back of u_ν is $u_\nu^{-1} = (1 + u_\nu - 1)^{-1} = 1 - (u_\nu - 1)$, we see that the inverse is given by $x \otimes 1 \mapsto 1 \otimes x - \sum_\nu (u_\nu - 1) \otimes \partial_\nu(x)$. Put $\mathcal{D}'_{X/B,1} := \mathcal{H}om_{\mathcal{O}_X}(p_{2*}\mathcal{P}^1_{X/B}, \mathcal{O}_X)$. Let $1, \partial'_\nu \in \mathcal{D}'_{X/B,1}$ denote the dual basis of the basis $1, u_\nu^{-1} - 1 = -(u_\nu - 1)$ of $p_{2*}\mathcal{P}^1_{X/B}$. Then, for a right $\mathcal{D}_{X/B}$-module $\mathcal{L}$, the associated isomorphism

$$p_1^\natural\mathcal{L} \cong \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/B,1} \xrightarrow{\cong} \mathcal{D}'_{X/B,1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong p_2^\natural\mathcal{L}$$

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is given by $y \otimes 1 \mapsto 1 \otimes y$ and $y \otimes \partial_\nu \mapsto 1 \otimes (y\partial_\nu) - \partial'_\nu \otimes y$. Its inverse coincides with the pull-back by the isomorphism inv above, which is given by $1 \otimes y \mapsto y \otimes 1$ and $\partial'_\nu \otimes y \mapsto (y\partial_\nu) \otimes 1 - y \otimes \partial_\nu$.

Now the image of $p_2^*(m \otimes n)$ by the isomorphism

$$p_2^*(\mathcal{M} \otimes \mathcal{N}) \cong p_1^*(\mathcal{M}) \otimes_{\mathcal{P}^1_{X/B}} p_1^*(\mathcal{N})$$

is

$$\Big(1 \otimes m + \sum_\nu (u_\nu - 1) \otimes \partial_\nu m\Big) \otimes \Big(1 \otimes n + \sum_\nu (u_\nu - 1) \otimes \partial_\nu n\Big),$$

whose image in $p_1^*(\mathcal{M} \otimes \mathcal{N})$ is

$$p_1^*(m \otimes n) + \sum_\nu (u_\nu - 1) p_1^*((\partial_\nu m) \otimes n + m \otimes (\partial_\nu m)).$$

This implies the first equality. The image of $p_2^*(\varphi)$ by the isomorphism

$$p_2^*(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) \cong \mathcal{H}\mathrm{om}_{\mathcal{P}^1_{X/B}}(p_2^\natural \mathcal{M}, p_2^\natural \mathcal{N})$$

is given by

$$\kappa \mapsto \{p_{2*}\mathcal{P}^1_{X/B} \to \mathcal{N}; a \mapsto \varphi(\kappa(a))\}$$

(cf. the proof of Lemma 2.4.4). Its image under the isomorphism

$$\mathcal{H}\mathrm{om}_{\mathcal{P}^1_{X/B}}(p_2^\natural \mathcal{M}, p_2^\natural \mathcal{N}) \xrightarrow{\cong} \mathcal{H}\mathrm{om}_{\mathcal{P}^1_{X/B}}(p_1^\natural \mathcal{M}, p_1^\natural \mathcal{N})$$

is given by

$$\begin{array}{llll}
\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/B,1} & \xrightarrow{\cong} \mathcal{D}'_{X/B,1} \otimes_{\mathcal{O}_X} \mathcal{M} & \longrightarrow \mathcal{D}'_{X/B,1} \otimes_{\mathcal{O}_X} \mathcal{N} & \xleftarrow{\cong} \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/B,1} \\
m \otimes 1 & \mapsto 1 \otimes m & \mapsto 1 \otimes \varphi(m) & \mapsto \varphi(m) \otimes 1 \\
m \otimes \partial_\nu & \mapsto 1 \otimes (m\partial_\nu) & \mapsto 1 \otimes \varphi(m\partial_\nu) & \mapsto (\varphi(m\partial_\nu) - \varphi(m)\partial_\nu) \otimes 1 \\
& \quad - \partial'_\nu \otimes m & \quad - \partial'_\nu \otimes \varphi(m) & \quad + \varphi(m) \otimes \partial_\nu
\end{array}$$

This coincides with the image of

$$\varphi \otimes 1 + \sum_\nu \varphi_\nu \otimes (u_\nu - 1) \in \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \otimes_{\mathcal{O}_X} P^1_{X/B},$$

where $\varphi_\nu(m) = \varphi(m\partial_\nu) - \varphi(m)\partial_\nu$. Thus, we obtain the second equality. The image of

$$m \otimes e \otimes \partial_\nu \in \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/B,1} = p_1^\natural(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E})$$

in

$$(\mathcal{D}'_{X/B,1} \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathcal{P}^1_{X/B}} (p_{2*}\mathcal{P}^1_{X/B} \otimes_{\mathcal{O}_X} \mathcal{E}) = p_2^\natural \mathcal{M} \otimes_{\mathcal{P}^1_{X/B}} p_2^* \mathcal{E}$$

is

$$(1 \otimes m\partial_\nu - \partial'_\nu \otimes m) \otimes \Big(1 \otimes e - \sum_\mu (u_\mu - 1) \otimes \partial_\mu e\Big),$$

whose image in $\mathcal{D}'_{X/B,1} \otimes_{\mathcal{O}_X} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}$ is given by $1 \otimes (m\partial_\nu \otimes e - m \otimes \partial_\nu e) - \partial'_\nu \otimes m \otimes e$. This implies the third equality. □

Proposition 2.4.6. *Assume that there exists a right $\mathcal{D}_{X/B}$-module $\mathcal{L}$ such that $\mathcal{L}$ is invertible as an $\mathcal{O}_X$-module. Then the functor from the category of left $\mathcal{D}_{X/B}$-modules to the category of*

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right $\mathcal{D}_{X/B}$-modules defined by $\mathcal{E} \mapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E}$ is an equivalence of categories. A quasi-inverse is given by $\mathcal{M} \mapsto \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{M} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M})$.

Proof. It suffices to prove that the natural isomorphisms $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \cong \mathcal{E}$ and $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{M} \cong \mathcal{M}$ are compatible with the actions of $\mathcal{D}_{X/B}$. By a straightforward computation, one can verify that the composites of the following sequences of homomorphisms are the identity maps for $i = 1$ and 2, which implies the desired compatibility:

$$p_i^{n*}\mathcal{E} \to [p_i^{n\natural}\mathcal{L}, p_i^{n\natural}\mathcal{L} \otimes_{\mathcal{P}^n} p_i^{n*}\mathcal{E}]_{\mathcal{P}^n} \cong [p_i^{n\natural}\mathcal{L}, p_i^{n\natural}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E})]_{\mathcal{P}^n} \cong p_i^{n*}[\mathcal{L}, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E}]_{\mathcal{O}_X} \cong p_i^{n*}\mathcal{E},$$

$$p_i^{n\natural}\mathcal{M} \cong p_i^{n\natural}(\mathcal{L} \otimes_{\mathcal{O}_X} [\mathcal{L}, \mathcal{M}]_{\mathcal{O}_X}) \cong p_i^{n\natural}\mathcal{L} \otimes_{\mathcal{P}^n} p_i^{n*}[\mathcal{L}, \mathcal{M}]_{\mathcal{O}_X} \cong p_i^{n\natural}\mathcal{L} \otimes_{\mathcal{P}^n} [p_i^{n\natural}\mathcal{L}, p_i^{n\natural}\mathcal{M}]_{\mathcal{P}^n} \to p_i^{n\natural}\mathcal{M}.$$

Here $\mathcal{P}^n$ denotes $\mathcal{P}^n_{X/B}$, and $[-, \sim]_{\mathcal{A}}$ denotes $\mathcal{H}om_{\mathcal{A}}(-, \sim)$ for $\mathcal{A} = \mathcal{O}_X$, $\mathcal{P}^n$. □

If M_X is saturated, $f\colon X \to T$ is universally saturated, $\mathring{f}$ is of finite type, $I_f = M_X$, and $\Omega^1_{X/T}$ has a constant rank d, then one can apply Proposition 2.4.6 above to $\mathcal{L} = \Omega^d_{X/T}$.

LEMMA 2.4.7. *Let $\mathcal{E}$ and $\mathcal{F}$ (respectively $\mathcal{M}$ and $\mathcal{N}$) be left (respectively right) $\mathcal{D}_{X/S}$-modules. Let $\mathcal{E}'$ and $\mathcal{F}'$ (respectively $\mathcal{M}'$ and $\mathcal{N}'$) be $\mathcal{E}$ and $\mathcal{F}$ (respectively $\mathcal{M}$ and $\mathcal{N}$) regarded as left (respectively right) $\mathcal{D}_{X/T}$-modules via $\mathcal{D}_{X/T} \to \mathcal{D}_{X/S}$ (2.1.8). Then the natural isomorphisms $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \cong \mathcal{E}' \otimes_{\mathcal{O}_X} \mathcal{F}'$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}')$, and $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E} \cong \mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{E}'$ are compatible with the actions of $\mathcal{D}_{X/S}$ and $\mathcal{D}_{X/T}$ via (2.1.8).*

Proof. This follows from the proof of Proposition 2.4.1, Lemmas 2.2.9 and 2.4.4, and the functoriality of (2.4.2) and (2.4.3) with respect to $\mathcal{F}$ and $\mathcal{G}$. □

2.5 Inverse images

Let X' be another fine log scheme smooth over T and let $g\colon X' \to X$ be a morphism over T. Then g induces PD morphisms $P^n_{X'/B}(r) \to P^n_{X/B}(r)$ ($r, n \in \mathbb{N}$) compatible with the projections. By pulling back stratifications by the above PD morphisms for $r = 1$ (cf. Theorem 2.2.1), we see that, for a left $\mathcal{D}_{X/B}$-module $\mathcal{F}$, the inverse image $g^*_{/B}(\mathcal{F}) = \mathcal{O}_{X'} \otimes_{g^{-1}(\mathcal{O}_X)} g^{-1}(\mathcal{F})$ as an $\mathcal{O}_X$-module is canonically regarded as a left $\mathcal{D}_{X'/B}$-module. Thus, we obtain a functor $g^*_{/B}$ from the category $\mathcal{D}_{X/B}$-**Mod** of left $\mathcal{D}_{X/B}$-modules to the category $\mathcal{D}_{X'/B}$-**Mod** of left $\mathcal{D}_{X'/B}$-modules. For another morphism $h\colon X'' \to X'$ of smooth fine log schemes over T, we have a natural isomorphism of functors $(g \circ h)^*_{/B} \cong h^*_{/B} \circ g^*_{/B}$. By Lemma 2.2.9(1), the following diagram is commutative up to canonical isomorphisms.

$$\begin{array}{ccc} \mathcal{D}_{X/S}\text{-}\mathbf{Mod} & \xrightarrow{r_X} & \mathcal{D}_{X/T}\text{-}\mathbf{Mod} \\ {\scriptstyle g^*_{/S}}\downarrow & & \downarrow{\scriptstyle g^*_{/T}} \\ \mathcal{D}_{X'/S}\text{-}\mathbf{Mod} & \xrightarrow{r_{X'}} & \mathcal{D}_{X'/T}\text{-}\mathbf{Mod} \end{array} \tag{2.5.1}$$

Here the horizontal arrows are the functors induced by the ring homomorphisms (2.1.8) for X and X'.

The functor $g^*_{/B}$ is determined by the reduction mod p of g up to canonical isomorphisms as follows.

PROPOSITION 2.5.2 (cf. [Ber00, Proposition 2.1.5]). *Assume that $p > 2$. Let X and X' be as above. Then, for two morphisms $g, g' : X' \to X$ over T which coincide modulo p, there exists a*

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canonical isomorphism $c^*_{g,g'/B}\colon g'^*_{/B} \xrightarrow{\cong} g^*_{/B}$ *of functors from* $\mathcal{D}_{X/B}$-**Mod** *to* $\mathcal{D}_{X'/B}$-**Mod**. *This isomorphism satisfies the following properties.*

(1) *For three morphisms* $g, g', g''\colon X' \to X$ *which coincide modulo* p, *we have*

$$c^*_{g,g'/B} \circ c^*_{g',g''/B} = c^*_{g,g''/B}.$$

(2) *For two morphisms* $g, g'\colon X' \to X$ *over* T *which coincide modulo* p *and a morphism* $h\colon X \to X''$ *over* T, *the following diagram is commutative.*

$$\begin{array}{ccc} (h\circ g')^*_{/B} & \xrightarrow[c^*_{h\circ g, h\circ g'/B}]{\cong} & (h\circ g)^*_{/B} \\ \downarrow{\cong} & & {\cong}\downarrow \\ g'^*_{/B}\circ h^*_{/B} & \xrightarrow[c^*_{g,g'/B}\circ h^*_{/B}]{\cong} & g^*_{/B}\circ h^*_{/B} \end{array}$$

(3) *For two morphisms* $g, g'\colon X' \to X$ *over* T *which coincide modulo* p *and a morphism* $h\colon X'' \to X'$ *over* T, *the following diagram is commutative.*

$$\begin{array}{ccc} (g'\circ h)^*_{/B} & \xrightarrow[c^*_{g\circ h, g'\circ h/B}]{\cong} & (g\circ h)^*_{/B} \\ \downarrow{\cong} & & {\cong}\downarrow \\ h^*_{/B}\circ g'^*_{/B} & \xrightarrow[h^*_{/B}\circ c^*_{g,g'/B}]{\cong} & h^*_{/B}\circ g^*_{/B} \end{array}$$

Proof. By the assumption that$p > 2$, the canonical PD structure of the ideal $p\mathcal{O}_X$ is nilpotent. Hence, the morphism $(g, g')\colon X' \to X \times_B X$ factors through a unique PD morphism $\rho\colon X' \to P^n_{X/B}$ for a sufficiently large n. Hence, for any left $\mathcal{D}_{X/S}$-module $\mathcal{E}$, pulling back the associated isomorphism $\varepsilon_n\colon p_2^{n*}\mathcal{E} \xrightarrow{\cong} p_1^{n*}\mathcal{E}$ (Theorem 2.2.1) by ρ, we obtain an isomorphism $c^*_{g,g'/B}(\mathcal{E})\colon g'^*_{/B}(\mathcal{E}) \xrightarrow{\cong} g^*_{/B}(\mathcal{E})$ as $\mathcal{O}_{X'}$-modules, which is obviously independent of the choice of n and is functorial on $\mathcal{E}$. The property (iii) in Theorem 2.2.1 implies the property (1). The property (3) is obvious and the property (2) follows from the fact that the functor $h^*_{/B}$ is defined by pulling back PD stratifications by the PD morphism $P^n_{X/B} \to P^n_{X''/B}$ induced by h. It remains to prove that the isomorphism $c^*_{g,g'/B}(\mathcal{E})$ is $\mathcal{D}_{X'/B}$-linear. For each n, there exists an $n' \geqslant n$ for which we have a commutative diagram for $i = 1, 2$ and $j = 1, 2$.

$$\begin{array}{ccccc} P^n_{X'/B} & \xrightarrow{\tilde{\rho}} & P^{n'}_{X/B}(3) & \xrightarrow{\widetilde{p}_j} & P^{n'}_{X/B} \\ {\scriptstyle p_i}\downarrow & & {\scriptstyle q_i}\downarrow & & {\scriptstyle p_i}\downarrow \\ X' & \xrightarrow{\rho} & P^{n'}_{X/B} & \xrightarrow{p_j} & X \end{array}$$

Here the morphism q_i (respectively $\widetilde{p}_j$) is the PD morphism induced by the projection to the ith component $X^4_{/B} = X^2_{/B} \times_B X^2_{/B} \to X^2_{/B}$ (respectively $X^4_{/B} = X^2_{/B} \times_B X^2_{/B} \xrightarrow{p_i \times p_i} X \times_B X$), and the morphism $\tilde{\rho}$ is induced by $(g, g') \times (g, g')\colon X \times_B X \mapsto X^2_{/B} \times_B X^2_{/B}$. We obtain the desired compatibility by pulling back by $\tilde{\rho}$ the diagram

$$\begin{array}{ccccccc}
q_2^*p_2^*\mathcal{E} & = & \widetilde{p}_2^*p_2^*\mathcal{E} & \xrightarrow[\widetilde{p}_2^*(\varepsilon)]{\cong} & \widetilde{p}_2^*p_1^*\mathcal{E} & = & q_1^*p_2^*\mathcal{E} \\
{\scriptstyle\cong}\big\downarrow{\scriptstyle q_2^*(\varepsilon)} & & & & & & {\scriptstyle\cong}\big\downarrow{\scriptstyle q_1^*(\varepsilon)} \\
q_2^*p_1^*\mathcal{E} & = & \widetilde{p}_1^*p_2^*\mathcal{E} & \xrightarrow[\widetilde{p}_1^*(\varepsilon)]{\cong} & \widetilde{p}_1^*p_1^*\mathcal{E} & = & q_1^*p_1^*\mathcal{E}
\end{array}$$

which is proven to be commutative by using the property (iii) in Theorem 2.2.1. □

The isomorphisms $c^*_{g,g'/S}$ and $c^*_{g,g'/T}$ are compatible with (2.5.1) as follows.

LEMMA 2.5.3. *Assume that $p>2$ and let X, X', g, and g' be the same as in the beginning of Proposition 2.5.2. Let r_X and $r_{X'}$ be as in the diagram (2.5.1). Then the following diagram of functors is commutative.*

$$\begin{array}{ccc}
r_{X'}\circ g'^*_{/S} & \xrightarrow[r_{X'}\circ c^*_{g,g'/S}]{\cong} & r_{X'}\circ g^*_{/S} \\
{\scriptstyle\cong}\big\downarrow{\scriptstyle (2.5.1)} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle (2.5.1)} \\
g'^*_{/T}\circ r_X & \xrightarrow[c^*_{g,g'/T}\circ r_X]{\cong} & g^*_{/T}\circ r_X
\end{array}$$

Proof. This immediately follows from the construction of $c^*_{g,g'/B}$ in Proposition 2.5.2 and Lemma 2.2.9(1). □

2.6 Direct images

Throughout this subsection, let $f\colon X\to T$ denote a smooth and universally saturated morphism of fine and saturated log schemes such that $\mathring{f}$ is of finite type, $\Omega^1_{X/T}$ has constant rank d, and $I_f=M_X$ (cf. § 2.3). Similarly for $f'\colon X'\to T$, d' and $f''\colon X''\to T$, d''.

Let g be a morphism $X'\to X$ over T. We define the direct image functor

$$g_{+/B}\colon D^-(\mathcal{D}_{X'/B}\text{-}\mathbf{Mod})\to D^-(\mathcal{D}_{X/B}\text{-}\mathbf{Mod})$$

and study its properties. We also prove that the direct image functors $g_{+/S}$ and $g_{+/T}$ are compatible with the restriction of scalars by the homomorphisms (2.1.8) for X and X' (Corollary 2.6.10). Put $\omega_X:=\Omega^d_{X/T}$ and $\omega_{X'}:=\Omega^{d'}_{X'/T}$. We define $\mathcal{D}_{X\leftarrow X'/B}$ to be $g^*_{/B}(\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{\mathcal{O}_{X'}}\omega_{X'}$, which is a $(g^{-1}(\mathcal{D}_{X/B}),\mathcal{D}_{X'/B})$-bimodule (Proposition 2.4.1). Here $g^*_{/B}$ denotes the inverse image with respect to the left action of $\mathcal{D}_{X/B}$ on $\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\omega_X^{-1}$ induced by the right action of $\mathcal{D}_{X/B}$ on $\mathcal{D}_{X/B}$. The left action of $\mathcal{D}_{X/B}$ on $\mathcal{D}_{X/B}$, which commutes with its right action, induces the left action of $g^{-1}(\mathcal{D}_{X/B})$ on $\mathcal{D}_{X\leftarrow X'/B}$. We define the direct image functor $g_{+/B}$ by

$$g_{+/B}(\mathcal{K}):=\mathbb{R}g_*(\mathcal{D}_{X\leftarrow X'/B}\otimes^{\mathbb{L}}_{\mathcal{D}_{X'/B}}\mathcal{K}).$$

Note that any complex of $(g^{-1}(\mathcal{D}_{X/B}),\mathcal{D}_{X'/B})$-bimodules bounded above has a resolution by $(g^{-1}(\mathcal{D}_{X/B}),\mathcal{D}_{X'/B})$-bimodules flat over both $g^{-1}(\mathcal{D}_{X/B})$ and $\mathcal{D}_{X'/B}$, since $\mathcal{D}_{X/B}$ and $\mathcal{D}_{X'/B}$ are flat over W_N.

Assume that the morphism g is smooth. For a left $\mathcal{D}_{X'/B}$-module $\mathcal{E}$, let $d_{\mathcal{E}}\colon\mathcal{E}\to\mathcal{E}\otimes_{\mathcal{O}_{X'}}\Omega^1_{X'/X}$ be the composite of

$$\mathcal{E}\to\mathcal{E}\otimes_{\mathcal{O}_{X'}}\mathcal{P}^1_{X'/B}\cong\mathcal{E}\otimes_{\mathcal{O}_{X'}}(p_1^{-1}\mathcal{O}_{X'}\oplus\Omega^1_{X'/B})\xrightarrow{\text{proj}}\mathcal{E}\otimes_{\mathcal{O}_{X'}}\Omega^1_{X'/B}\to\mathcal{E}\otimes_{\mathcal{O}_{X'}}\Omega^1_{X'/X}.$$

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Then we have $d_{\mathcal{E}}(ae) = e \otimes da + a\, d_{\mathcal{E}}(e)$ for $a \in \mathcal{O}_{X'}$ and $e \in \mathcal{E}$, and obtain the de Rham complex $\mathcal{E} \otimes_{\mathcal{O}_{X'}} \Omega^{\bullet}_{X'/X}$ whose differential maps are defined by $d^q(e \otimes \omega) = d_{\mathcal{E}}(e) \wedge \omega + e \otimes d^q\omega$ for $e \in \mathcal{E}$ and $\omega \in \Omega^q_{X'/X}$ (cf. Proposition 2.1.5(1)). Note that we have a canonical isomorphism

$$\Omega^{d'-d}_{X'/X} \cong g^{-1}(\omega_X^{-1}) \otimes_{g^{-1}(\mathcal{O}_X)} \omega_{X'}.$$

Proposition 2.6.1. *Assume that g is smooth. Then the de Rham complex $\Omega^{\bullet}_{X'/X} \otimes_{\mathcal{O}_{X'}} \mathcal{D}_{X'/B}$ of $\mathcal{D}_{X'/B}$ regarded as a left $\mathcal{D}_{X'/B}$-module gives a resolution*

$$(\Omega^{\bullet}_{X'/X} \otimes_{\mathcal{O}_{X'}} \mathcal{D}_{X'/B})[d'-d] \xrightarrow{\varepsilon} \mathcal{D}_{X\leftarrow X'/B}$$

of $\mathcal{D}_{X\leftarrow X'/B}$ as a right $\mathcal{D}_{X'/B}$-module. Here the right $\mathcal{D}_{X'/B}$-linear homomorphism

$$\varepsilon\colon \Omega^{d'-d}_{X'/X} \otimes_{\mathcal{O}_{X'}} \mathcal{D}_{X'/B} \longrightarrow \mathcal{D}_{X\leftarrow X'/B} \cong g^{-1}(\mathcal{D}_{X/B}) \otimes_{g^{-1}(\mathcal{O}_X)} \Omega^{d'-d}_{X'/X}$$

is defined by $\omega \otimes 1 \mapsto 1 \otimes \omega$ for $\omega \in \Omega^{d'-d}_{X'/X}$.

Proof. Let $\mathcal{C}^{\bullet}$ denote the de Rham complex. Note that the differential maps of $\mathcal{C}^{\bullet}$ are right $\mathcal{D}_{X'/B}$-linear because the right and left actions of $\mathcal{D}_{X'/B}$ on $\mathcal{D}_{X'/B}$ commute. Since the question is étale local on X and X', we may assume that there exist $t_1, \ldots, t_d \in \Gamma(X, M_X)$ and $t_{d+1}, \ldots, t_{d'} \in \Gamma(X', M_{X'})$ such that $\{d \log t_\nu; 1 \leqslant \nu \leqslant d\}$ is a basis of $\Omega^d_{X/T}$ and $\{d \log t_\nu; d+1 \leqslant \nu \leqslant d'\}$ is a basis of $\Omega^{d'-d}_{X'/X}$. We define $\partial_\nu \in \mathcal{D}_{X/B}$ $(\nu_0 \leqslant \nu \leqslant d)$ (respectively $\partial'_\nu \in \mathcal{D}_{X'/B}$ $(\nu_0 \leqslant \nu \leqslant d')$) as in Proposition 2.1.5 using $t_1, \ldots, t_d$ (respectively $g^*(t_1), \ldots, g^*(t_d), t_{d+1}, \ldots, t_{d'}$). Let $\xi_\nu \in T_{X/B} := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/B}, \mathcal{O}_X)$ (respectively $\xi'_\nu \in T_{X'/B} := \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^1_{X'/B}, \mathcal{O}_{X'})$) be the corresponding sections. Then the differential maps of $\mathcal{C}^{\bullet}$ are given by $d^q(\omega \otimes P) = d\omega \otimes P + \sum_{d+1 \leqslant \nu \leqslant d'} d \log t_\nu \wedge \omega \otimes \partial'_\nu P$ for $\omega \in \Omega^q_{X'/X}$ and $P \in \mathcal{D}_{X'/B}$. For $d+1 \leqslant \nu \leqslant d'$, the direct image of ξ'_ν in $g^*(T_{X/B})$ is 0. Hence, by using Propositions 2.3.5 and 2.4.5, we see that the composite of ε with $d^{d'-d-1}$ is 0. We can define the increasing filtration $F_n\mathcal{C}^{\bullet}$ of the complex $\mathcal{C}^{\bullet}$ by putting $F_n\mathcal{C}^q = \Omega^q_{X'/X} \otimes_{\mathcal{O}_{X'}} \mathcal{D}_{X'/B, n+q-(d'-d)}$. Its graded quotient is $\Omega^{\bullet}_{X'/X} \otimes_{\mathcal{O}_{X'}} \mathrm{Sym}^{\bullet}_{\mathcal{O}_{X'}} T_{X'/B}$ with differential maps $\omega \otimes x \mapsto \sum_{d+1 \leqslant \nu \leqslant d'} d \log t_\nu \wedge \omega \otimes \xi'_\nu x$. Hence, $\mathrm{gr}^F_{\bullet}\, \mathcal{C}^{\bullet}$ is isomorphic to the Koszul complex of $\mathrm{Sym}^{\bullet}_{\mathcal{O}_{X'}} T_{X'/B}$ with respect to the regular sequence $\xi'_{d+1}, \ldots, \xi'_{d'}$. Put

$$F_n\mathcal{D}_{X\leftarrow X'/B} := g^{-1}(\mathcal{D}_{X/B,n} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \otimes_{g^{-1}(\mathcal{O}_X)} \omega_{X'}.$$

Then, by Propositions 2.3.5 and 2.4.5, we see that the homomorphism ε is compatible with the filtrations, $\mathrm{gr}^F_{\bullet}\, \mathcal{D}_{X\leftarrow X'/B} \cong \mathrm{Sym}^{\bullet}_{\mathcal{O}_{X'}} g^*(T_{X/B}) \otimes_{\mathcal{O}_{X'}} \Omega^{d'-d}_{X'/X}$, and $\mathrm{gr}^F_{\bullet}\, \varepsilon$ is induced by the homomorphism $\mathrm{Sym}^{\bullet}_{\mathcal{O}_{X'}} T_{X'/B} \to \mathrm{Sym}_{\mathcal{O}_{X'}} g^*(T_{X/B})$ associated to $g_*\colon T_{X'/B} \to g^*(T_{X/B})$. Hence, $\mathrm{gr}^{\bullet}_F\, \varepsilon\colon\ \mathrm{gr}^{\bullet}_F\, \mathcal{C}^{\bullet}[d'-d] \to \mathrm{gr}^{\bullet}_F(\mathcal{D}_{X\leftarrow X'/B})$ is a resolution. By induction on n, we see that $F_n\varepsilon\colon F_n\mathcal{C}^{\bullet}[d'-d] \to F_n\mathcal{D}_{X\leftarrow X'/B}$ is also a resolution. By taking the inductive limit with respect to n, we obtain the proposition. □

Corollary 2.6.2. *Assume that g is smooth. Then, for a left $\mathcal{D}_{X'/B}$-module $\mathcal{E}$, we have a canonical isomorphism*

$$g_{+/B}(\mathcal{E}) \cong \mathbb{R}g_*(\mathcal{E} \otimes_{\mathcal{O}_{X'}} \Omega^{\bullet}_{X'/X})[d'-d]$$

in $D^-(\mathcal{O}_X\text{-}\mathbf{Mod})$.

The direct image functors are compatible with compositions as follows. Let $h\colon X'' \to X'$ be another morphism over T. Using Propositions 2.3.5 and 2.4.5, we see that the natural isomorphism

$$\begin{aligned} h^{-1}(\mathcal{D}_{X\leftarrow X'/B})\otimes_{h^{-1}(\mathcal{D}_{X'/B})}\mathcal{D}_{X'\leftarrow X''/B} &= h^{-1}(g^{-1}(\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{g^{-1}(\mathcal{O}_X)}\omega_{X'}) \\ &\quad \otimes_{h^{-1}(\mathcal{D}_{X'/B})} h^{-1}(\mathcal{D}_{X'/B}\otimes_{\mathcal{O}_{X'}}\omega_{X'}^{-1})\otimes_{h^{-1}(\mathcal{O}_{X'})}\omega_{X''} \\ &\cong h^{-1}g^{-1}(\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{h^{-1}g^{-1}(\mathcal{O}_X)}\omega_{X''} \\ &\cong \mathcal{D}_{X\leftarrow X''/B} \end{aligned} \tag{2.6.3}$$

is $((g\circ h)^{-1}(\mathcal{D}_{X/B}),\mathcal{D}_{X''/B})$-bilinear.

LEMMA 2.6.4. *Under the notation and assumptions as above, the homomorphism in the derived category of sheaves of $((g\circ h)^{-1}(\mathcal{D}_{X/B}),\mathcal{D}_{X''/B})$-bimodules:*

$$h^{-1}(\mathcal{D}_{X\leftarrow X'/B})\otimes^{\mathbb{L}}_{h^{-1}(\mathcal{D}_{X'/B})}\mathcal{D}_{X'\leftarrow X''/B}\longrightarrow\mathcal{D}_{X\leftarrow X''/B},$$

induced by the isomorphism (2.6.3), is an isomorphism.

Proof. It suffices to prove that $\mathcal{H}^q$ of the LHS is 0 for $q\neq 0$. Let $\mathcal{K}^\bullet$ be a flat resolution of $\mathcal{D}_{X\leftarrow X'/B}$ as a right $\mathcal{D}_{X'/B}$-module. Then the LHS of the homomorphism in the lemma is isomorphic to

$$\begin{aligned} h^{-1}(\mathcal{K}^\bullet)\otimes_{h^{-1}(\mathcal{D}_{X'/B})}\mathcal{D}_{X'\leftarrow X''/B} &\cong h^{-1}(\mathcal{K}^\bullet)\otimes_{h^{-1}(\mathcal{O}_{X'})}(h^{-1}(\omega_{X'}^{-1})\otimes_{h^{-1}(\mathcal{O}_{X'})}\omega_{X''}) \\ &\xrightarrow{\cong} h^{-1}(\mathcal{D}_{X\leftarrow X'/B})\otimes_{h^{-1}(\mathcal{O}_{X'})}(h^{-1}(\omega_{X'}^{-1})\otimes_{h^{-1}(\mathcal{O}_{X'})}\omega_{X''}). \end{aligned}$$

For the second isomorphisms, note that $\mathcal{K}^q$ and $\mathcal{D}_{X\leftarrow X'/B}$ are flat as right $\mathcal{O}_{X'}$-modules. □

PROPOSITION 2.6.5. *Under the notation and assumptions as before Lemma 2.6.4, we have a canonical isomorphism $g_{+/B}\circ h_{+/B}\cong(g\circ h)_{+/B}$.*

Proof. Since $\mathcal{D}_{X\leftarrow X'/B}$ is quasi-coherent as a right $\mathcal{O}_{X'}$-module and $\mathcal{D}_{X'/B}$ is a quasi-coherent $\mathcal{O}_{X'}$-algebra, there exists a free resolution of $\mathcal{D}_{X\leftarrow X'/B}$ as a right $\mathcal{D}_{X'/B}$-module Zariski locally on X'. Since X'' is noetherian, this implies that we have an isomorphism

$$\begin{aligned} &\mathcal{D}_{X\leftarrow X'/B}\otimes^{\mathbb{L}}_{\mathcal{D}_{X'/B}}\mathbb{R}h_*(\mathcal{D}_{X'\leftarrow X''/B}\otimes^{\mathbb{L}}_{\mathcal{D}_{X''/B}}\mathcal{E}) \\ &\xrightarrow{\cong}\mathbb{R}h_*(h^{-1}(\mathcal{D}_{X\leftarrow X'/B})\otimes^{\mathbb{L}}_{h^{-1}(\mathcal{D}_{X'/B})}\mathcal{D}_{X'\leftarrow X''/B}\otimes^{\mathbb{L}}_{\mathcal{D}_{X''/B}}\mathcal{E}) \end{aligned}$$

for $\mathcal{E}\in D^-(\mathcal{D}_{X''/B}\text{-}\mathbf{Mod})$. By taking $\mathbb{R}g_*$, and using Lemma 2.6.4, we obtain the desired isomorphism. □

PROPOSITION 2.6.6. *For three morphisms $g\colon X'\to X$, $h\colon X''\to X'$, and $i\colon X'''\to X''$ over T, the following diagram is commutative.*

$$\begin{array}{ccc} g_{+/B}\circ h_{+/B}\circ i_{+/B} & \xrightarrow{\cong} & (g\circ h)_{+/B}\circ i_{+/B} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ g_{+/B}\circ(h\circ i)_{+/B} & \xrightarrow{\cong} & (g\circ h\circ i)_{+/B} \end{array}$$

Proof. Exercise. □

By using Proposition 2.5.2, we see that the direct image functor $g_{+/B}$ is determined by the reduction mod p of g up to canonical isomorphisms as follows.

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PROPOSITION 2.6.7 (cf. [Ber00, 3.4.1(b), 3.4.3]). *Assume that* $p > 2$. *For any two morphisms* $g, g' \colon X' \to X$ *over* T *which coincide modulo* p, *there exists a canonical isomorphism*

$$c_{g,g',+/B} \colon g'_{+/B} \cong g_{+/B}$$

of functors from $D^-(\mathcal{D}_{X'/B}\text{-}\mathbf{Mod})$ *to* $D^-(\mathcal{D}_{X/B}\text{-}\mathbf{Mod})$. *This isomorphism satisfies the following properties.*

(1) *For three morphisms* $g, g', g'' \colon X' \to X$ *over* T *which coincide modulo* p, *we have*

$$c_{g,g',+/B} \circ c_{g',g'',+/B} = c_{g,g'',+/B}.$$

(2) *For two morphisms* $g, g' \colon X' \to X$ *over* T *which coincide modulo* p *and a morphism* $h \colon X \to X''$ *over* T, *the following diagram is commutative.*

$$\begin{array}{ccc} (h\circ g')_{+/B} & \xrightarrow[c_{h\circ g, h\circ g', +/B}]{\cong} & (h\circ g)_{+/B} \\ \text{Proposition 2.6.5} \downarrow \cong & & \cong \downarrow \text{Proposition 2.6.5} \\ h_{+/B}\circ g'_{+/B} & \xrightarrow[h_{+/B}\circ c_{g,g',+/B}]{\cong} & h_{+/B}\circ g_{+/B} \end{array}$$

(3) *For two morphisms* $g, g' \colon X' \to X$ *over* T *which coincide modulo* p *and a morphism* $h \colon X'' \to X'$ *over* T, *the following diagram is commutative.*

$$\begin{array}{ccc} (g'\circ h)_{+/B} & \xrightarrow[c_{g\circ h, g'\circ h, +/B}]{\cong} & (g\circ h)_{+/B} \\ \text{Proposition 2.6.5} \downarrow \cong & & \cong \downarrow \text{Proposition 2.6.5} \\ g'_{+/B}\circ h_{+/B} & \xrightarrow[c_{g,g',+/B}\circ h_{+/B}]{\cong} & g_{+/B}\circ h_{+/B} \end{array}$$

Proof. By Proposition 2.5.2, we have an isomorphism

$$c^*_{g,g'/B}(\mathcal{D}_{X/B} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \otimes \mathrm{id}_{\omega_{X'}} \colon \mathcal{D}_{X \xleftarrow{g'} X'/B} \xrightarrow{\cong} \mathcal{D}_{X \xleftarrow{g} X'/B}$$

as right $\mathcal{D}_{X'/B}$-modules, which we denote by $c_{g,g',\mathcal{D}/B}$. By the functoriality of $c^*_{g,g'/B}$, it is also compatible with the left action of $g^{-1}(\mathcal{D}_{X/B}) = g'^{-1}(\mathcal{D}_{X/B})$ and induces the desired isomorphism $c_{g,g',+/B}$. The property (1) follows from Proposition 2.5.2(1). The property (2) is reduced to showing that the following diagram is commutative.

$$\begin{array}{ccc} g'^{-1}(\mathcal{D}_{X''\leftarrow X/B}) \otimes_{g'^{-1}(\mathcal{D}_{X/B})} \mathcal{D}_{X \overset{g'}{\leftarrow} X'/B} & \xrightarrow[(2.6.3)]{\cong} & \mathcal{D}_{X'' \overset{h\circ g'}{\leftarrow} X/B} \\ \mathrm{id}\otimes c_{g,g',\mathcal{D}/B} \downarrow \cong & & c_{h\circ g, h\circ g', \mathcal{D}/B} \downarrow \cong \\ g^{-1}(\mathcal{D}_{X''\leftarrow X/B}) \otimes_{g^{-1}(\mathcal{D}_{X/B})} \mathcal{D}_{X \overset{g}{\leftarrow} X'/B} & \xrightarrow[(2.6.3)]{\cong} & \mathcal{D}_{X'' \overset{h\circ g}{\leftarrow} X/B} \end{array}$$

which is verified by using Proposition 2.5.2(2). One can prove the property (3) similarly by using Proposition 2.5.2(3). □

Finally, we will show that the direct image functors for $B = T$ and $B = S$ are compatible. By definition, we have a natural homomorphism $\mathcal{D}_{X\leftarrow X'/T} \to \mathcal{D}_{X\leftarrow X'/S}$ induced by (2.1.8). Since the right action of $\mathcal{D}_{X/T}$ on ω_X is induced by that of $\mathcal{D}_{X/S}$ via (2.1.8), the commutativity of (2.5.1) and Lemma 2.4.7 imply that the above homomorphism is compatible with the right actions of $\mathcal{D}_{X'/T} \to \mathcal{D}_{X'/S}$ and the left actions of $g^{-1}(\mathcal{D}_{X/T}) \to g^{-1}(\mathcal{D}_{X/S})$. Hence, it induces a

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homomorphism of $(g^{-1}(\mathcal{D}_{X/T}), \mathcal{D}_{X'/S})$-bimodules:

$$\mathcal{D}_{X\leftarrow X'/T} \otimes_{\mathcal{D}_{X'/T}} \mathcal{D}_{X'/S} \longrightarrow \mathcal{D}_{X\leftarrow X'/S}. \tag{2.6.8}$$

Proposition 2.6.9. *The homomorphism (2.6.8) is an isomorphism.*

Corollary 2.6.10. *The following diagram is commutative up to canonical isomorphisms.*

$$\begin{array}{ccc} D^-(\mathcal{D}_{X'/S}\text{-}\mathbf{Mod}) & \xrightarrow{r_{X'}} & D^-(\mathcal{D}_{X'/T}\text{-}\mathbf{Mod}) \\ {\scriptstyle g_{+/S}}\downarrow & & \downarrow{\scriptstyle g_{+/T}} \\ D^-(\mathcal{D}_{X/S}\text{-}\mathbf{Mod}) & \xrightarrow{r_X} & D^-(\mathcal{D}_{X/T}\text{-}\mathbf{Mod}) \end{array}$$

Here the horizontal arrows are the functors induced by (2.1.8) for X and X'.

Proof. By Proposition 2.6.9, the following diagram is commutative up to a canonical isomorphism.

$$\begin{array}{ccc} \mathcal{D}_{X'/S}\text{-}\mathbf{Mod} & \longrightarrow & \mathcal{D}_{X'/T}\text{-}\mathbf{Mod} \\ {\scriptstyle \mathcal{D}_{X\leftarrow X'/S}\otimes_{\mathcal{D}_{X'/S}}-}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_{X\leftarrow X'/T}\otimes_{\mathcal{D}_{X'/T}}-} \\ g^{-1}(\mathcal{D}_{X/S})\text{-}\mathbf{Mod} & \longrightarrow & g^{-1}(\mathcal{D}_{X/T})\text{-}\mathbf{Mod} \end{array}$$

Furthermore, the two horizontal functors are exact and preserve flat modules because $\mathcal{D}_{X'/S}$ (respectively $\mathcal{D}_{X/S}$) is flat as a left $\mathcal{D}_{X'/T}$ (respectively $\mathcal{D}_{X/T}$)-module. Hence, for $\mathcal{K} \in D^-(\mathcal{D}_{X'/S}\text{-}\mathbf{Mod})$, we have

$$\mathcal{D}_{X\leftarrow X'/S} \otimes^{\mathbb{L}}_{\mathcal{D}_{X'/S}} \mathcal{K} \cong \mathcal{D}_{X\leftarrow X'/T} \otimes^{\mathbb{L}}_{\mathcal{D}_{X/T}} \mathcal{K}$$

in $D^-(g^{-1}(\mathcal{D}_{X/T}))$. By taking $\mathbb{R}g_*$, we obtain the desired commutative diagram. $\square$

The isomorphisms in Proposition 2.6.5 are compatible with the diagram in Corollary 2.6.10 as follows.

Lemma 2.6.11. *Under the notation and assumptions as before Lemma 2.6.4, the following diagram of functors is commutative.*

$$\begin{array}{ccccc} r_X \circ g_{+/S} \circ h_{+/S} & \xrightarrow[\cong]{\text{Corollary 2.6.10}} & g_{+/T} \circ r_{X'} \circ h_{+/S} & \xrightarrow[\cong]{\text{Corollary 2.6.10}} & g_{+/T} \circ h_{+/T} \circ r_{X''} \\ \cong\downarrow{\scriptstyle \text{Proposition 2.6.5}} & & & & \cong\downarrow{\scriptstyle \text{Proposition 2.6.5}} \\ r_X \circ (g\circ h)_{+/S} & & \xrightarrow[\cong]{\text{Corollary 2.6.10}} & & (g\circ h)_{+/T} \circ r_{X''} \end{array}$$

Here r_X denotes the functor $D^-(\mathcal{D}_{X/S}\text{-}\mathbf{Mod}) \to D^-(\mathcal{D}_{X/T}\text{-}\mathbf{Mod})$ induced by (2.1.8), and $r_{X'}$ and $r_{X''}$ are defined similarly using X' and X'' instead of X.

Proof. This follows from the following commutative diagram.

$$\begin{array}{ccc} h^{-1}(\mathcal{D}_{X\leftarrow X'/S}) \otimes_{h^{-1}(\mathcal{D}_{X'/S})} \mathcal{D}_{X'\leftarrow X''/S} & \xrightarrow[\cong]{(2.6.3)} & \mathcal{D}_{X\leftarrow X''/S} \\ {\scriptstyle (2.6.8)}\uparrow & & \uparrow{\scriptstyle (2.6.8)} \\ h^{-1}(\mathcal{D}_{X\leftarrow X'/T}) \otimes_{h^{-1}(\mathcal{D}_{X'/T})} \mathcal{D}_{X'\leftarrow X''/T} & \xrightarrow[\cong]{(2.6.3)} & \mathcal{D}_{X\leftarrow X''/T} \end{array}$$

$\square$

The isomorphisms $c_{g,g',+/S}$ and $c_{g,g',+/T}$ in Proposition 2.6.7 are compatible with the diagram in Corollary 2.6.10 as follows.

Lemma 2.6.12. *Assume that $p>2$ and let X, X', g, and g' be the same as in the beginning of Proposition 2.6.7. Let r_X and $r_{X'}$ be as in Corollary 2.6.10. Then the following diagram of functors is commutative.*

$$\begin{array}{ccc} r_X \circ g'_{+/S} & \xrightarrow[r_X \circ c_{g,g',+/S}]{\cong} & r_X \circ g_{+/S} \\ {\scriptstyle \cong}\big\downarrow {\scriptstyle \text{Corollary 2.6.10}} & & {\scriptstyle \cong}\big\downarrow {\scriptstyle \text{Corollary 2.6.10}} \\ g'_{+/T} \circ r_{X'} & \xrightarrow[c_{g,g',+/T} \circ r_{X'}]{\cong} & g_{+/T} \circ r_{X'} \end{array}$$

Proof. We are reduced to the following commutative diagram, which follows from Lemma 2.5.3.

$$\begin{array}{ccc} \mathcal{D}_{X \overset{g'}{\leftarrow} X'/S} & \xrightarrow[c_{g,g',\mathcal{D}/S}]{\cong} & \mathcal{D}_{X \overset{g}{\leftarrow} X'/S} \\ \big\uparrow & & \big\uparrow \\ \mathcal{D}_{X \overset{g'}{\leftarrow} X'/T} & \xrightarrow[c_{g,g',\mathcal{D}/T}]{\cong} & \mathcal{D}_{X \overset{g}{\leftarrow} X'/T} \end{array}$$

Here $c_{g,g',\mathcal{D}/B}$ is the isomorphism defined in the proof of Proposition 2.6.7. □

In the rest of § 2.6, we prove Proposition 2.6.9. Since the question is étale local on $\mathring{X}$, we may assume that there exist $t_1, \ldots, t_d$ as in Proposition 2.1.1. Choose such t_ν and define $\partial_\nu \in \mathcal{D}_{X/T}$ $(1 \leqslant \nu \leqslant d)$ and $\partial_0 \in \mathcal{D}_{X/S}$ as in Proposition 2.1.5.

Lemma 2.6.13. *For $n \in \mathbb{N}$, we have $\sum_{r \leqslant n} \mathcal{D}_{X/T}\partial_0^r = \sum_{r \leqslant n} \partial_0^r \mathcal{D}_{X/T}$ in $\mathcal{D}_{X/S}$.*

Proof. Let $\mathcal{I}_n$ (respectively $\mathcal{I}'_n$) denote the LHS (respectively RHS). We prove that $\mathcal{I}_n = \mathcal{I}'_n$ by induction on n. The claim is trivial for $n=0$. Let n be a positive integer and assume that $\mathcal{I}_{n-1} = \mathcal{I}'_{n-1}$. It suffices to prove that $\mathcal{D}_{X/T}\partial_0^n \subset \mathcal{I}'_n$ and $\partial_0^n \mathcal{D}_{X/T} \subset \mathcal{I}_n$. Since $\mathcal{I}_n$ (respectively $\mathcal{I}'_n$) are stable under the left (respectively right) action of $\mathcal{D}_{X/T}$ and $\partial_\nu \partial_0 = \partial_0 \partial_\nu$ $(1 \leqslant \nu \leqslant d)$, the claim is reduced to $x\partial_0^n = [x, \partial_0]\partial_0^{n-1} + \partial_0 x \partial_0^{n-1} = -\partial_0(x)\partial_0^{n-1} + \partial_0 x \partial_0^{n-1} \in \mathcal{I}_{n-1} + \partial_0 \mathcal{I}_{n-1} = \mathcal{I}'_{n-1} + \partial_0 \mathcal{I}'_{n-1} \subset \mathcal{I}'_n$ (respectively $\partial_0^n x = \partial_0^{n-1}[\partial_0, x] + \partial_0^{n-1} x \partial_0 = \partial_0^{n-1}\partial_0(x) + \partial_0^{n-1} x \partial_0 \in \mathcal{I}'_{n-1} + \mathcal{I}'_{n-1}\partial_0 = \mathcal{I}_{n-1} + \mathcal{I}_{n-1}\partial_0 \subset \mathcal{I}_n$) for $x \in \mathcal{O}_X$. □

Put

$$\mathcal{I}_n = \sum_{r \leqslant n} \mathcal{D}_{X/T}\partial_0^r = \sum_{r \leqslant n} \partial_0^r \mathcal{D}_{X/T}.$$

Note that these are stable under both the left and right actions of $\mathcal{D}_{X/T}$. Put $\mathcal{I}_{-1} = 0$.

Lemma 2.6.14. *For $n \in \mathbb{N}$, the homomorphism $\mathcal{I}_n \to \mathcal{I}_{n+1}; P \mapsto P\partial_0$ induces an isomorphism $\mathcal{I}_n/\mathcal{I}_{n-1} \xrightarrow{\cong} \mathcal{I}_{n+1}/\mathcal{I}_n$. Furthermore, it is $\mathcal{O}_X$-linear for the right $\mathcal{O}_X$-action.*

Proof. The homomorphism $\mathcal{D}_{X/T} \to \mathcal{I}_m; P \mapsto P\partial_0^m$ induces an isomorphism

$$\mathcal{D}_{X/T} \xrightarrow{\cong} \mathcal{I}_m/\mathcal{I}_{m-1} \quad \text{for } m \in \mathbb{N}.$$

This implies the first claim. The second claim follows from $P\partial_0 x - Px\partial_0 = P\partial_0(x) \in \mathcal{I}_n$ for $P \in \mathcal{I}_n$ and $x \in \mathcal{O}_X$. □

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Since $\mathcal{D}_{X/S}=\mathcal{I}_n\oplus(\sum_{r>n}\partial_0^r\mathcal{D}_{X/T})$, we see that the homomorphism $g^*(\mathcal{I}_n\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{\mathcal{O}_{X'}}\omega_{X'}\to\mathcal{D}_{X\leftarrow X'/S}$ is injective. Let $\mathcal{J}_n$ denote the LHS of the above homomorphism.

Since the question is étale local also on $\mathring{X}'$, we assume that there exist $t'_1,\ldots,t'_{d'}\in\Gamma(X',M_{X'})$ such that $\{d\log(t'_\mu)\}_{1\leqslant\mu\leqslant d'}$ is a basis of $\Omega^1_{X'/T}$. Choose such t'_μ and define $\partial'_\mu\in\mathcal{D}_{X'/T}$ and $\partial'_0\in\mathcal{D}_{X'/S}$ in the same way as in Proposition 2.1.5.

Lemma 2.6.15.

(1) *For $n\in\mathbb{N}$, we have $\mathcal{J}_n\cdot\partial'_0\subset\mathcal{J}_{n+1}$.*

(2) *For $n\in\mathbb{N}$, the homomorphism $\mathcal{J}_n\to\mathcal{J}_{n+1};P'\mapsto P'\cdot\partial'_0$ induces an isomorphism $\mathcal{J}_n/\mathcal{J}_{n-1}\xrightarrow{\cong}\mathcal{J}_{n+1}/\mathcal{J}_n$.*

Proof. Put

$$g^*(d\log(t_\nu))=\sum_{0\leqslant\mu\leqslant d'}a_{\nu\mu}\,d\log(t'_\mu)$$

in $\Omega^1_{X'/S}$ for $1\leqslant\nu\leqslant d$. For $P\in\mathcal{I}_n$ and local bases $\omega\in\omega_X$, $\omega'\in\omega_{X'}$, we have

$$(g^*(P\otimes\omega^{-1})\otimes\omega')\partial'_0=g^*(P\otimes\omega^{-1})\otimes(\omega'\partial'_0)-\{\partial'_0(g^*(P\otimes\omega^{-1}))\}\otimes\omega'$$

by Proposition 2.4.5. The first term of the RHS is contained in $\mathcal{J}_n$. For the second term, by Proposition 2.4.5 again, we have

$$\begin{aligned}\partial'_0(g^*(P\otimes\omega^{-1}))&=g^*(\partial_0(P\otimes\omega^{-1}))+\sum_{1\leqslant\nu\leqslant d}a_{\nu0}\cdot g^*(\partial_\nu(P\otimes\omega^{-1}))\\&=g^*((Pb_0-P\partial_0)\otimes\omega^{-1})+\sum_{1\leqslant\nu\leqslant d}a_{\nu0}\cdot g^*(P(b_\nu-\partial_\nu)\otimes\omega^{-1}),\end{aligned}$$

where $b_\nu\in\mathcal{O}_X$ $(0\leqslant\nu\leqslant d)$ is defined by $\omega\partial_\nu=b_\nu\omega$. Since $P(b_\nu-\partial_\nu)\in\mathcal{I}_n$ for $1\leqslant\nu\leqslant d$, $Pb_0\in\mathcal{I}_n$, and $P\partial_0\in\mathcal{I}_{n+1}$, the above computation implies that $\mathcal{J}_n\cdot\partial'_0\subset\mathcal{J}_{n+1}$ and the homomorphism $\mathcal{J}_n/\mathcal{J}_{n-1}\to\mathcal{J}_{n+1}/\mathcal{J}_n$ in question coincides with the isomorphism

$$g^*(\mathcal{I}_n/\mathcal{I}_{n-1}\otimes_{\mathcal{O}_X}\omega_X)\otimes_{\mathcal{O}_{X'}}\omega_{X'}\xrightarrow{\cong}g^*(\mathcal{I}_{n+1}/\mathcal{I}_n\otimes_{\mathcal{O}_X}\omega_X)\otimes_{\mathcal{O}_{X'}}\omega_{X'}$$

induced by the isomorphism in Lemma 2.6.14. □

Proof of Proposition 2.6.9. By Lemma 2.6.15, the homomorphism $\mathcal{J}_0\to\mathcal{J}_n/\mathcal{J}_{n+1};P'\mapsto P'\cdot\partial_0'^n$ is an isomorphism for $n\in\mathbb{N}$. This implies, by induction on n, that the homomorphism

$$\bigoplus_{0\leqslant r\leqslant n}\mathcal{J}_0\to\mathcal{J}_n;\quad(x_r)\mapsto\sum_{0\leqslant r\leqslant n}x_r(\partial'_0)^r$$

is an isomorphism for $n\in\mathbb{N}$. By taking the inductive limit with respect to n, we obtain an isomorphism

$$\bigoplus_{r\in\mathbb{N}}\mathcal{D}_{X\leftarrow X'/T}\xrightarrow{\cong}\mathcal{D}_{X\leftarrow X'/S};\quad(P'_r)\mapsto\sum_{r\in\mathbb{N}}P'_r\partial_0'^r.$$

Since $\mathcal{D}_{X'/S}$ is a free left $\mathcal{D}_{X'/T}$-module whose basis is given by $\{\partial_0'^r\}_{r\in\mathbb{N}}$, this implies that (2.6.8) is an isomorphism. □

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3. Nearby cycles: local case

Let X be a fine log scheme smooth over T such that $\mathring{X}$ is of finite type over S and $\Omega^1_{X/T}$ has constant rank d. Throughout this section, we assume that, étale local on $\mathring{X}$, there exist $t_1, \ldots, t_{d+1} \in \Gamma(X, M_X)$ and an integer c, $1 \leqslant c \leqslant d+1$, such that $t_c \cdots t_{d+1}$ is the image of $t \in \Gamma(T, M_T)$, $t_1, \ldots, t_{c-1}$ is invertible,

$$\mathbb{N}_X^{d+1} \to M_X; \quad (n_1, \ldots, n_{d+1}) \mapsto t_1^{n_1} \cdots t_{d+1}^{n_{d+1}}$$

is a chart, and the morphism

$$\mathring{X} \to \mathrm{Spec}(W_N[s_1, \ldots, s_{d+1}]/(s_c \cdots s_{d+1}))$$

defined by the W_N-homomorphism

$$W_N[s_1, \ldots, s_{d+1}]/(s_c \cdots s_{d+1}) \to \Gamma(X, \mathcal{O}_X)$$

sending s_i to t_i is étale. By replacing t_{d+1} with $t_{d+1}(t_1 \cdots t_{c-1})^{-1}$ if $c \geqslant 2$, we always consider $t_1, \ldots, t_d$ as above with $c = 1$ in the following. The assumption implies that M_X is saturated, the structure morphism $f \colon X \to T$ is universally saturated, and $I_f = M_X$ (cf. § 2.3).

We also assume that we are given a morphism $\alpha \colon X \to Y$ of fine log schemes over T such that the underlying morphism of schemes $\mathring{\alpha} \colon \mathring{X} \to \mathring{Y}$ is a closed immersion, $\mathring{Y}$ is smooth over S, the log structure M_Y is the inverse image of M_T, and $\Omega^1_{Y/T}$ has constant rank e.

3.1 Definition of nearby cycles: local case

Definition 3.1.1. For a left $\mathcal{D}_{X/S}$-module $\mathcal{E}$ such that the underlying $\mathcal{O}_X$-module is locally free of finite type, we call the direct image $\alpha_{+/S}(\mathcal{E}) \in D^-(\mathcal{D}_{Y/S}\text{-}\mathbf{Mod})$ of $\mathcal{E}$ under α (cf. § 2.6) the *nearby cycles of $\mathcal{E}$ realized on Y*.

Note that, by Proposition 2.1.11 and (2.1.10), a left $\mathcal{D}_{\mathring{Y}/S}$-module endowed with an endomorphism $\partial_t^{\log}$ is interpreted as a left $\mathcal{D}_{Y/S}$-module. If we regard the nearby cycles $\alpha_{+/S}(\mathcal{E})$ as an object of $D^-(\mathcal{D}_{\mathring{Y}/S}\text{-}\mathbf{Mod}) = D^-(\mathcal{D}_{Y/T}\text{-}\mathbf{Mod})$ by forgetting the action of $\partial_t^{\log}$, then it is canonically isomorphic to $\alpha_{+/T}(\mathcal{E}')$ by Corollary 2.6.10. Here $\mathcal{E}'$ denotes $\mathcal{E}$ with the action of $\mathcal{D}_{X/T}$ via (2.1.8).

By Theorem 3.1.2 below, $\alpha_{+/S}(\mathcal{E})$ (respectively $\alpha_{+/T}(\mathcal{E}')$) may be regarded as a $\mathcal{D}_{Y/S}$-module (respectively $\mathcal{D}_{\mathring{Y}/S} = \mathcal{D}_{Y/T}$-module). In §§ 3.4 and 3.5, we will also apply Theorem 3.1.2 for $B = T$ to the intersections of 'smooth components of X' endowed with the inverse images of M_T.

As in § 2, let B be S or T.

Theorem 3.1.2. *Let $\mathcal{E}$ be a left $\mathcal{D}_{X/B}$-module $\mathcal{E}$ such that the underlying $\mathcal{O}_X$-module is locally free of finite type. Then:*

(1) *the natural homomorphism $\alpha_*(\mathcal{D}_{Y \leftarrow X/B} \otimes_{\mathcal{D}_{X/B}} \mathcal{E}) \leftarrow \alpha_{+/B}(\mathcal{E})$ is an isomorphism; i.e. we have $\mathcal{H}^q(\alpha_{+/B}(\mathcal{E})) = 0$ for $q \neq 0$;*

(2) *the object $\alpha_{+/B}(\mathcal{E})$ of $D(\mathcal{D}_{Y/S}\text{-}\mathbf{Mod})$ is perfect (cf. [Ill71, Définition 4.7 and Exemple 4.8]).*

We need some lemmas. For a left $\mathcal{D}_{X/B}$-module $\mathcal{F}$, we can construct a de Rham complex $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/B}$ in the same way as before Proposition 2.6.1. Applying this to $\mathcal{D}_{X/B}$ regarded as a left $\mathcal{D}_{X/B}$-module, we obtain a complex $\Omega^\bullet_{X/B} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/B}$, whose differential maps are right $\mathcal{D}_{X/B}$-linear because the right and left actions of $\mathcal{D}_{X/B}$ on $\mathcal{D}_{X/B}$ mutually commute.

Let $\mathcal{E}$ be as in Theorem 3.1.2. Since $\mathcal{D}_{X/B}$ is a $(\mathcal{D}_{X/B},\mathcal{D}_{X/B})$-bimodule, we may regard $\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}$ as a $(\mathcal{D}_{X/B},\mathcal{D}_{X/B})$-bimodule by Proposition 2.4.1. Let $T_{X/B}$ denote $\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/B},\mathcal{O}_X)$. Then, by applying $\mathcal{H}om_{\mathcal{D}_{X/B},r}(-,\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E})$ to the complex of right $\mathcal{D}_{X/B}$-modules $\Omega^\bullet_{X/B}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B}$ and using isomorphisms

$$\mathcal{H}om_{\mathcal{D}_{X/B},r}(\Omega^q_{X/B}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B},\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E})\cong(\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E})\otimes_{\mathcal{O}_X}\wedge^qT_{X/B}$$

of left $\mathcal{D}_{X/B}$-modules, we obtain a complex of left $\mathcal{D}_{X/B}$-modules:

$$\cdots\to\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^2T_{X/B}\to\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}T_{X/B}\to\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}\to 0\to\cdots.$$

Here $\mathcal{H}om_{\mathcal{D}_{X/B},r}$ denotes $\mathcal{H}om$ with respect to right $\mathcal{D}_{X/B}$-actions.

LEMMA 3.1.3. *Put $\mathcal{C}^{-q}=\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^qT_{X/B}$ for $q\in\mathbb{N}$ and let $\mathcal{C}^\bullet$ denote the complex constructed above.*

(1) *Suppose that there exist $t_1,\dots,t_{d+1}\in\Gamma(X,M_X)$ as in the beginning of § 3 and define the differential operators $\partial_\nu\in\mathcal{D}_{X/B}$ ($\nu_0\leqslant\nu\leqslant d$) using $t_0=t,t_1,\dots,t_d$ as in Proposition 2.1.5. Let ξ_ν denote the image of ∂_ν in $\mathcal{D}_{X/B,1}/\mathcal{D}_{X/B,0}\cong T_{X/B}$. Then, for $q\in\mathbb{N}$, $q>0$, the differential map $d_{\mathcal{C}}^{-q}\colon\mathcal{C}^{-q}\to\mathcal{C}^{-q+1}$ is given by the formula*

$$d_{\mathcal{C}}^{-q}(x\otimes\xi_{\nu_1}\wedge\cdots\wedge\xi_{\nu_q})=\sum_{r=1}^q(-1)^{r-1}x\cdot\partial_{\nu_r}\otimes\xi_{\nu_1}\wedge\cdots\wedge\hat{\xi_{\nu_r}}\wedge\cdots\wedge\xi_{\nu_q}$$

for $x\in\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}$ and $\nu_0\leqslant\nu_1<\nu_2<\cdots<\nu_q\leqslant d$.

(2) *The morphism $\mathcal{C}^0=\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}\to\mathcal{E}:P\otimes e\mapsto P\cdot e$ defines a resolution $\mathcal{C}^\bullet\to\mathcal{E}$ of the left $\mathcal{D}_{X/B}$-module $\mathcal{E}$ by locally free left $\mathcal{D}_{X/B}$-modules of finite type.*

Proof. (1) The connection $\mathcal{D}_{X/B}\to\Omega^1_{X/B}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B}$ induced by the left $\mathcal{D}_{X/B}$-action on $\mathcal{D}_{X/B}$ maps P to $\sum_{\nu=\nu_0}^d d\log(t_\nu)\otimes\partial_\nu P$. Hence, the differential map

$$\Omega^{q-1}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B}\to\Omega^q_{X/B}\otimes_{\mathcal{O}_X}\mathcal{D}_{X/B}$$

maps

$$(d\log t_{\mu_1}\wedge\cdots\wedge d\log t_{\mu_{q-1}})\otimes 1$$

to

$$\sum_{\nu=\nu_0}^d(d\log t_\nu\wedge d\log t_{\mu_1}\wedge\cdots\wedge d\log t_{\mu_{q-1}})\otimes\partial_\nu$$

for $\nu_0\leqslant\mu_1<\cdots<\mu_{q-1}\leqslant d$. This immediately implies the formula in (1).

(2) The question is étale local on $\mathring{X}$ and we may assume that there exist $t_1,\dots,t_{d+1}\in\Gamma(X,M_X)$ as in the beginning of § 3. Let ∂_ν and ξ_ν be as in (1). We put $\mathcal{C}^1=\mathcal{E}$ and $\mathcal{C}^q=0$ for $q>1$, and define $d^0\colon\mathcal{C}^0\to\mathcal{C}^1$ to be the morphism $P\otimes e\mapsto P\cdot e$. By using the formula $(P\otimes e)\partial_\nu=P\partial_\nu\otimes e-P\otimes\partial_\nu e$ for $P\in\mathcal{D}_{X/B}$ and $e\in\mathcal{E}$ (Proposition 2.4.5), we see that $d^0\circ d^{-1}=0$. Now it remains to show that the complex $\mathcal{C}^\bullet$ is acyclic. We define an increasing filtration F_n ($n\in\mathbb{Z}$) of $\mathcal{C}^\bullet$ by $F_n\mathcal{C}^{-q}=\mathcal{D}_{X/B,n-q}\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^qT_{X/B}$ for $q\in\mathbb{N}$ and $F_n\mathcal{C}^1=\mathcal{E}$ (if $n\geqslant 0$), 0 (if $n<0$). Here we put $\mathcal{D}_{X/B,m}=0$ for $m<0$. Note that $F_{-1}\mathcal{C}^\bullet=0$. Using the above explicit description of the right action of ∂_ν on $\mathcal{D}_{X/B}\otimes_{\mathcal{O}_X}\mathcal{E}$, one can verify that this filtration

is compatible with the differential maps and its graded quotients are as follows:

$$\mathrm{gr}^F\,\mathcal{C}^{-q} = (\mathrm{Sym}^\bullet_{\mathcal{O}_X} T_{X/B} \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathcal{O}_X} \wedge^q T_{X/B}, \quad \mathrm{gr}^F\,\mathcal{C}^1 = \mathcal{E},$$

$$\mathrm{gr}^F\,d^{-q}(y \otimes \xi_{\nu_1} \wedge \cdots \wedge \xi_{\nu_q}) = \sum_{r=1}^{q} (-1)^{r-1} \xi_{\nu_r} \cdot y \otimes \xi_{\nu_1} \wedge \cdots \wedge \hat{\xi}_{\nu_r} \wedge \cdots \wedge \xi_{\nu_q},$$

and $\mathrm{gr}^F_\bullet\,d^0$ is the homomorphism induced by the projection $\mathrm{Sym}^\bullet_{\mathcal{O}_X} T_{X/B} \to \mathrm{Sym}^0_{\mathcal{O}_X} T_{X/B} = \mathcal{O}_X$. Hence, the non-positive degree part of $\mathrm{gr}^F\,\mathcal{C}^\bullet$ is isomorphic to the Koszul complex of the $\mathrm{Sym}^\bullet_{\mathcal{O}_X} T_{X/B}$-module $\mathrm{Sym}^\bullet_{\mathcal{O}_X} T_{X/B} \otimes_{\mathcal{O}_X} \mathcal{E}$ with respect to the sequence $\xi_{\nu_0}, \ldots, \xi_d \in \mathrm{Sym}^\bullet_{\mathcal{O}_X} T_{X/B}$, which is a regular sequence on the module. Thus, we see that $\mathrm{gr}^F\,\mathcal{C}^\bullet$ is acyclic, and that $F_n\mathcal{C}^\bullet$ is also acyclic by induction on n. Taking the inductive limit with respect to n, we obtain the desired acyclicity. □

Let $T_{Y/B}$ denote $\mathcal{H}om_{\mathcal{O}_Y}(\Omega^1_{Y/B}, \mathcal{O}_Y)$ and let $\alpha_*\colon T_{X/B} \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} T_{Y/B}$ denote the the dual of the morphism $\alpha^*\colon \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Omega^1_{Y/B} \to \Omega^1_{X/B}$ induced by α.

Lemma 3.1.4. *Let the notation and assumptions be the same as in Lemma 3.1.3(1). Then the sequence* $\alpha_*(\xi_{\nu_0}), \alpha_*(\xi_{\nu_0+1}), \ldots, \alpha_*(\xi_d)$ *in the sheaf of commutative rings* $\mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{\mathcal{O}_Y} T_{Y/B})$ *is a regular sequence.*

Proof. Let $\alpha'\colon X \to Y'$ be another morphism of fine log schemes over T satisfying the same conditions as α. We first prove that the lemma holds for Y if and only if it holds for Y'. By considering $\alpha''\colon X \to Y \times_T Y'$ induced by α and α', we are reduced to the case where there exists a smooth morphism $h\colon Y' \to Y$ such that $\alpha = h \circ \alpha'$. Since the question is Zariski local, we may assume that there exist $w_1, \ldots, w_e \in \Gamma(Y, \mathcal{O}_Y)$ (respectively $w_{e+1}, \ldots, w_{e'} \in \Gamma(Y', \mathcal{O}_{Y'})$) such that dw_μ $(1 \leqslant \mu \leqslant e)$ (respectively dw_μ $(e+1 \leqslant \mu \leqslant e')$) is a basis of $\Omega^1_{Y/T}$ (respectively $\Omega^1_{Y'/Y}$). By subtracting a lifting of $\alpha'^*(w_\mu)$ in $\Gamma(Y, \mathcal{O}_Y)$ from w_μ, we may assume that $\alpha'^*(w_\mu) = 0$ for $e+1 \leqslant \mu \leqslant e'$. When $B = S$, let $\eta_0, \eta_1, \ldots, \eta_e \in T_{Y/S}$ (respectively $\eta'_0, \eta'_1, \ldots, \eta'_{e'} \in T_{Y'/S}$) be the dual basis of $d\log t, dw_1, \ldots, dw_e \in \Omega^1_{Y/S}$ (respectively $d\log t, dh^*(w_1), \ldots, dh^*(w_e), dw_{e+1}, \ldots, dw_{e'} \in \Omega^1_{Y'/S}$). When $B = T$, let $\eta_1, \ldots, \eta_e \in T_{Y/T}$ (respectively $\eta'_1, \ldots, \eta'_{e'} \in T_{Y'/T}$) be the dual basis of $dw_1, \ldots, dw_e \in \Omega^1_{Y/T}$ (respectively $dh^*(w_1), \ldots, dh^*(w_e), dw_{e+1}, \ldots, dw_{e'} \in \Omega^1_{Y'/T}$). Then, for $\nu_0 \leqslant \nu \leqslant d$, we have

$$\alpha_*(\xi_\nu) = \sum_{\nu_0 \leqslant \mu \leqslant e} a_{\nu\mu} \otimes \eta_\mu \quad \text{and} \quad \alpha'_*(\xi_\nu) = \sum_{\nu_0 \leqslant \mu \leqslant e} a_{\nu\mu} \otimes \eta'_\mu.$$

Here $a_{\nu\mu} \in \Gamma(X, \mathcal{O}_X)$ for $\mu \geqslant 1$ is given by $a_{\nu\mu} = \partial_\nu(\alpha^*(w_\mu))$. When $B = S$, $a_{00} = 1$ and $a_{\nu 0} = 0$ for $\nu \geqslant 1$. This implies the desired claim.

Since the question is Zariski local on X, we may assume that there exists an étale lifting $\mathring{Z} \to \mathrm{Spec}(W_N[s_1, \ldots, s_{d+1}])$ of the étale morphism $\mathring{X} \to \mathrm{Spec}(W_N[s_1, \ldots, s_{d+1}]/(s_1 \cdots s_{d+1}))$ defined by $t_1, \ldots, t_{d+1} \in \Gamma(X, M_X)$. Choose such a $\mathring{Z}$, and let Z be $\mathring{Z}$ with the inverse image of M_T. Then the natural morphism $\beta\colon X \to Z$ satisfies the conditions on α. We prove the lemma for β. Let $\theta_{\nu_0}, \ldots, \theta_{d+1} \in T_{Z/B}$ be the dual basis of $d\log t, ds_1, \ldots, ds_{d+1} \in \Omega^1_{Z/S}$ (respectively $ds_1, \ldots, ds_{d+1} \in \Omega^1_{Z/T}$) when $B = S$ (respectively $B = T$). Then we have $\beta_*(\xi_\nu) = t_\nu \otimes \theta_\nu - t_{d+1} \otimes \theta_{d+1}$ for $1 \leqslant \nu \leqslant d$. When $B = S$, we have $\beta_*(\xi_0) = t_{d+1} \otimes \theta_{d+1} + 1 \otimes \theta_0$. Hence, it suffices to prove that the sequence

$$s_1 \cdots s_{d+1}, s_1 V_1 - s_{d+1} V_{d+1}, \ldots, s_d V_d - s_{d+1} V_{d+1}$$

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is a regular sequence in $W_N[s_1,\dots,s_{d+1},V_1,\dots,V_{d+1}]$. Put $f_\nu:=s_\nu V_\nu-s_{d+1}V_{d+1}$ for $1\leqslant\nu\leqslant d$. Since f_ν and $s_1,\dots s_{d+1}$ are homogeneous of positive degrees, it is equivalent to saying that the sequence $f_1\dots,f_{d+1},s_1\cdots s_{d+1}$ is regular [Mat80, (15.B) Theorem 27 and Remark]. Since the W_N homomorphism

$$W_N[U_1,\dots,U_{d+1}]\to W_N[s_1,\dots,s_{d+1},V_1,\dots,V_{d+1}]$$

defined by $U_\nu\mapsto s_\nu V_\nu$ is flat, the last claim follows from the regularity of the sequence $U_1-U_{d+1},\dots,U_d-U_{d+1},U_1\cdots U_{d+1}$ in $W_N[U_1,\dots,U_{d+1}]$. □

We assume that there exist $t_1,\dots,t_{d+1}\in\Gamma(X,M_X)$ as in the beginning of § 3 and define ∂_ν and ξ_μ as in Lemma 3.1.3(1). We also assume that $\mathring{Y}$ is affine and there exist $z_1,\dots,z_e\in\Gamma(Y,\mathcal{O}_Y^\times)$ such that $\{d\log(z_\mu)=z_\mu^{-1}dz_\mu\}$ is a basis of $\Omega^1_{Y/T}$. We define the differential operator $\partial^Y_\mu\in\mathcal{D}_{Y/B}$ $(\nu_0\leqslant\mu\leqslant e)$ using t, $z_1,\dots,z_e$ as in Proposition 2.1.5. Let ξ^Y_μ denote the image of ∂^Y_μ in $T_{Y/B}\cong\mathcal{D}_{Y/B,1}/\mathcal{D}_{Y/B,0}$. The image of $d\log(z_\mu)\otimes1$ $(1\leqslant\mu\leqslant e)$ under the homomorphism $\alpha^*\colon\Omega^1_{Y/B}\otimes_{\mathcal{O}_Y}\mathcal{O}_X\to\Omega^1_{X/B}$ is $\sum_{\nu_0\leqslant\nu\leqslant d}\partial_\nu\log(\alpha^*(z_\mu))\,d\log t_\nu$, where $\partial_\nu\log(f)=f^{-1}\partial_\nu(f)$ for $f\in\mathcal{O}_X^\times$. When $B=S$, we have $\alpha^*(d\log t)=d\log t$. Hence, the images of ξ_ν $(1\leqslant\nu\leqslant d)$ under the homomorphism $\alpha_*\colon T_{X/B}\to T_{Y/B}\otimes_{\mathcal{O}_Y}\mathcal{O}_X$ are $\sum_{1\leqslant\mu\leqslant e}\xi^Y_\mu\otimes\partial_\nu\log(\alpha^*(z_\mu))$. When $B=S$, we have

$$\alpha_*(\xi_0)=\xi^Y_0\otimes1+\sum_{1\leqslant\mu\leqslant e}\xi^Y_\mu\otimes\partial_0\log(\alpha^*(z_\mu)).$$

For $\nu_0\leqslant\nu\leqslant d$ and $1\leqslant\mu\leqslant e$, choose $a_{\nu\mu}\in\Gamma(Y,\mathcal{O}_Y)$ such that $\alpha^*(a_{\nu\mu})=\partial_\nu\log(\alpha^*(z_\mu))$. Put

$$d\log\underline{t}:=d\log t_1\wedge\dots\wedge d\log t_d\in\omega_X,\quad d\log\underline{z}:=d\log z_1\wedge\dots\wedge d\log z_e\in\omega_Y,$$

and $P_\nu:=\sum_{1\leqslant\mu\leqslant e}\partial^Y_\mu\cdot a_{\nu\mu}\in\mathcal{D}_{Y/B}$ for $1\leqslant\nu\leqslant d$. When $B=S$, put $P_0:=\partial^Y_0+\sum_{1\leqslant\mu\leqslant e}\partial^Y_\mu\cdot a_{0\mu}\in\mathcal{D}_{Y/S}$.

Lemma 3.1.5. *Under the notation and assumptions as above, the action of $\partial_\nu\in\mathcal{D}_{X/B}$ $(\nu_0\leqslant\nu\leqslant d)$ on $\mathcal{D}_{Y\leftarrow X/B}=(\mathcal{D}_{Y/B}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_Y}\omega_X$ is given by the following formula:*

$$((P\otimes(d\log\underline{z})^{-1})\otimes d\log\underline{t})\cdot\partial_\nu=(P\cdot P_\nu\otimes(d\log\underline{z})^{-1})\otimes d\log\underline{t},\quad P\in\mathcal{D}_{Y/B}.$$

Proof. We have the following equalities in $\alpha^*(\mathcal{D}_{Y/B}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_X}\omega_X$:

$$\begin{aligned}\alpha^*(P\otimes(d\log\underline{z})^{-1})\otimes d\log\underline{t})\cdot\partial_\nu&=-\partial_\nu\alpha^*(P\otimes(d\log\underline{z})^{-1})\otimes d\log\underline{t}\\&=-\alpha^*(Q_\nu(P\otimes(d\log\underline{z})^{-1}))\otimes d\log\underline{t}\\&=\alpha^*(P\cdot P_\nu\otimes(d\log\underline{z})^{-1})\otimes d\log\underline{t},\end{aligned}$$

where $Q_\nu:=\sum_{1\leqslant\mu\leqslant e}a_{\nu\mu}\partial^Y_\mu\in\mathcal{D}_{Y/B}$ $(1\leqslant\nu\leqslant d)$ and, when $B=S$, $Q_0:=\partial^Y_0+\sum_{1\leqslant\mu\leqslant e}a_{0\mu}\partial^Y_\mu\in\mathcal{D}_{Y/S}$. The first equality follows from $(d\log\underline{t})\partial_\nu=0$ (Proposition 2.3.5) and the third formula in Proposition 2.4.5. The third equality follows from $(d\log\underline{z})\partial^Y_\mu=0$ and the second formula in Proposition 2.4.5. □

Proof of Theorem 3.1.2. (1) Let $\mathcal{C}^\bullet$ be the complex as in Lemma 3.1.3. Then what we want to prove is $\mathcal{H}^{-q}(\mathcal{D}_{Y\leftarrow X/B}\otimes_{\mathcal{D}_{X/B}}\mathcal{C}^\bullet)=0$ for $q\geqslant1$. Since the question is étale local on $\mathring{Y}$, we may assume that the assumption before Lemma 3.1.5 is satisfied. Then the isomorphisms $\mathcal{O}_X\cong\omega_X; a\mapsto a\,d\log\underline{t}$ and $\mathcal{O}_Y\cong\omega_Y: b\mapsto b\,d\log\underline{z}$ induce isomorphisms of left $\mathcal{D}_{Y/B}$-modules:

$$\mathcal{D}_{Y\leftarrow X/B}\otimes_{\mathcal{D}_{X/B}}\mathcal{C}^{-q}\cong\mathcal{D}_{Y\leftarrow X/B}\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^qT_{X/B}\cong\mathcal{D}_{Y/B}\otimes_{\mathcal{O}_Y}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^qT_{X/B}.$$

By Lemma 3.1.5, the differential maps are explicitly described as

$$\begin{aligned}\mathcal{D}_{Y/B}\otimes_{\mathcal{O}_Y}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^q T_{X/B} &\longrightarrow \mathcal{D}_{Y/B}\otimes_{\mathcal{O}_Y}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^{q-1}T_{X/B}\\ P\otimes e\otimes \xi_{\nu_1}\wedge\cdots\wedge\xi_{\nu_q} &\mapsto \sum_{r=1}^{q}(-1)^{r-1}(P\cdot P_{\nu_r}\otimes e - P\otimes\partial_{\nu_r}e)\otimes\xi_{\nu_1}\wedge\cdots\wedge\hat{\xi}_{\nu_r}\wedge\cdots\wedge\xi_{\nu_q}.\end{aligned}$$

Similarly as the proof of Lemma 3.1.3(2), we can define an increasing filtration F_n $(n\in\mathbb{Z})$ of the complex by putting

$$F_n(\mathcal{D}_{Y/B}\otimes_{\mathcal{O}_Y}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^q T_{X/B}) = \mathcal{D}_{Y/B,n-q}\otimes_{\mathcal{O}_Y}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^q T_{X/B}\quad \text{for } q\in\mathbb{N}.$$

Its graded quotient is $\mathrm{Sym}^\bullet_{\mathcal{O}_X}\alpha^*(T_{Y/B})\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^q T_{X/B}$ with the differential maps

$$\begin{aligned}\mathrm{Sym}^\bullet_{\mathcal{O}_X}\alpha^*(T_{Y/B})\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^q T_{X/B} &\longrightarrow \mathrm{Sym}^\bullet_{\mathcal{O}_X}\alpha^*(T_{Y/B})\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^{q-1}T_{X/B}\\ x\otimes e\otimes\xi_{\nu_1}\wedge\cdots\wedge\xi_{\nu_q} &\mapsto \sum_{r=1}^{q}(-1)^{r-1}\alpha_*(\xi_{\nu_r})\cdot x\otimes e\otimes\xi_{\nu_1}\wedge\cdots\wedge\hat{\xi}_{\nu_r}\wedge\cdots\wedge\xi_{\nu_q},\end{aligned}$$

which is isomorphic to the Koszul complex of the locally free $\mathrm{Sym}^\bullet_{\mathcal{O}_X}\alpha^*(T_{Y/B})$-module of finite type $\mathrm{Sym}^\bullet_{\mathcal{O}_X}\alpha^*(T_{Y/B})\otimes_{\mathcal{O}_X}\mathcal{E}$ with respect to the sequence $\alpha_*(\xi_{\nu_0}),\ldots,\alpha_*(\xi_d)$. Hence, from Lemma 3.1.4, we obtain $\mathcal{H}^{-q}(\mathcal{D}_{Y\leftarrow X/B}\otimes_{\mathcal{D}_{X/B}}\mathcal{C}^\bullet)=0$ for $q>0$ in the same way as the last argument of the proof of Lemma 3.1.3(2).

(2) It suffices to prove that

$$\mathcal{D}_{Y\leftarrow X/B}\otimes_{\mathcal{D}_{X/B}}\mathcal{C}^{-q}\cong(\mathcal{D}_{Y/B}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_Y}\omega_X\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^q T_{X/B}$$

is perfect as an object of $D(\mathcal{D}_{Y/B}\text{-}\mathbf{Mod})$ (cf. [Ill71, Proposition 4.10]). Since $\omega_X\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_X}\wedge^q T_{X/B}$ is a locally free $\mathcal{O}_X$-module of finite type and $\mathcal{D}_{Y/B}\otimes_{\mathcal{O}_Y}\omega_Y^{-1}$ is flat as a right $\mathcal{O}_Y$-module, it suffices to prove that $\mathcal{O}_X$ is perfect as an object of $D(\mathcal{O}_Y\text{-}\mathbf{Mod})$. Since $\mathring{X}\to S$ is flat and locally a complete intersection and $\mathring{Y}\to S$ is smooth, the closed immersion $\mathring{\alpha}\colon\mathring{X}\to\mathring{Y}$ is regular. Hence, the Koszul complex associated to a regular system of generators of the ideal defining $\mathring{\alpha}$, which exists Zariski locally, gives a resolution of $\mathcal{O}_X$ of finite length by locally free $\mathcal{O}_Y$-modules of finite type. □

We define the increasing filtration $F_n\mathcal{D}_{Y\leftarrow X/B}$ $(n\in\mathbb{Z})$ of $\mathcal{D}_{Y\leftarrow X/B}$ by

$$F_n\mathcal{D}_{Y\leftarrow X/B} := \alpha^*(\mathcal{D}_{Y/B,n}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_X}\omega_X.$$

We have

$$\alpha^{-1}(\mathcal{D}_{Y/B,m})\cdot F_n\mathcal{D}_{Y\leftarrow X/B}\subset F_{n+m}\mathcal{D}_{Y\leftarrow X/B}.$$

For a left $\mathcal{D}_{X/B}$-module $\mathcal{E}$ such that the underlying $\mathcal{O}_X$-module is locally free of finite type, we define the increasing filtration $F_n\alpha_{+/B}(\mathcal{E})$ of $\alpha_{+/B}(\mathcal{E})$ to be the image of $\alpha_*(F_n\mathcal{D}_{Y\leftarrow X/B}\otimes_{\mathcal{O}_X}\mathcal{E})$. We have

$$\mathcal{D}_{Y/B,n}\cdot F_m(\alpha_{+/B}(\mathcal{E}))\subset F_{n+m}(\alpha_{+/B}(\mathcal{E})).$$

Hence, $\mathrm{gr}^F_\bullet(\alpha_{+/B}(\mathcal{E}))$ is a graded $\mathrm{gr}_\bullet\,\mathcal{D}_{Y/B}=\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/B}$-module.

We have an isomorphism

$$\mathrm{gr}^F_\bullet\,\mathcal{D}_{Y\leftarrow X/B}\cong\alpha^*(\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/B}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_X}\omega_X.$$

Hence, by definition, we have a natural epimorphism of $\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/B}$-modules:

$$\alpha_*(\alpha^*(\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/B}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_X}\omega_X\otimes_{\mathcal{O}_X}\mathcal{E})\longrightarrow\mathrm{gr}^F_\bullet(\alpha_{+/B}(\mathcal{E})).\tag{3.1.6}$$

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Proposition 3.1.7. *Under the notation and assumptions as above, the homomorphism (3.1.6) induces an isomorphism*

$$\alpha_*((\alpha^*(\mathrm{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/B}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_X}\omega_X)\otimes_{\mathrm{Sym}^\bullet_{\mathcal{O}_X}T_{X/B}}\mathcal{E})\xrightarrow{\cong}\mathrm{gr}^F_\bullet(\alpha_{+/B}(\mathcal{E})).$$

Here we regard $\alpha^*(\mathrm{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/B}\otimes_{\mathcal{O}_Y}\omega_Y^{-1})\otimes_{\mathcal{O}_X}\omega_X$ *(respectively* $\mathcal{E}$*) as a* $\mathrm{Sym}^\bullet_{\mathcal{O}_X}T_{X/B}$*-module via the homomorphism* $\alpha_*\colon \mathrm{Sym}^\bullet_{\mathcal{O}_X}T_{X/B}\to\alpha^*(\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/B})$ *(respectively the projection* $\mathrm{Sym}^\bullet_{\mathcal{O}_X}T_{X/B}\to\mathcal{O}_X$*).*

Proof. The question is étale local on $\mathring{Y}$. Let $\mathcal{C}^\bullet$ be the complex defined in Lemma 3.1.3 and define the filtration F_n of the complex $\mathcal{C}^\bullet_Y:=\mathcal{D}_{Y\leftarrow X/B}\otimes_{\mathcal{D}_{X/B}}\mathcal{C}^\bullet$ as in the proof of Theorem 3.1.2(1). The vanishing of $\mathcal{H}^{-1}(\mathrm{gr}^F_m\,\mathcal{C}^\bullet_Y)$ shown in the proof of Theorem 3.1.2(1) implies that $\mathcal{H}^0(F_{m-1}\mathcal{C}^\bullet_Y))\to\mathcal{H}^0(F_m\mathcal{C}^\bullet_Y)$ is injective. Hence,

$$\mathcal{H}^0(F_n\mathcal{C}^\bullet_Y)\to\varinjlim_m\mathcal{H}^0(F_m\mathcal{C}^\bullet_Y)=\mathcal{H}^0(\mathcal{C}^\bullet_Y)=\alpha_{+/B}(\mathcal{E})$$

is injective. Since $F_n(\alpha_{+/B}(\mathcal{E}))$ is the image of $F_n\mathcal{C}^0_Y$, this implies that

$$\mathcal{H}^0(F_n\mathcal{C}^\bullet_Y)\xrightarrow{\cong}F_n(\alpha_{+/B}(\mathcal{E}))\quad\text{and}\quad\mathcal{H}^0(\mathrm{gr}^F_n\,\mathcal{C}^\bullet_Y)\xrightarrow{\cong}\mathrm{gr}^F_n(\alpha_{+/B}(\mathcal{E})).$$

From the explicit description of the differential map of $\mathrm{gr}^F\,\mathcal{C}^\bullet_Y$ in the proof of Theorem 3.1.2(1), we obtain the desired isomorphism. □

Example 3.1.8. Let X be a scheme $\mathrm{Spec}(W_N[t_1,t_2,\dots,t_{d+1}]/(t_1t_2\cdots t_{d+1}))$ endowed with the fine log structure associated to

$$\mathbb{N}^{d+1}\to W_N[t_1,t_2,\dots,t_{d+1}]/(t_1t_2\cdots t_{d+1});\quad(n_1,n_2,\dots,n_{d+1})\mapsto t_1^{n_1}t_2^{n_2}\cdots t_{d+1}^{n_{d+1}}$$

and define a smooth morphism of fine log schemes $X\to T$ by $\mathbb{N}\to\mathbb{N}^{d+1};n\mapsto(n,n,\dots,n)$. Let Y be $\mathrm{Spec}(W_N[s_1,s_2,\dots,s_{d+1}])$ endowed with the inverse image of M_T and define a T-morphism $\alpha\colon X\to Y$ by $s_i\mapsto t_i$ $(1\leqslant i\leqslant d+1)$. Then X and α satisfy the condition in the beginning of § 3.

Let $d\log(\underline{t})$ (respectively $d\underline{s}$) denote the basis $d\log t_1\wedge d\log t_2\wedge\cdots\wedge d\log t_d$ (respectively $ds_1\wedge ds_2\wedge\cdots\wedge ds_{d+1}$) of $\omega_X=\Omega^d_{X/T}$ (respectively $\omega_Y=\Omega^{d+1}_{Y/T}$). Let $\xi_0,\xi_1,\dots,\xi_d\in T_{X/S}$ (respectively $\xi^{Y,\log}_t,\xi^Y_1,\dots,\xi^Y_{d+1}\in T_{Y/S}$) be the dual basis of $d\log t,d\log t_1,\dots,d\log t_d\in\Omega^1_{X/S}$ (respectively $d\log t,ds_1,\dots,ds_{d+1}\in\Omega^1_{Y/S}$) and let $\partial_0,\partial_1,\dots,\partial_d\in\mathcal{D}_{X/S}$ (respectively $\partial^{Y,\log}_t,\partial^Y_1,\dots,\partial^Y_{d+1}\in\mathcal{D}_{Y/S}$) denote the corresponding elements. Note that $\partial_1,\dots,\partial_d\in\mathcal{D}_{X/T}$ (respectively $\partial^Y_1,\dots,\partial^Y_{d+1}\in\mathcal{D}_{Y/T}$) and $\partial^{Y,\log}_t$ coincides with $\partial^{\log}_t$ considered in Proposition 2.1.11.

We have an isomorphism

$$\mathcal{D}_{X/S}\Big/\sum_{i=0}^d\mathcal{D}_{X/S}\partial_i\xrightarrow{\cong}\mathcal{O}_X;P\mapsto P\cdot1,$$
$$\mathcal{D}_{X/T}\Big/\sum_{i=1}^d\mathcal{D}_{X/T}\partial_i\xrightarrow{\cong}\mathcal{O}_X;P\mapsto P\cdot1.$$

Hence, we have

$$\alpha_{+/S}\mathcal{O}_X=\mathcal{D}_{Y\leftarrow X/S}\Big/\sum_{i=0}^d\mathcal{D}_{Y\leftarrow X/S}\partial_i,\quad\alpha_{+/T}\mathcal{O}_X=\mathcal{D}_{Y\leftarrow X/T}\Big/\sum_{i=1}^d\mathcal{D}_{Y\leftarrow X/T}\partial_i.$$

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Since the images of ξ_0, ξ_i $(1 \leqslant i \leqslant d)$ under the homomorphism $\alpha_*\colon T_{X/S} \to T_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_X$ are

$$\xi_t^{Y,\log} \otimes 1 + s_{d+1}\xi_{d+1}^Y \otimes 1, \quad s_i\xi_i^Y \otimes 1 - s_{d+1}\xi_{d+1}^Y \otimes 1,$$

the isomorphism of left $\mathcal{D}_{Y/B}$-modules

$$\mathcal{D}_{Y/B} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{D}_{Y\leftarrow X/B} = \mathcal{D}_{Y/B} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \omega_X; P \otimes 1 \mapsto P \otimes (d\underline{s})^{-1} \otimes d\log \underline{t}$$

induces isomorphisms (cf. Lemma 3.1.5)

$$\alpha_{+/S}\mathcal{O}_X \cong \mathcal{D}_{Y/S}\Big/\bigg(\mathcal{D}_{Y/S}s_1\cdots s_{d+1} + \mathcal{D}_{Y/S}(\partial_t^{\log} + \partial_{d+1}^Y s_{d+1}) + \sum_{i=1}^d \mathcal{D}_{Y/S}(\partial_i^Y s_i - \partial_{d+1}^Y s_{d+1})\bigg),$$

$$\alpha_{+/T}\mathcal{O}_X \cong \mathcal{D}_{Y/T}\Big/\bigg(\mathcal{D}_{Y/T}s_1\cdots s_{d+1} + \sum_{i=1}^d \mathcal{D}_{Y/T}(\partial_i^Y s_i - \partial_{d+1}^Y s_{d+1})\bigg).$$

The natural isomorphism $\alpha_{+/T}\mathcal{O}_X \cong \alpha_{+/S}\mathcal{O}_X$ as $\mathcal{D}_{Y/T}$-modules (Corollary 2.6.10) corresponds to the morphism induced by the canonical morphism $\mathcal{D}_{Y/T} \to \mathcal{D}_{Y/S}$ (cf. (2.1.8)).

The description of $\alpha_{+/T}\mathcal{O}_X$ above is of the same form as the case of complex analytic varieties. The description of $\alpha_{+/S}\mathcal{O}_X$ implies that the action of $\partial_t^{\log}$ on the class $[P]$ of $P \in \mathcal{D}_{Y/T} = \mathcal{D}_{\mathring{Y}/S}$ is given by

$$\partial_t^{\log}([P]) = -[P\partial_{d+1}^Y s_{d+1}] \quad (= -[P\partial_i^Y s_i], 1 \leqslant i \leqslant d).$$

3.2 Another local explicit description of $\mathcal{D}_{X/S}$

In this subsection, we assume that there exist $t_1, \ldots, t_{d+1} \in \Gamma(X, M_X)$ as in the beginning of § 3. Define ∂_ν $(0 \leqslant \nu \leqslant d)$ as in Proposition 2.1.5.

Since $d\log(t) = \sum_{1\leqslant\nu\leqslant d+1} d\log(t_d)$, the set

$$\{d\log(t_\nu) \mid 1 \leqslant \nu \leqslant d+1\}$$

is a basis of $\Omega^1_{X/S}$. Let

$$\tilde{\xi}_\nu \in T_{X/S} = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) \quad (1 \leqslant \nu \leqslant d+1)$$

be its dual basis, and let $\tilde{\partial}_\nu \in \mathcal{D}_{X/S,1}$ be the composite of

$$\mathcal{P}_{X/S,1} = p_1^{-1}(\mathcal{O}_X) \oplus \Omega^1_{X/S} \xrightarrow{\text{proj}} \Omega^1_{X/S} \xrightarrow{\tilde{\xi}_\nu} \mathcal{O}_X.$$

From the equality $d\log(t) = \sum_{1\leqslant\nu\leqslant d+1} d\log(t_\nu)$, we obtain

$$\begin{cases}\tilde{\partial}_\nu = \partial_\nu + \partial_0 & (1 \leqslant \nu \leqslant d)\\ \tilde{\partial}_{d+1} = \partial_0, \end{cases} \qquad \begin{cases}\partial_0 = \tilde{\partial}_{d+1} \\ \partial_\nu = \tilde{\partial}_\nu - \tilde{\partial}_{d+1} & (1 \leqslant \nu \leqslant d).\end{cases} \tag{3.2.1}$$

Hence, Proposition 2.1.5 and Corollary 2.1.6 imply the following analogues for $\tilde{\partial}_\nu$.

Proposition 3.2.2. *Let the notation and assumptions be as above.*

(1) *We have* $\tilde{\partial}_\nu\tilde{\partial}_\mu = \tilde{\partial}_\mu\tilde{\partial}_\nu$ *for* $1 \leqslant \nu, \mu \leqslant d+1$.

(2) *We have* $\tilde{\partial}_\nu \cdot x = \tilde{\partial}_\nu(x) + x \cdot \tilde{\partial}_\nu$ *for* $x \in \mathcal{O}_X$.

(3) *For* $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$ *and* $x \in \mathcal{O}_X$, *we have* $d(x) = \sum_{1\leqslant\nu\leqslant d+1} \tilde{\partial}_\nu(x)\, d\log(t_\nu)$.

(4) *For either of the left $\mathcal{O}_X$-action or the right $\mathcal{O}_X$-action on $\mathcal{D}_{X/S,n}$, $\mathcal{D}_{X/S,n}$ is a free $\mathcal{O}_X$-module with a basis*

$$\left\{ \prod_{1\leqslant\nu\leqslant d+1} \widetilde{\partial}_\nu^{n_\nu} \,\middle|\, n_\nu \in \mathbb{N}, \sum_{1\leqslant\nu\leqslant d+1} n_\nu \leqslant n \right\}.$$

We can also derive the following analogues from Propositions 2.3.5 and 2.4.5.

PROPOSITION 3.2.3. *Under the notation and assumptions as above, we have*

$$(x\, d\log t_1 \wedge d\log t_2 \wedge \cdots \wedge d\log t_d)\widetilde{\partial}_\nu = -\widetilde{\partial}_\nu(x)\, d\log t_1 \wedge d\log t_2 \wedge \cdots \wedge d\log t_d$$

for $x \in \mathcal{O}_X$ and $1 \leqslant \nu \leqslant d+1$.

PROPOSITION 3.2.4. *Under the notation and assumptions as above, Proposition 2.4.5 still holds for $B = S$ and $\widetilde{\partial}_\nu$ $(1 \leqslant \nu \leqslant d+1)$.*

3.3 Weight filtration

We define the increasing filtration $P_n\omega_X$ $(n \in \mathbb{Z})$ of $\omega_X = \Omega^d_{X/T}$ by

$$P_n\omega_X := \{\omega \wedge \omega_1 \wedge \cdots \wedge \omega_{d-n} \mid \omega \in \Omega^n_{X/T}, \omega_1, \ldots, \omega_{d-n} \in \Omega^1_{\mathring{X}/S}\} \quad (0 \leqslant n \leqslant d)$$

and $P_{-1}\omega_X = 0$. For a local coordinate $t_1, \ldots, t_{d+1} \in \Gamma(X, M_X)$ as in the beginning of § 3, we have

$$P_n\omega_X = \sum_{\substack{J\subset\{1,2,\ldots,d+1\} \\ |J|=d-n}} \underline{t}_J \cdot \omega_X \tag{3.3.1}$$

for $-1 \leqslant n \leqslant d$. Here $\underline{t}_J = \prod_{\mu\in J} t_\mu$ if $J \neq \emptyset$ and $\underline{t}_\emptyset = 1$.

Since $\mathcal{D}_{Y/S}$ is locally free for the right action of $\mathcal{O}_Y$, the natural homomorphism

$$\mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} P_n\omega_X \longrightarrow \mathcal{D}_{Y\leftarrow X/S}$$

is injective. We define $Q_n\mathcal{D}_{Y\leftarrow X/S}$ to be its image. Using (3.3.1) and the formula $t_\mu \cdot \widetilde{\partial}_\nu = (\widetilde{\partial}_\nu - \delta_{\nu\mu}) \cdot t_\mu$, where $\delta_{\nu\nu} = 1$ and $\delta_{\nu\mu} = 0$ for $\nu \neq \mu$, we see that $Q_n\mathcal{D}_{Y\leftarrow X/S}$ $(n \in \mathbb{Z})$ is an increasing filtration of $\mathcal{D}_{Y\leftarrow X/S}$ by $(\mathcal{D}_{Y/S}, \mathcal{D}_{X/S})$-subbimodules.

Let $\mathcal{E}$ be a left $\mathcal{D}_{X/S}$-module such that the underlying $\mathcal{O}_X$-module is locally free of finite type, and let $\mathcal{F}$ denote its nearby cycles realized on Y: $\alpha_{+/S}(\mathcal{E}) = \mathcal{D}_{Y\leftarrow X/S} \otimes_{\mathcal{D}_{X/S}} \mathcal{E}$ (cf. Theorem 3.1.2(1)). We define the increasing filtration $Q_n\mathcal{F}$ $(n \in \mathbb{Z})$ of $\mathcal{F}$ by left $\mathcal{D}_{Y/S}$-submodules to be the image of $Q_n\mathcal{D}_{Y\leftarrow X/S} \otimes_{\mathcal{D}_{X/S}} \mathcal{E}$. We have $Q_{-1}\mathcal{F} = 0$ and $Q_d\mathcal{F} = \mathcal{F}$. We define the increasing filtration $F_n\mathcal{F}$ $(n \in \mathbb{Z})$ of $\mathcal{F} = \alpha_{+/S}(\mathcal{E})$ as before Proposition 3.1.7 and the increasing filtration $Q_n \operatorname{gr}^F_\bullet \mathcal{F}$ of $\operatorname{gr}^F_\bullet \mathcal{F}$ by graded $\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/S}$-submodules to be the image of

$$\alpha_*((\alpha^*(\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/S} \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_X} P_n\omega_X) \otimes_{\operatorname{Sym}^\bullet_{\mathcal{O}_X} T_{X/S}} \mathcal{E}).$$

DEFINITION 3.3.2. We say that $\mathcal{E}$ *has trivial monodromy* if the composite

$$\mathcal{E} \xrightarrow{\theta_1} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S} \cong \mathcal{E} \otimes_{\mathcal{O}_X} (p_1^{-1}(\mathcal{O}_X) \oplus \Omega^1_{X/S}) \xrightarrow{\text{proj}} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

factors through the image of $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{\mathring{X}/S}$. Here θ_1 denotes the homomorphism associated to the left $\mathcal{D}_{X/S}$-action on $\mathcal{E}$ (Proposition 2.2.3).

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Let $\partial_t^{\log} \in \mathcal{D}_{Y/S}$ be the differential operator obtained by applying to Y/S the construction before Proposition 2.1.11 and let $\xi_t^{\log}$ be the corresponding section of $T_{Y/S}$.

Proposition 3.3.3. *Assume that $\mathcal{E}$ has trivial monodromy.*

(1) *We have $\partial_t^{\log}(Q_n\mathcal{F}) \subset Q_{n-1}\mathcal{F}$ for $n \in \mathbb{Z}$. Especially, $(\partial_t^{\log})^{d+1}(\mathcal{F}) = 0$.*

(2) *We have $\xi_t^{\log} \cdot Q_n \operatorname{gr}^F_\bullet \mathcal{F} \subset Q_{n-1} \operatorname{gr}^F_\bullet \mathcal{F}$ for $n \in \mathbb{Z}$. Especially, $(\xi_t^{\log})^{d+1} \operatorname{gr}^F_\bullet \mathcal{F} = 0$.*

Proof. Since the question is étale local on $\overset{\circ}{Y}$, we may assume that there exist $t_1, \dots, t_{d+1}$ as in the beginning of § 3, $\overset{\circ}{Y}$ is affine, and there exist $z_1, \dots, z_e \in \Gamma(Y, \mathcal{O}_Y^\times)$ as before Lemma 3.1.5. We define $\widetilde{\partial}_\nu$, $\widetilde{\xi}_\nu$ $(1 \leqslant \nu \leqslant d+1)$ as in § 3.2 and ∂_μ^Y, ξ_μ^Y $(0 \leqslant \mu \leqslant e)$ as before Lemma 3.1.5 for $B = S$. Note that $\partial_0^Y = \partial_t^{\log}$ and $\xi_0^Y = \xi_t^{\log}$. Using $\alpha^*(d\log t) = d\log t_1 + \dots + d\log t_{d+1}$, we obtain $\alpha_*(\widetilde{\xi}_\nu) = 1 \otimes \xi_0^Y + \sum_{1 \leqslant \mu \leqslant e} \widetilde{\partial}_\nu \log(\alpha^*(z_\mu)) \otimes \xi_\mu^Y$. Choose $\widetilde{a}_{\nu\mu} \in \Gamma(Y, \mathcal{O}_Y)$ such that $\widetilde{\partial}_\nu \log(\alpha^*(z_\mu)) = \alpha^*(\widetilde{a}_{\nu\mu})$ for $1 \leqslant \nu \leqslant d+1$ and $1 \leqslant \mu \leqslant e$, and put $\widetilde{P}_\nu := \partial_0^Y + \sum_{1 \leqslant \mu \leqslant e} \partial_\mu^Y \cdot \widetilde{a}_{\nu\mu}$. Let $d\log \underline{t}$ and $d\log \underline{z}$ be as before Lemma 3.1.5 and put $\omega_{\underline{t}/\underline{z}} := (d\log \underline{z})^{-1} \otimes\ d\log \underline{t} \in \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \omega_X$ to simplify the notation.

(1) By using Propositions 3.2.3 and 3.2.4, we see that the action of $\widetilde{\partial}_\nu$ on $\mathcal{D}_{Y \leftarrow X/S}$ is given by the formulae $(P \otimes \omega_{\underline{t}/\underline{z}})\widetilde{\partial}_\nu = P \cdot \widetilde{P}_\nu \otimes \omega_{\underline{t}/\underline{z}}$, $P \in \mathcal{D}_{Y/S}$ (cf. Lemma 3.1.5). By (3.3.1), $Q_n\mathcal{F}$ $(-1 \leqslant n \leqslant d)$ coincides with the sum of the images of the submodules $\mathcal{D}_{Y \leftarrow X/S} \otimes_{\mathcal{O}_X} \underline{t}_J \mathcal{E}$ of $\mathcal{D}_{Y \leftarrow X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ for $J \subset \{1, 2, \dots, d+1\}$ such that $|J| = d - n$. For $0 \leqslant n \leqslant d$, $J \subset \{1, 2, \dots, d+1\}$ such that $|J| = d - n$, $P \in \mathcal{D}_{Y/S}$, and $\nu \in \{1, \dots, d+1\} \backslash J$, we have the following equalities in $\mathcal{F}$:

$$\begin{aligned}
\partial_t^{\log}(P \otimes \omega_{\underline{t}/\underline{z}} \otimes \underline{t}_J e) &= P\partial_Y^0 \otimes \omega_{\underline{t}/\underline{z}} \otimes \underline{t}_J e \\
&= P\biggl(\widetilde{P}_\nu - \sum_{1 \leqslant \mu \leqslant e} \partial_\mu^Y \widetilde{a}_{\nu\mu}\biggr) \otimes \omega_{\underline{t}/\underline{z}} \otimes \underline{t}_J e \\
&= -\sum_{1 \leqslant \mu \leqslant e} P\partial_\mu^Y \otimes \omega_{\underline{t}/\underline{z}} \otimes \widetilde{\partial}_\nu \log(\alpha^*(z_\mu)) \cdot \underline{t}_J e + P \otimes \omega_{\underline{t}/\underline{z}} \otimes \widetilde{\partial}_\nu(\underline{t}_J e).
\end{aligned}$$

Since $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$ (respectively $d\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$) factors through the image of $\Omega^1_{\overset{\circ}{X}/S}$ (respectively $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{\overset{\circ}{X}/S}$) and $\Omega^1_{\overset{\circ}{X}/S}$ is generated by $dt_1, \dots, dt_{d+1}$, we have $\widetilde{\partial}_\nu(\mathcal{O}_X) \subset t_\nu \mathcal{O}_X$ (respectively $\widetilde{\partial}_\nu(\mathcal{E}) \subset t_\nu \mathcal{E}$). Hence, $\partial_t^{\log}(P \otimes \omega_{\underline{t}/\underline{z}} \otimes \underline{t}_J e)$ is contained in the image of $\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{O}_X} \underline{t}_J \cdot t_\nu \mathcal{E}$. Note that $\widetilde{\partial}_\nu \cdot \underline{t}_J = \underline{t}_J \cdot \widetilde{\partial}_\nu$.

(2) By (3.3.1), $Q_n \operatorname{gr}^F_\bullet \mathcal{E}$ $(-1 \leqslant n \leqslant d)$ is the sum of the images of $\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/S} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \omega_X \otimes_{\mathcal{O}_X} \underline{t}_J \mathcal{E}$ for $J \subset \{1, 2, \dots, d+1\}$ such that $|J| = d - n$. For $0 \leqslant n \leqslant d$, $J \subset \{1, 2, \dots, d+1\}$ such that $|J| = d - n$, $x \in \operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/S}$, and $\nu \in \{1, \dots, d+1\} \backslash J$, we have the following equalities in $\operatorname{gr}^F_\bullet \mathcal{F}$ by Proposition 3.1.7:

$$0 = (\alpha_*(\widetilde{\xi}_\nu)(x \otimes \omega_{\underline{t}/\underline{z}})) \otimes \underline{t}_J e = (\xi_0^Y x \otimes \omega_{\underline{t}/\underline{z}}) \otimes \underline{t}_J e + \sum_{1 \leqslant \mu \leqslant e} (\xi_\mu^Y x \otimes \omega_{\underline{t}/\underline{z}}) \otimes \widetilde{\partial}_\nu \log(\alpha^*(z_\mu)) \cdot \underline{t}_J e.$$

Hence, $\widetilde{\partial}_\nu(\mathcal{O}_X) \subset t_\nu \mathcal{O}_X$ implies that $\xi_t^{\log}(x \otimes \omega_{\underline{t}/\underline{z}} \otimes \underline{t}_J e)$ is contained in the image of

$$\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/S} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \omega_X \otimes_{\mathcal{O}_X} \underline{t}_J \cdot t_\nu \mathcal{E}. \qquad \square$$

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We define the $\mathcal{D}_{Y/S}$-submodules $W_n\mathcal{F}$ $(-d\leqslant n\leqslant d)$ of $\mathcal{F}$ by

$$W_n\mathcal{F}=\sum_{\substack{b-2a=n\\0\leqslant a\leqslant b\leqslant d}}(\partial_t^{\log})^aQ_b\mathcal{F}. \tag{3.3.4}$$

We have $W_d\mathcal{F}=Q_d\mathcal{F}=\mathcal{F}$. We put $W_n\mathcal{F}=0$ for $n\leqslant -d-1$ and $W_n\mathcal{F}=\mathcal{F}$ for $n\geqslant d+1$. Similarly, we define the graded $\mathrm{Sym}^{\bullet}_{\mathcal{O}_Y}T_{Y/S}$-submodules $W_n\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}$ $(-d\leqslant n\leqslant d)$ of $\mathrm{gr}^F_{\bullet}\,\mathcal{F}$ by

$$W_n\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}=\sum_{\substack{b-2a=n\\0\leqslant a\leqslant b\leqslant d}}(\xi_t^{\log})^aQ_b\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}. \tag{3.3.5}$$

We have $W_d\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}=Q_d\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}=\mathrm{gr}^F_{\bullet}\,\mathcal{F}$. We put $W_n\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}=0$ for $n\leqslant -d-1$ and $W_n\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}=\mathcal{F}$ for $n\geqslant d+1$.

LEMMA 3.3.6. *Assume that $\mathcal{E}$ has trivial monodromy.*

(1) *We have $W_{n-1}\mathcal{F}\subset W_n\mathcal{F}$ and $\partial_t^{\log}(W_n\mathcal{F})\subset W_{n-2}\mathcal{F}$ for $n\in\mathbb{Z}$.*

(2) *We have $W_{n-1}\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}\subset W_n\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}$ and $\xi_t^{\log}(W_n\,\mathrm{gr}^F_{\bullet}\,\mathcal{F})\subset W_{n-2}\,\mathrm{gr}^F_{\bullet}\,\mathcal{F}$ for $n\in\mathbb{Z}$.*

Proof. (1) For the first claim, we may assume that $-d+1\leqslant n\leqslant d$. Let a and b be integers such that $b-2a=n-1$ and $0\leqslant a\leqslant b\leqslant d$. If $a\geqslant 1$, we have

$$(b-1)-2(a-1)=n,\quad 0\leqslant a-1\leqslant b-1\leqslant d$$

and

$$(\partial_t^{\log})^aQ_b\mathcal{F}\subset(\partial_t^{\log})^{a-1}Q_{b-1}\mathcal{F}\subset W_n\mathcal{F}$$

by Proposition 3.3.3(1). If $a=0$, we have $0\leqslant b+1=n\leqslant d$ and $Q_b\mathcal{F}\subset Q_{b+1}\mathcal{F}\subset W_n\mathcal{F}$. Hence, $W_{n-1}\mathcal{F}\subset W_n\mathcal{F}$. For the second claim, we may assume that $-d\leqslant n\leqslant d$. Let a and b be integers such that $b-2a=n$ and $0\leqslant a\leqslant b\leqslant d$. If $a+1\leqslant b$, then $0\leqslant a+1\leqslant b\leqslant d$, $n-2=b-2(a+1)(\geqslant -d)$, and

$$\partial_t^{\log}((\partial_t^{\log})^aQ_b\mathcal{F})=(\partial_t^{\log})^{a+1}Q_b\mathcal{F}\subset W_{n-2}\mathcal{F}.$$

If $a=b$, then $\partial_t^{\log}((\partial_t^{\log})^aQ_b\mathcal{F})=0$ by Proposition 3.3.3(1). Hence, $\partial_t^{\log}(W_n\mathcal{F})\subset W_{n-2}\mathcal{F}$.

(2) Just replace $\partial_t^{\log}$, $\mathcal{F}$, and Proposition 3.3.3(1) with $\xi_t^{\log}$, $\mathrm{gr}^F_{\bullet}\,\mathcal{F}$, and Proposition 3.3.3(2) respectively in the proof of (1) above. □

3.4 The graded quotient of the weight filtration

Let $X_{0,\lambda}$ $(\lambda\in\Lambda)$ be the irreducible components of $\mathring{X}\times_S S_0$. We choose and fix a total order of the finite set Λ. We assume that $X_{0,\lambda}$ are smooth over S_0 for all $\lambda\in\Lambda$. We define the sheaf of ideals $\mathcal{I}_\lambda$ of $\mathcal{M}_X$ to be the inverse image of 0 under the morphism $M_X\to\mathcal{O}_X\to\mathcal{O}_{X_{0,\lambda}}$.

LEMMA 3.4.1. *Let $U\to\mathring{X}$ be an étale morphism and assume that there exist $t_1,\dots,t_{d+1}\in\Gamma(U,M_X)$ satisfying the conditions in the beginning of § 3 for $(U,M_X|_U)$ such that $U_{0,\lambda}=X_{0,\lambda}\times_{\mathring{X}}U$ is defined by the ideal $t_1\mathcal{O}_{U_0}$, where $U_0=U\times_S S_0$. Then $\mathcal{I}_\lambda|_U=t_1\cdot M_X|_U$.*

Proof. The inclusion $t_1\cdot M_X|_U\subset\mathcal{I}_\lambda$ is trivial. Let x be an element of $\Gamma(U,\mathcal{I}_\lambda)$. Choose an étale covering $\{U_\alpha\to U\}$ of U such that $a|_{U_\alpha}$ is written in the form $t_1^{n_{1\alpha}}\cdots t_{d+1}^{n_{d+1\alpha}}u_\alpha$, $n_{\nu\alpha}\in\mathbb{N}$, $u_\alpha\in\Gamma(U_\alpha,\mathcal{O}^\times_{U_\alpha})$. Put $U_{\alpha,0,\lambda}:=U_\alpha\times_U U_{0,\lambda}$. Since $U_{\alpha,0,\lambda}$ is étale over $\mathrm{Spec}(k[s_1,\dots,s_{d+1}]/(s_1\cdots s_{d+1},s_1))=\mathrm{Spec}(k[s_2,\dots,s_{d+1}])$, we have $n_{1\alpha}>0$ (respectively $t_1\in\Gamma(U_\alpha,\mathcal{O}^\times_{U_\alpha})$) if $U_{\alpha,0,\lambda}\neq\emptyset$ (respectively $U_{\alpha,0,\lambda}=\emptyset$). Hence, $a|_{U_\alpha}\in t_1\cdot\Gamma(U_\alpha,M_X)$. □

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By Lemma 3.4.1, $\mathcal{I}_\lambda \cdot \mathcal{O}_X$ is a coherent ideal of $\mathcal{O}_X$ and the closed subscheme X_λ of $\mathring{X}$ defined by $\mathcal{I}_\lambda \cdot \mathcal{O}_X$ is a smooth lifting of $X_{0,\lambda}$ over S. For a non-empty subset I of Λ, we define $X_{0,I}$ (respectively X_I) to be the fiber product of $X_{0,\lambda}$ (respectively X_λ) $(\lambda \in I)$ over $\mathring{X}_0$ (respectively $\mathring{X}$) endowed with the inverse image of M_{T_0} (respectively M_T). By Lemma 3.4.1, we see that X_I/T is a smooth lifting of $X_{0,I}/T_0$. Let ι_I denote the closed immersion $\mathring{X}_I \to \mathring{X}$.

We define the increasing filtration $P_n\Omega^{d+1}_{X/S}$ $(n \in \mathbb{Z})$ of $\Omega^{d+1}_{X/S}$ by

$$P_n\Omega^{d+1}_{X/S} := \{\omega \wedge \omega_1 \wedge \cdots \wedge \omega_{d+1-n} \mid \omega \in \Omega^n_{X/S}, \omega_1, \ldots, \omega_{d+1-n} \in \Omega^1_{\mathring{X}/S}\} \quad (0 \leqslant n \leqslant d+1).$$

We have $P_{d+1}\Omega^{d+1}_{X/S} = \Omega^{d+1}_{X/S}$ and $P_0\Omega^{d+1}_{X/S} = 0$.

Proposition 3.4.2. *We have the following canonical isomorphism of $\mathcal{O}_X$-modules for an integer $1 \leqslant n \leqslant d+1$:*

$$\bigoplus_{I \subset \Lambda, |I|=n} \Omega^{d+1-n}_{\mathring{X}_I/S} \xrightarrow{\cong} \mathrm{gr}^P_n \Omega^{d+1}_{X/S}.$$

For a non-empty finite subset I of Λ, we define $X_I^{\log}$ to be $\mathring{X}_I$ endowed with the inverse image of the log structure M_X. Since the ideal of $\mathcal{O}_X$ defining the closed subscheme $\mathring{X}_I$ of $\mathring{X}$ is locally generated by elements of M_X, we have an isomorphism $\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_I} \xrightarrow{\cong} \Omega^1_{X_I^{\log}/S}$. Especially, $\Omega^1_{X_I^{\log}/S}$ is a locally free $\mathcal{O}_{X_I}$-module of rank $d+1$. We define $\Omega^q_{X_I^{\log}/S}$ to be $\wedge^q\Omega^1_{X_I^{\log}/S}$. We define $P_0\Omega^q_{X/S}$ to be the $\mathcal{O}_X$-submodule of $\Omega^q_{X/S}$ generated by $\omega_1 \wedge \cdots \wedge \omega_q$ $(\omega_i \in \Omega^1_{\mathring{X}/S})$.

Lemma 3.4.3. *Let I be a non-empty subset of Λ of cardinality $\leqslant d+1$ and put $r = \dim X_I$ $(= d+1-|I|)$. The homomorphism $\Omega^r_{\mathring{X}_I/S} \to \Omega^r_{X_I^{\log}/S}$ is injective, and the homomorphism $\Omega^r_{X/S} \to \Omega^r_{X_I^{\log}/S}$ induces an epimorphism $P_0\Omega^r_{X/S} \to \Omega^r_{\mathring{X}_I/S}$.*

Proof. The question is étale local on $\mathring{X}$ and we may assume that there exist $t_1, \ldots, t_{d+1} \in \Gamma(X, M_X)$ as in the beginning of § 3, and $\mathring{X}_I$ is defined by the ideal of $\mathcal{O}_X$ generated by the images of $t_{r+1}, \ldots, t_{d+1}$. Then $\{d\log t_{\nu_1} \wedge \cdots \wedge d\log t_{\nu_r} \mid 1 \leqslant \nu_1 < \cdots < \nu_r \leqslant d+1\}$ is a basis of $\Omega^r_{X_I^{\log}/S}$ and $dt_1 \wedge \cdots \wedge dt_r$ is a basis of $\Omega^r_{\mathring{X}_I/S}$. Hence, the first claim follows from the fact that $t_1 \cdots t_r$ is not a zero divisor on $\mathcal{O}_{X_I}$. Note that $\mathring{X}_I$ is étale over

$$\mathrm{Spec}(W_N[s_1, \ldots, s_{d+1}]/(s_1 \cdots s_{d+1}, s_{r+1}, \ldots, s_{d+1})).$$

The second claim follows from the fact that the composite of $\Omega^1_{\mathring{X}/S} \to \Omega^1_{X/S} \to \Omega^1_{X_I^{\log}/S}$ coincides with the composite of $\Omega^1_{\mathring{X}/S} \to \Omega^1_{\mathring{X}_I/S} \to \Omega^1_{X_I^{\log}/S}$. □

For a non-empty subset $I = \{i(1), i(2), \ldots, i(n)\}$, $i(1) < i(2) < \cdots < i(n)$ of Λ such that $n \leqslant d+1$, we can verify that an $\mathcal{O}_X$-linear homomorphism

$$P_0\Omega^{d+1-n}_{X/S} \longrightarrow \mathrm{gr}^P_n \Omega^{d+1}_{X/S} \tag{3.4.4}$$

is well defined by the correspondence $\omega \mapsto d\log t_{i(1)} \wedge \cdots \wedge d\log t_{i(n)} \wedge \omega$, where $t_{i(m)}$ is a local section of M_X such that $\mathcal{I}_{i(m)} = t_{i(m)} M_X$.

Proof of Proposition 3.4.2. We prove that the homomorphism (3.4.4) factors through the epimorphism $P_0\Omega^{d+1-n}_{X/S} \to \Omega^{d+1-n}_{\mathring{X}_I/S}$ in Lemma 3.4.3 and induces the isomorphism in the

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proposition. Since the question is étale local on $\mathring{X}$, we may assume that there exist $t_1, \dots, t_{d+1} \in \Gamma(X, M_X)$ as in the beginning of § 3, $\Lambda = \{1, \dots, d+1\}$, and $\mathcal{I}_\lambda = t_\lambda M_X$ for $\lambda \in \Lambda$. For a subset $J = \{j(1), j(2), \dots, j(m)\}$, $j(1) < j(2) < \dots < j(m)$ of Λ, we define $d\log \underline{t}_J := d\log t_{j(1)} \wedge d\log t_{j(2)} \wedge \dots \wedge d\log t_{j(m)}$, $\underline{t}_J := t_{j(1)} t_{j(2)} \cdots t_{j(m)}$ and $J^c := \Lambda \backslash J$. The $\mathcal{O}_X$-module $P_0\Omega^{d+1-n}_{X/S}$ is the direct sum of the submodules $\underline{t}_{J^c}\mathcal{O}_X\, d\log \underline{t}_{J^c}$ for $J \subset \Lambda$ such that $|J| = n$, whose images in $\Omega^{d+1-n}_{\mathring{X}_I/S}$ and $\mathrm{gr}^P_n\, \Omega^{d+1}_{X/S}$ are 0 unless $J = I$. Since $\Omega^{d+1-n}_{\mathring{X}_I/S} = \underline{t}_{I^c}\mathcal{O}_{X_I}\, d\log \underline{t}_{I^c}$, the kernel of

$$\underline{t}_{I^c}\mathcal{O}_X\, d\log \underline{t}_{I^c} \longrightarrow \Omega^{d+1-n}_{\mathring{X}_I/S}$$

is $\sum_{i\in I} t_i \underline{t}_{I^c}\mathcal{O}_X\, d\log \underline{t}_{I^c}$, whose image in $\mathrm{gr}^P_n\, \Omega^{d+1}_{X/S}$ is 0. Note that we have an étale morphism $\mathring{X} \to \mathrm{Spec}(W_N[s_1, \dots, s_{d+1}]/(s_1 \cdots s_{d+1}))$ defined by t_ν. Thus, we obtain the first claim. We have

$$P_m\Omega^{d+1}_{X/S} = \sum_{I\subset\Lambda, |I|=m} \underline{t}_{I^c}\mathcal{O}_X\, d\log \underline{t}_\Lambda \quad \text{for } 0 \leqslant m \leqslant d+1.$$

Hence, for the second claim, it is enough to show that the kernel of the epimorphism

$$\bigoplus_{I\subset\Lambda, |I|=n} \mathcal{O}_X \longrightarrow \frac{\sum_{I\subset\Lambda,|I|=n} \underline{t}_{I^c}\mathcal{O}_X}{\sum_{I'\subset\Lambda,|I'|=n-1} \underline{t}^{I'^c}\mathcal{O}_X}; \quad (a_I)_I \mapsto \sum_I \underline{t}_{I^c} a_I$$

is $\bigoplus_{I\subset\Lambda,|I|=n}(\sum_{i\in I} t_i\mathcal{O}_X)$. This is obvious when $n = d+1$. When $n \leqslant d$, this follows from the fact that the kernel of $\mathcal{O}_X \mapsto \mathcal{O}_X/(\sum_{\widetilde{I}\subset\Lambda, \widetilde{I}\neq I, |\widetilde{I}|=n} \underline{t}_{\widetilde{I}^c}\mathcal{O}_X); a \mapsto \underline{t}_{I^c}a$ is $\sum_{i\in I} t_i\mathcal{O}_X$, which is verified being reduced to the case $X = \mathrm{Spec}(W_N[t_1, \dots, t_{d+1}]/(t_1 \cdots t_{d+1}))$. ☐

Let $\mathscr{E}$ be a crystal of $\mathcal{O}_{\mathring{X}/S}$-modules locally free of finite type on the nilpotent crystalline site $(\mathring{X}/S)_{\mathrm{Ncrys}}$. Since the PD thickenings $X \to P^n_{X/S}(r)$ and the projections among them are objects and morphisms of the site $(X/S)_{\mathrm{Ncrys}}$, the inverse image of $\mathscr{E}$ to $(X/S)_{\mathrm{Ncrys}}$ defines a left $\mathcal{D}_{X/S}$-action on its evaluation $\mathscr{E}_X$ on id: $X \to X$ via the equivalence of categories in Theorem 2.2.1. The homomorphism $\mathscr{E}_X \to \mathscr{E}_X \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ considered in Definition 3.3.2 is nothing but the connection associated to the crystal. Let D be the PD envelope of $\mathring{X} \to \mathring{Y}$, and let $\mathscr{E}_D$ be the evaluation of $\mathscr{E}$ on $\mathring{X} \to D$. Then the above connection of $\mathscr{E}_X$ is the pull-back of the connection $\mathscr{E}_D \to \mathscr{E}_D \otimes_{\mathcal{O}_Y} \Omega^1_{\mathring{Y}/S}$ associated to the crystal $\mathscr{E}$. This implies that the left $\mathcal{D}_{X/S}$-module $\mathscr{E}_X$ has trivial monodromy (Definition 3.3.2). Similarly, the inverse image of $\mathscr{E}$ to $(X_I/T)_{\mathrm{Ncrys}}$ defines a left $\mathcal{D}_{X_I/T}$-action on its evaluation $\mathscr{E}_{X_I} = \iota_I^*(\mathscr{E}_X)$ on id: $X_I \to X_I$.

Let α_I denote the exact closed immersion $X_I \to Y$ over T induced by the composite of $\mathring{X}_I \xrightarrow{\iota_I} \mathring{X} \xrightarrow{\mathring{\alpha}} \mathring{Y}$. We define the left $\mathcal{D}_{Y/S} = \mathcal{D}_{Y/T}[\partial_t^{\log}]$-module $\mathcal{F}$ to be $\alpha_{+/S}(\mathscr{E}_X)$ and the left $\mathcal{D}_{Y/T}$-module $\mathcal{F}_I$ to be $\alpha_{I+/T}(\mathscr{E}_{X_I})$ (cf. Theorem 3.1.2(1)). Since $\mathring{X}_I$ and $\mathring{Y}$ are smooth over S, we may regard $\mathscr{E}_{X_I}$ and $\mathcal{F}_I$ as a $\mathcal{D}_{\mathring{X}_I/S}$-module and a $\mathcal{D}_{\mathring{Y}/S}$-module respectively by (2.1.10). Then $\mathcal{F}_I$ is canonically isomorphic to the direct image $\mathring{\alpha}_{I+}(\mathscr{E}_{X_I})$ by Proposition 2.3.4 and the proofs of Proposition 2.4.1, [Ber96, Corollaire 2.3.3], and [Ber00, Proposition 1.1.7]. We define the increasing filtration $F_n\mathcal{F}$ (respectively $F_n\mathcal{F}_I$) of $\mathcal{F}$ (respectively $\mathcal{F}_I$) as before Proposition 3.1.7. Let $W_n\mathcal{F}$ and $W_n\, \mathrm{gr}^F_\bullet\, \mathcal{F}$ be the filtrations defined in § 3.3. In the rest of § 3.4, we will prove the following theorem.

Theorem 3.4.5.

(1) *There exists a canonical homomorphism of $\mathcal{D}_{Y/T}$-modules $\kappa_I\colon \mathcal{F}_I \to \mathrm{gr}^W_{|I|-1}\, \mathcal{F}$ for each non-empty subset $I \subset \Lambda$ of cardinality $\leqslant d+1$, and they induce the following isomorphism*

for $-d \leqslant n \leqslant d$*:*

$$\bigoplus_{\substack{b-2a=n\\0\leqslant a\leqslant b\leqslant d}}\ \bigoplus_{\substack{I\subset\Lambda\\|I|=b+1}} \mathcal{F}_I \xrightarrow{\cong} \operatorname{gr}^W_n \mathcal{F};\ (x_I)_{a,b,I} \mapsto \sum_{a,b,I} (\partial_t^{\log})^a(\kappa_I(x_I)).$$

(2) *There exists a canonical homomorphism of graded* $\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/T}$*-modules* $\tau_I\colon \operatorname{gr}^F_\bullet \mathcal{F}_I \to \operatorname{gr}^W_{|I|-1} \operatorname{gr}^F_\bullet \mathcal{F}$ *for each non-empty subset* $I \subset \Lambda$ *of cardinality* $\leqslant d+1$*, and they induce the following isomorphism for* $-d \leqslant n \leqslant d$*:*

$$\bigoplus_{\substack{b-2a=n\\0\leqslant a\leqslant b\leqslant d}}\ \bigoplus_{\substack{I\subset\Lambda\\|I|=b+1}} \operatorname{gr}^F_{\bullet-a} \mathcal{F}_I \xrightarrow{\cong} \operatorname{gr}^W_n \operatorname{gr}^F_\bullet \mathcal{F};\ (y_I)_{a,b,I} \mapsto \sum_{a,b,I} (\xi_t^{\log})^a(\tau_I(x_I)).$$

Corollary 3.4.6. *For* $0 \leqslant n \leqslant d$*, the homomorphism* $(\partial_t^{\log})^n : \operatorname{gr}^W_n \mathcal{F} \to \operatorname{gr}^W_{-n} \mathcal{F}$ *is an isomorphism.*

Proof. We have a bijective map from

$$\{(a,b) \in \mathbb{N}^2 \mid b-2a=n, 0 \leqslant a \leqslant b \leqslant d\} \quad \text{to} \quad \{(a,b) \in \mathbb{N}^2 \mid b-2a=-n, 0 \leqslant a \leqslant b \leqslant d\}$$

defined by $(a,b) \mapsto (a+n, b)$. Hence, the claim follows from Theorem 3.4.5(1). □

Let I be a non-empty subset of Λ such that $|I| \leqslant d+1$. Put $n = |I|-1$. We have an isomorphism $\omega_X \xrightarrow{\cong} \Omega^{d+1}_{X/S}; \omega \mapsto \widetilde{\omega} \wedge d\log t$, which induces an isomorphism $P_m\omega_X \xrightarrow{\cong} P_{m+1}\Omega^{d+1}_{X/S}$. Here $\tilde{\omega}$ denotes a lifting of ω to $\Omega^d_{X/S}$. Hence, the homomorphism $\mathcal{D}_{Y/T} \to \mathcal{D}_{Y/S}$ (cf. Proposition 2.1.9), the isomorphism $\Omega^1_{\mathring{X}_I/S} \xrightarrow{\cong} \Omega^1_{X_I/T}$, and the isomorphism in Proposition 3.4.2 induce a homomorphism of $(\mathcal{D}_{Y/T}, \mathcal{O}_X)$-bimodules $\mathcal{D}_{Y\leftarrow X_I/T} \to \operatorname{gr}^Q_n \mathcal{D}_{Y\leftarrow X/S}$. By taking $\otimes_{\mathcal{O}_X} \mathscr{E}_X$ and composing with the natural homomorphisms

$$\operatorname{gr}^Q_n \mathcal{D}_{Y\leftarrow X/S} \otimes_{\mathcal{O}_X} \mathscr{E}_X \to \operatorname{gr}^Q_n \mathcal{F} \to \operatorname{gr}^W_n \mathcal{F},$$

we obtain a homomorphism of left $\mathcal{D}_{Y/T}$-modules:

$$\mathcal{D}_{Y\leftarrow X_I/T} \otimes_{\mathcal{O}_{X_I}} \mathscr{E}_{X_I} \longrightarrow \operatorname{gr}^W_n \mathcal{F}. \tag{3.4.7}$$

Similarly, we have a homomorphism of $(\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/T}, \mathcal{O}_X)$-bimodules:

$$\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/T} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \omega_{X_I} \to \operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/S} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \operatorname{gr}^P_n \omega_X.$$

By taking $\otimes_{\mathcal{O}_X} \mathscr{E}_X$ and composing with the natural homomorphisms

$$\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/S} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \operatorname{gr}^P_n \omega_X \otimes_{\mathcal{O}_X} \mathscr{E}_X \to \operatorname{gr}^Q_n \operatorname{gr}^F_\bullet \mathcal{F} \to \operatorname{gr}^W_n \operatorname{gr}^F_\bullet \mathcal{F},$$

we obtain a homomorphism of graded $\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/T}$-modules:

$$\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/T} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \omega_{X_I} \otimes_{\mathcal{O}_{X_I}} \mathscr{E}_{X_I} \longrightarrow \operatorname{gr}^W_n \operatorname{gr}^F_\bullet \mathcal{F}. \tag{3.4.8}$$

Lemma 3.4.9.

(1) *The homomorphism* (3.4.7) *factors through the natural epimorphism* $\mathcal{D}_{Y\leftarrow X_I/T} \otimes_{\mathcal{O}_{X_I}} \mathscr{E}_{X_I} \to \mathcal{F}_I$ *and defines a homomorphism of left* $\mathcal{D}_{Y/T}$*-modules* $\mathcal{F}_I \to \operatorname{gr}^W_n \mathcal{F}$*.*

(2) *The homomorphism* (3.4.8) *factors through the natural epimorphism*

$$\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/T} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \omega_{X_I} \otimes_{\mathcal{O}_{X_I}} \mathscr{E}_{X_I} \to \operatorname{gr}^F_\bullet \mathcal{F}_I$$

and defines a homomorphism of graded $\operatorname{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/T}$*-modules* $\operatorname{gr}^F_\bullet \mathcal{F}_I \to \operatorname{gr}^W_n \operatorname{gr}^F_\bullet \mathcal{F}$*.*

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We define the homomorphism κ_I (respectively τ_I) in Theorem 3.4.5(1) (respectively (2)) to be the homomorphism constructed in Lemma 3.4.9(1) (respectively (2)) above.

Proof. Since the question is étale local on $\mathring{Y}$, we may assume that $\mathring{Y}$ is affine and there exist $t_1, \ldots, t_{d+1}$ as in the beginning of § 3, $z_1, \ldots, z_e$ as before Lemma 3.1.5, and $\Lambda = \{1, 2, \ldots, d+1\}$. We define $\partial_\mu^Y \in \mathcal{D}_{Y/S}$, $\xi_\mu^Y \in T_{Y/S}$ $(0 \leqslant \mu \leqslant e)$, $d\log \underline{z}$, and $d\log \underline{t}$ as before Lemma 3.1.5 for $B = S$, and $\widetilde{\partial}_\nu \in \mathcal{D}_{X/S}$ and $\widetilde{\xi}_\nu \in T_{X/S}$ $(1 \leqslant \nu \leqslant d+1)$ as in § 3.2. For a subset J of Λ, we define $d\log \underline{t}_J$, $\underline{t}_J$, and J^c as in the proof of Proposition 3.4.2. We define the basis $d\underline{t}_{I^c}$ of ω_{X_I} to be $dt_{j(1)} \wedge \cdots \wedge dt_{j(d-n)}$, where $1 \leqslant j(1) < \cdots < j(d-n) \leqslant d+1$ and $I^c = \{j(r) \mid 1 \leqslant r \leqslant d-n\}$. We regard $T_{Y/T}$ (respectively $\mathcal{D}_{Y/T}$) as an $\mathcal{O}_Y$-submodule (respectively a sheaf of subrings) of $T_{Y/S}$ (respectively $\mathcal{D}_{Y/S}$) by the canonical injective homomorphism $T_{Y/T} \to T_{Y/S}$ (respectively $\mathcal{D}_{Y/T} \to \mathcal{D}_{Y/S}$). Then we have $\partial_\mu^Y \in \mathcal{D}_{Y/T}$ and $\xi_\mu^Y \in T_{Y/T}$ for $1 \leqslant \mu \leqslant e$. Let $\theta_\nu^I \in T_{X_I/T}$ $(\nu \in I^c)$ be the dual basis of $dt_\nu \in \Omega^1_{X_I/T}$ $(\nu \in I^c)$ and let $D_\nu^I \in \mathcal{D}_{X_I/T}$ $(\nu \in I^c)$ be the corresponding differential operators.

(1) It is enough to prove that, for $P \in \mathcal{D}_{Y/T}$, $e \in \mathscr{E}_X$, and $\nu \in I^c$, the images of the two elements

$$x_1 := (P \otimes (d\log \underline{z})^{-1} \otimes d\underline{t}_{I^c}) D_\nu^I \otimes \iota_I^*(e)$$

and

$$x_2 := (P \otimes (d\log \underline{z})^{-1} \otimes d\underline{t}_{I^c}) \otimes D_\nu^I(\iota_I^*(e))$$

of $\mathcal{D}_{Y \leftarrow X_I/T} \otimes_{\mathcal{O}_{X_I}} \mathscr{E}_{X_I}$ under the homomorphism (3.4.7) coincide. For $1 \leqslant \mu \leqslant e$, put

$$\mathring{\alpha}^*(dz_\mu) = \sum_{1 \leqslant \nu \leqslant d+1} b_{\nu\mu}\, dt_\nu, \quad b_{\nu\mu} \in \mathcal{O}_X.$$

Then we have $\widetilde{\partial}_\nu(\alpha^*(z_\mu)) = t_\nu b_{\nu\mu}$ for $1 \leqslant \nu \leqslant d+1$ and $D_\nu^I(\alpha_I^*(z_\mu)) = \iota_I^*(b_{\nu\mu})$ for $\nu \in I^c$. We define $a_{\nu\mu}$ (respectively $\mathring{a}_{\nu\mu}$) to be a lifting of $t_\nu b_{\nu\mu}$ (respectively $b_{\nu\mu}$) to $\Gamma(Y, \mathcal{O}_Y)$ multiplied by z_μ^{-1}, and define $\widetilde{P}_\nu \in \mathcal{D}_{Y/S}$ (respectively $\mathring{P}_\nu \in \mathcal{D}_{Y/T}$) to be $\partial_0^Y + \sum_{1 \leqslant \mu \leqslant e} \partial_\mu^Y a_{\nu\mu}$ (respectively $\sum_{1 \leqslant \mu \leqslant e} \partial_\mu^Y \mathring{a}_{\nu\mu}$). Then the right action of $\widetilde{\partial}_\nu \in \mathcal{D}_{X/S}$ $(1 \leqslant \nu \leqslant d+1)$ (respectively $D_\nu^I \in \mathcal{D}_{X_I/T}$ $(\nu \in I^c)$) on $\mathcal{D}_{Y \leftarrow X/S}$ (respectively $\mathcal{D}_{Y \leftarrow X_I/T}$) is given by $(Q \otimes (d\log \underline{z})^{-1} \otimes d\log \underline{t})\widetilde{\partial}_\nu = Q \cdot \widetilde{P}_\nu \otimes (d\log \underline{z})^{-1} \otimes d\log \underline{t}$ (respectively $(P \otimes (d\log \underline{z})^{-1} \otimes d\underline{t}_{I^c}) D_\nu^I = P \cdot \mathring{P}_\nu \otimes (d\log \underline{z})^{-1} \otimes d\underline{t}_{I^c}$) (cf. Lemma 3.1.5). Note that we have $d\underline{t}_{I^c} \cdot D_\nu^I = 0$ by Proposition 2.3.4 and [Ber00, Théorème 1.2.3]. Choose a lifting $e_D \in \mathscr{E}_D$ of e and let $\sum_{1 \leqslant \nu \leqslant d+1} e_\nu\, dt_\nu$ be the image of e_D by $\mathscr{E}_D \to \mathscr{E}_D \otimes_{\mathcal{O}_Y} \Omega^1_{\mathring{Y}/S} \to \mathscr{E}_X \otimes_{\mathcal{O}_X} \Omega^1_{\mathring{X}/S}$. Then we have $\widetilde{\partial}_\nu(e) = t_\nu e_\nu$ for $1 \leqslant \nu \leqslant d+1$ and $D_\nu^I(\iota_I^*(e)) = \iota_I^*(e_\nu)$ for $\nu \in I^c$. Hence, the images of the elements x_1 and x_2 in $\mathrm{gr}_n^W \mathcal{F}$ are represented by the elements $\varepsilon_I(P \cdot \mathring{P}_\nu \otimes (d\log \underline{z})^{-1} \otimes \underline{t}_{I^c}\, d\log \underline{t}) \otimes e$ and $\varepsilon_I(P \otimes (d\log \underline{z})^{-1} \otimes \underline{t}_{I^c}\, d\log \underline{t}) \otimes e_\nu$ of $Q_n\mathcal{F}$. Here $\varepsilon_I \in \{\pm 1\}$ is defined by $\varepsilon_I d\log \underline{t}_\Lambda = d\log \underline{t}_I \wedge d\log \underline{t}_{I^c}$ in $\Omega^{d+1}_{X/S}$. The latter element coincides with

$$\begin{aligned}\varepsilon_I(P \otimes (d\log \underline{z})^{-1} \otimes \underline{t}_{I^c\setminus\{\nu\}}\, d\log \underline{t}) \otimes \widetilde{\partial}_\nu(e) &= \varepsilon_I(P \otimes (d\log \underline{z})^{-1} \otimes \underline{t}_{I^c\setminus\{\nu\}}\, d\log \underline{t})\widetilde{\partial}_\nu \otimes e \\ &= \varepsilon_I(P \cdot \widetilde{P}_\nu \otimes (d\log \underline{z})^{-1} \otimes \underline{t}_{I^c\setminus\{\nu\}}\, d\log \underline{t}) \otimes e.\end{aligned}$$

For the second equality, note that $\widetilde{\partial}_\nu(\underline{t}_{I^c\setminus\{\nu\}}) = 0$. Now, using $\alpha^*(a_{\nu\mu}) = t_\nu \alpha^*(\mathring{a}_{\nu\mu})$, we see that the difference of the two elements of $Q_n\mathcal{F}$ is

$$\varepsilon_I P \partial_0^Y \otimes (d\log \underline{z})^{-1} \otimes \underline{t}_{I^c\setminus\{\nu\}}\, d\log \underline{t} \otimes e = \partial_0^{\log}(\varepsilon_I P \otimes (d\log \underline{z})^{-1} \otimes \underline{t}_{I^c\setminus\{\nu\}}\, d\log \underline{t} \otimes e),$$

which is contained in $\partial_0^{\log}(Q_{n+1}\mathcal{F}) \subset W_{n-1}\mathcal{F}$.

(2) By Proposition 3.1.7 for $B=T$, it is enough to prove that, for a non-empty subset I of Λ, $x\in \mathrm{Sym}^\bullet_{\mathcal{O}_Y} T_{Y/T}$, $e\in \mathscr{E}_X$, and $\nu\in I^c$, the image of the element $(x\otimes (d\log \underline{z})^{-1}\otimes d\underline{t}_{I^c})\theta^I_\nu\otimes \iota_I^*(e)$ under the homomorphism (3.4.8) is 0. Let $b_{\nu\mu}$ and ε_I be as in the proof of (1). Then we have

$$(x\otimes (d\log \underline{z})^{-1}\otimes d\underline{t}_{I^c})\theta^I_\nu=\sum_{1\leqslant\mu\leqslant e} x\xi^Y_\mu\otimes (d\log \underline{z})^{-1}\otimes \alpha_I^*(z_\mu)^{-1}\iota_I^*(b_{\nu\mu})\, d\underline{t}_{I^c}$$

and the image in question is represented by the element

$$\varepsilon_I\sum_{1\leqslant\mu\leqslant e} x\xi^Y_\mu\otimes (d\log \underline{z})^{-1}\otimes \alpha^*(z_\mu)^{-1}b_{\nu\mu}\underline{t}_{I^c}\, d\log \underline{t}\otimes e\in Q_n\,\mathrm{gr}^F_\bullet \mathcal{F}.$$

Since

$$\alpha_*(\widetilde{\xi}_\nu)=1\otimes \xi^Y_0+\sum_{1\leqslant\mu\leqslant e} t_\nu\alpha^*(z_\mu)^{-1}b_{\nu\mu}\otimes \xi^Y_\mu\in \mathcal{O}_X\otimes_{\mathcal{O}_Y} T_{Y/S},$$

Proposition 3.1.7 for $B=S$ implies that the above element coincides with $-\varepsilon_I x\xi^Y_0\otimes (d\log \underline{z})^{-1}\otimes \underline{t}_{(I\cup\{\nu\})^c}\, d\log \underline{t}\otimes e$, which is contained in $\xi^{\log}_t Q_{n+1}\,\mathrm{gr}^F_\bullet \mathcal{F}\subset W_{n-1}\,\mathrm{gr}^F_\bullet \mathcal{F}$. □

Proof of Theorem 3.4.5(2). Since the question is étale local on $\mathring{X}$, we may assume that there exist $t_1,\dots,t_{d+1}$ as in the beginning of § 3 and $\Lambda=\{1,2,\dots,d+1\}$. Let $\widetilde{\xi}_\nu\in T_{X/S}$ be the dual basis of $d\log t_\nu\in \Omega^1_{X/S}$ as in § 3.2. For a subset I of Λ, we define $d\log \underline{t}_I$, $\underline{t}_I$, and I^c as in the proof of Proposition 3.4.2 and, for a non-empty I, we define $d\underline{t}_{I^c}\in \omega_{X_I}$ and $\theta^I_\nu\in T_{X_I/T}$ $(\nu\in I^c)$ as in the proof of Lemma 3.4.9. We may also assume that there exist $w_1,\dots,w_e\in \Gamma(Y,\mathcal{O}_Y)$ such that dw_μ $(1\leqslant\mu\leqslant e)$ is a basis of $\Omega^1_{Y/T}$. We define $\eta_0,\eta_1,\dots,\eta_e\in T_{Y/S}$ to be the dual basis of $d\log t, dw_1,\dots,dw_e$. By trivializing ω_Y, ω_X, and ω_{X_I} by their bases $dw_1\wedge\cdots\wedge dw_e$, $d\log \underline{t}$, and $d\underline{t}_{I^c}$, we obtain the following isomorphisms from Proposition 3.1.7:

$$\mathrm{gr}^F_\bullet \mathcal{F}\cong \mathcal{O}_X[\eta_0,\eta_1,\dots,\eta_e]/(\alpha_*(\widetilde{\xi}_\nu), 1\leqslant\nu\leqslant d+1)\otimes_{\mathcal{O}_X}\mathscr{E}_X,$$
$$\mathrm{gr}^F_\bullet \mathcal{F}_I\cong \mathcal{O}_{X_I}[\eta_1,\dots,\eta_e]/(\alpha_{I*}(\theta^I_\nu), \nu\in I^c)\otimes_{\mathcal{O}_{X_I}}\mathscr{E}_{X_I}.$$

We have

$$Q_n\,\mathrm{gr}^F_\bullet \mathcal{F}=\sum_{I\subset\Lambda, |I|=n+1}\underline{t}_{I^c}\,\mathrm{gr}^F_\bullet \mathcal{F}\quad \text{for } 0\leqslant n\leqslant d,$$
$$W_n\,\mathrm{gr}^F_\bullet \mathcal{F}=\sum_{b-2a=n, 0\leqslant a\leqslant b\leqslant d}(\eta_0)^a\cdot Q_b\,\mathrm{gr}^F_\bullet \mathcal{F}\quad \text{for } -d\leqslant n\leqslant d,$$

and the homomorphism $\mathrm{gr}^F_\bullet \mathcal{F}_I\to \mathrm{gr}^W_{|I|-1}\mathrm{gr}^F_\bullet \mathcal{F}$ in Lemma 3.4.9(2) sends $\iota_I^*(f)\otimes\iota_I^*(e)$ to $\varepsilon_I f\otimes \underline{t}_{I^c}e$ for $f\in \mathcal{O}_X[\eta_1,\dots,\eta_e]$ and $e\in\mathscr{E}_X$. Here ε_I is defined as in the proof of Lemma 3.4.9. Hence, we may assume that $\mathscr{E}=\mathcal{O}_{\mathring{X}/S}$.

Let $\alpha'\colon X\to Y$ be another morphism of fine log schemes over T satisfying the same conditions as α. We first prove that the claim holds for α if and only if it holds for α'. As in the proof of Lemma 3.1.4, we are reduced to the case where there exists a smooth morphism $h\colon Y'\to Y$ such that $\alpha=h\circ\alpha'$ and $w_{e+1},\dots,w_{e'}\in \Gamma(Y',\mathcal{O}_{Y'})$ such that dw_μ $(e+1\leqslant\mu\leqslant e')$ is a basis of $\Omega^1_{Y'/Y}$ and $(\alpha')^*(w_\mu)=0$ $(e+1\leqslant\mu\leqslant e')$. Let $\eta'_0,\eta'_1,\dots,\eta'_{e'}\in T_{Y'/S}$ be the dual basis of the basis $d\log t, dh^*(w_1),\dots,dh^*(w_e), dw_{e+1},\dots,dw_{e'}$ of $\Omega^1_{Y'/S}$. Then we have

$$\alpha_*(\widetilde{\xi}_\nu)=\sum_{0\leqslant\mu\leqslant e} a_{\nu\mu}\otimes\eta_\mu,\quad \alpha'_*(\widetilde{\xi}_\nu)=\sum_{0\leqslant\mu\leqslant e} a_{\nu\mu}\otimes\eta'_\mu$$

and

$$\alpha_{I*}(\theta^I_\nu)=\sum_{1\leqslant\mu\leqslant e} a^I_{\nu\mu}\otimes\eta_\mu,\quad \alpha'_{I*}(\theta^I_\nu)=\sum_{1\leqslant\mu\leqslant e} a^I_{\nu\mu}\otimes\eta'_\mu$$

for the same $a_{\nu\mu}\in\mathcal{O}_X$ and $a^I_{\nu\mu}\in\mathcal{O}_{X_I}$. Hence, we have the following isomorphisms compatible with the homomorphisms constructed in Lemma 3.4.9(2) in the obvious sense:

$$\mathrm{gr}^F_\bullet\,\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_X[\eta'_{e+1},\dots,\eta'_{e'}]\xrightarrow{\cong}\mathrm{gr}^F_\bullet\,\mathcal{F}',$$
$$\mathrm{gr}^F_\bullet\,\mathcal{F}_I\otimes_{\mathcal{O}_{X_I}}\mathcal{O}_{X_I}[\eta'_{e+1},\dots,\eta'_{e'}]\xrightarrow{\cong}\mathrm{gr}^F_\bullet\,\mathcal{F}'_I.$$

Here $\mathcal{F}'=\alpha'_{+/S}\mathcal{O}_X$ and $\mathcal{F}'_I=\alpha'_{I+/T}\mathcal{O}_{X_I}$. This implies the desired claim.

Since the question is étale local on X, we may assume that

$$\mathring{X}=\mathrm{Spec}(W_N[t_1,\dots,t_{d+1}]/(t_1\cdots t_{d+1})).$$

Let Y be $\mathrm{Spec}(W_N[t_1,\dots,t_d])$ endowed with the inverse image of M_T. We will prove the isomorphism for the natural morphism of fine log schemes $\alpha\colon X\to Y$. Let $\theta_0,\theta_1,\dots,\theta_{d+1}\in T_{Y/S}$ be the dual basis of the basis $d\log t, dt_1,\dots,dt_{d+1}$ of $\Omega^1_{Y/S}$. Since

$$\alpha_*(\widetilde{\xi}_\nu)=t_\nu\otimes\theta_\nu+1\otimes\theta_0\in\mathcal{O}_X\otimes_{\mathcal{O}_Y}T_{Y/S}\quad\text{for }1\leqslant\nu\leqslant d+1$$

and

$$\alpha_{I*}(\theta^I_\nu)=1\otimes\theta_\nu\in\mathcal{O}_{X_I}\otimes_{\mathcal{O}_Y}T_{Y/T}\quad\text{for }\nu\in I^c,$$

we have isomorphisms

$$\begin{aligned}\Gamma(Y,\mathrm{gr}^F_\bullet\,\mathcal{F})&\cong W_N[t_1,\dots,t_{d+1},\theta_0,\theta_1,\dots,\theta_{d+1}]/(t_1\cdots t_{d+1},t_1\theta_1+\theta_0,\dots,t_{d+1}\theta_{d+1}+\theta_0),\\ \Gamma(Y,\mathrm{gr}^F_\bullet\,\mathcal{F}_I)&\cong W_N[t_1,\dots,t_{d+1},\theta_1,\dots,\theta_{d+1}]/(t_\nu,\nu\in I,\theta_\nu,\nu\in I^c)\\ &\cong W_N[t_\nu,\nu\in I^c,\theta_\nu,\nu\in I].\end{aligned}$$

The homomorphism in question has a lifting

$$\bigoplus_{\substack{b-2a=n\\0\leqslant a\leqslant b\leqslant d}}\ \bigoplus_{\substack{I\subset\Lambda\\|I|=b+1}} W_N[t_\nu,\nu\in I^c,\theta_\nu,\nu\in I]\longrightarrow\Gamma(Y,W_n\,\mathrm{gr}^F_\bullet\,\mathcal{F});\,(f_{a,b,I})\mapsto\sum f_{a,b,I}\underline{t}_{I^c}\theta_0^a$$

for $-d\leqslant n\leqslant d$, and the sum of them gives an isomorphism by Lemma 3.4.10 below. On the other hand, $\Gamma(Y,\mathrm{gr}^W_n\,\mathrm{gr}^F_\bullet\,\mathcal{F})$ is generated by $\underline{t}_{I^c}\theta_0^a$ for $b-2a=n$, $0\leqslant a\leqslant b\leqslant d$, $I\subset\Lambda$, and $|I|=b+1$ as a $W_N[t_1,\dots,t_{d+1},\theta_1,\dots,\theta_{d+1}]$-module since $\theta_0\cdot\mathrm{gr}^W_n\,\mathrm{gr}^F_\bullet\,\mathcal{F}=0$ by Lemma 3.3.6(2). Hence, the homomorphism in question is surjective. Combining with the above isomorphism, we see that it is an isomorphism by induction on n. □

LEMMA 3.4.10. *The following set is a basis of*

$$W_N[t_1,\dots,t_{d+1},\theta_0,\dots,\theta_{d+1}]/(t_1\cdots t_d,\theta_0+t_\nu\theta_\nu,1\leqslant\nu\leqslant d+1)$$

as a W_N-module:

$$\left\{\theta_0^a\cdot\underline{t}_{I^c}\prod_{\nu\in I^c}t_\nu^{n_\nu}\prod_{\nu\in I}\theta_\nu^{n_\nu}\ \middle|\ 0\leqslant a\leqslant b\leqslant d, I\subset\{1,\dots,d+1\},|I|=b+1,n_\nu\in\mathbb{N}\text{ for }1\leqslant\nu\leqslant d+1\right\}.$$

Proof. For each $1\leqslant\nu\leqslant d+1$,

$$W_N[\theta_0,t_\nu,\theta_\nu]/(\theta_0+t_\nu\theta_\nu)\cong W_N[t_\nu,\theta_\nu]$$

is a free $W_n[\theta_0]$-module with a basis $\{t_\nu^n\theta_\nu^m\mid n,m\in\mathbb{N},nm=0\}$. Hence,

$$W_N[t_1,\dots,t_{d+1},\theta_0,\dots,\theta_{d+1}]/(\theta_0+t_\nu\theta_\nu,1\leqslant\nu\leqslant d+1)$$

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is a free W_N-module with a basis

$$\left\{\theta_0^a \underline{t}_{I^c} \prod_{\nu\in I^c} t_\nu^{n_\nu} \prod_{\nu\in I} \theta_\nu^{n_\nu} \,\middle|\, a\in\mathbb{N}, I\subset\{1,\ldots,d+1\}, n_\nu\in\mathbb{N} \text{ for } 1\leqslant \nu\leqslant d+1\right\}.$$

For a, I, and n_ν as above, we have

$$t_1\cdots t_d\cdot \theta_0^a \underline{t}_{I^c}\prod_{\nu\in I^c} t_\nu^{n_\nu}\prod_{\nu\in I}\theta_\nu^{n_\nu} = (-1)^{|I_1|}\theta_0^{a+|I_1|}\underline{t}_{I_1^c}\prod_{\nu\in I^c} t_\nu^{n_\nu+1}\prod_{\nu\in I_1}\theta_\nu^{n_\nu-1},$$

where $I_1=\{\nu\in I \mid n_\nu\neq 0\}$. Conversely, if $a\geqslant |I|$, then we have

$$\theta_0^a\underline{t}_{I^c}\prod_{\nu\in I^c} t_\nu^{n_\nu}\prod_{\nu\in I}\theta_\nu^{n_\nu} = (-1)^{|I|} t_1\cdots t_d\cdot\theta_0^{a-|I|}\prod_{\nu\in I^c} t_\nu^{n_\nu}\prod_{\nu\in I}\theta_\nu^{n_\nu+1}.$$

Hence, $\theta_0^a\underline{t}_{I^c}\prod_{\nu\in I^c} t_\nu^{n_\nu}\prod_{\nu\in I}\theta_\nu^{n_\nu}$ for $a<|I|$ give a basis of the quotient by the ideal generated by $t_1\cdots t_{d+1}$. □

For $n\in\mathbb{Z}$, we define the filtration $F_r(W_n\mathcal{F})$ $(r\in\mathbb{Z})$ of $W_n\mathcal{F}$ to be $F_r\mathcal{F}\cap W_n\mathcal{F}$ and the filtration $F_r(\mathrm{gr}_n^W\mathcal{F})$ of $\mathrm{gr}_n^W\mathcal{F}$ to be the image of $F_r(W_n\mathcal{F})$. The graded quotients $\mathrm{gr}_\bullet^F(W_n\mathcal{F})$ and $\mathrm{gr}_\bullet^F(\mathrm{gr}_n^W\mathcal{F})$ are graded $\mathrm{gr}_\bullet\mathcal{D}_{Y/S}=\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/S}$-modules and we have a short exact sequence of graded $\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/S}$-modules:

$$0\longrightarrow \mathrm{gr}_\bullet^F(W_{n-1}\mathcal{F})\longrightarrow \mathrm{gr}_\bullet^F(W_n\mathcal{F})\longrightarrow \mathrm{gr}_\bullet^F(\mathrm{gr}_n^W\mathcal{F})\longrightarrow 0. \tag{3.4.11}$$

On the other hand, $\mathrm{gr}_\bullet^F(W_n\mathcal{F})$ is naturally regarded as a graded $\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/S}$-submodule of $\mathrm{gr}_\bullet^F\mathcal{F}$.

By the construction of κ_I in Lemma 3.4.9, we see that κ_I is compatible with the filtrations F_r and induces a homomorphism of $\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/T}$-modules $\mathrm{gr}_\bullet^F\kappa_I\colon \mathrm{gr}_\bullet^F\mathcal{F}_I\to\mathrm{gr}_\bullet^F\mathrm{gr}_{|I|-1}^W\mathcal{F}$. On the other hand, the homomorphism $\mathrm{gr}_n^W\mathcal{F}\to\mathrm{gr}_{n-2}^W\mathcal{F}$ defined by the multiplication by $\partial_t^{\log}$ sends F_r to F_{r+1} and induces a homomorphism $\mathrm{gr}_r^F(\mathrm{gr}_n^W\mathcal{F})\to\mathrm{gr}_{r+1}^F(\mathrm{gr}_{n-2}^W\mathcal{F})$. Hence, to prove Theorem 3.4.5(1), it suffices to show that the following homomorphism is an isomorphism for $-d\leqslant n\leqslant d$:

$$\bigoplus_{\substack{b-2a=n\\0\leqslant a\leqslant b\leqslant d}}\ \bigoplus_{\substack{I\subset\Lambda\\|I|=b+1}}\mathrm{gr}_{\bullet-a}^F\mathcal{F}_I\longrightarrow\mathrm{gr}_\bullet^F\mathrm{gr}_n^W\mathcal{F};\ (y_I)_{a,b,I}\mapsto\sum_{a,b,I}(\partial_t^{\log})^a\cdot\mathrm{gr}_{\bullet-a}^F\kappa_I(y_I). \tag{3.4.12}$$

Lemma 3.4.13. *Let n be an integer.*

(1) *As $\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/S}$-submodules of $\mathrm{gr}_\bullet^F\mathcal{F}$, $W_n(\mathrm{gr}_\bullet^F\mathcal{F})$ is contained in $\mathrm{gr}_\bullet^F(W_n\mathcal{F})$.*

(2) *For $I\subset\Lambda$ such that $|I|\leqslant d+1$, the following two diagrams are commutative.*

$$\begin{array}{ccc}
\mathrm{gr}_\bullet^F\mathcal{F}_I & \xrightarrow{\mathrm{gr}_\bullet^F\kappa_I} & \mathrm{gr}_\bullet^F(\mathrm{gr}_{|I|-1}^W\mathcal{F})\\
& {\searrow}^{\tau_I} & \uparrow\\
& & \mathrm{gr}_{|I|-1}^W(\mathrm{gr}_\bullet^F\mathcal{F})
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{gr}_{\bullet-1}^F(\mathrm{gr}_n^W\mathcal{F}) & \xrightarrow{\partial_t^{\log}} & \mathrm{gr}_\bullet^F(\mathrm{gr}_{n-2}^W\mathcal{F})\\
\uparrow & & \uparrow\\
\mathrm{gr}_n^W(\mathrm{gr}_{\bullet-1}^F\mathcal{F}) & \xrightarrow{\xi_t^{\log}} & \mathrm{gr}_{n-2}^W(\mathrm{gr}_\bullet^F\mathcal{F})
\end{array}$$

Here the three vertical maps are induced by the exact sequence (3.4.11) and (1) above.

Proof. The questions are étale local on $\mathring{Y}$ and we keep the notation and assumptions in the proof of Lemma 3.4.9.

(1) By definition, the $\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/S}$-module $W_n\,\mathrm{gr}_\bullet^F\mathcal{F}$ is generated by $(\xi_t^{\log})^a\otimes(d\log\underline{z})^{-1}\otimes\underline{t}_{I^c}\,d\log\underline{t}\otimes e$ for $0\leqslant a\leqslant d$, $I\subset\Lambda$, and $e\in\mathscr{E}_X$ such that $a\leqslant|I|-1$ and $|I|-1-2a=n$. They are

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the images of the sections $(\partial_t^{\log})^a\otimes(d\log\underline{z})^{-1}\otimes\underline{t}_{I^c}\,d\log\underline{t}\otimes e$ of $F_a(W_n\mathcal{F})$ in $\mathrm{gr}_a^F\mathcal{F}$ and hence contained in $\mathrm{gr}_a^F(W_n\mathcal{F})$.

(2) Every homomorphism in the left-hand diagram is $\mathrm{Sym}^\bullet_{\mathcal{O}_Y}T_{Y/T}$-linear. Hence, its commutativity can be verified by looking at the images of the sections $1\otimes(d\log\underline{z})^{-1}\otimes d\underline{t}_{I^c}\otimes\iota_I^*(e)$, $e\in\mathscr{E}_X$ of $\mathrm{gr}^F_\bullet\mathcal{F}_I$. The commutativity of the right-hand diagram is reduced to the coincidence of the homomorphism $\mathrm{gr}^F_{\bullet-1}W_n\mathcal{F}\to\mathrm{gr}^F_\bullet W_{n-2}\mathcal{F}$ induced by $\partial_t^{\log}$ with the multiplication by $\xi_t^{\log}$. □

LEMMA 3.4.14. *For $-d\leqslant n\leqslant d$, the homomorphism in Theorem 3.4.5(1) is an epimorphism.*

Proof. The question is étale local on $\mathring{Y}$ and we keep the notation and assumptions as in the proof of Lemma 3.4.9. Then the $\mathcal{D}_{Y/S}$-module $W_n\mathcal{F}$ is generated by $(\partial_t^{\log})^a\otimes(d\log\underline{z})^{-1}\otimes\underline{t}_{I^c}\,d\log\underline{t}\otimes e$ for $0\leqslant a\leqslant d$, $I\subset\Lambda$, and $e\in\mathscr{E}_X$ such that $a\leqslant|I|-1$ and $|I|-1-2a=n$. Since $\partial_t^{\log}\cdot W_n\mathcal{F}\subset W_{n-2}\mathcal{F}$, their images in $\mathrm{gr}_n^W\mathcal{F}$ generate $\mathrm{gr}_n^W\mathcal{F}$ regarded as a $\mathcal{D}_{Y/T}$-module. Hence, the claim follows from the fact that the image of the section $1\otimes(d\log\underline{z})^{-1}\otimes d\underline{t}_{I^c}\otimes\iota_I^*(e)$ of $\mathcal{F}_I$ by the $\mathcal{D}_{Y/T}$-linear homomorphism $(\partial_t^{\log})^a\kappa_I$ is $\varepsilon_I(\partial_t^{\log})^a\otimes(d\log\underline{z})^{-1}\otimes\underline{t}_{I^c}\,d\log\underline{t}\otimes e$. □

Proof of Theorem 3.4.5(1). By Lemma 3.4.13 and Theorem 3.4.5(2), it suffices to prove that the injective homomorphism $W_n(\mathrm{gr}^F_\bullet\mathcal{F})\hookrightarrow\mathrm{gr}^F_\bullet(W_n\mathcal{F})$ is an isomorphism for $n\in\mathbb{Z}$. The claim is obvious for $n\leqslant-d-1$; both sides are 0. For $-d\leqslant n\leqslant d$, suppose that the claim is true for $n-1$. Then the homomorphism $\mathrm{gr}_n^W(\mathrm{gr}^F_\bullet\mathcal{F})\to\mathrm{gr}^F_\bullet(\mathrm{gr}_n^W\mathcal{F})$ is injective. Hence, by Theorem 3.4.5(2) and Lemma 3.4.13(2), the homomorphism (3.4.12) is injective. Combining with Lemma 3.4.14, we see that the homomorphism (3.4.12) is an isomorphism. It implies that $\mathrm{gr}_n^W(\mathrm{gr}^F_\bullet\mathcal{F})\to\mathrm{gr}^F_\bullet(\mathrm{gr}_n^W\mathcal{F})$ is an isomorphism. Hence, $W_n(\mathrm{gr}^F_\bullet\mathcal{F})=\mathrm{gr}^F_\bullet(W_n\mathcal{F})$. □

Remark 3.4.15. By applying Proposition 3.4.16 below to $\mathcal{F}_x$, $\partial_t^{\log}$, $Q_b\mathcal{F}_x$, and $W_n\mathcal{F}_x$ for $x\in\mathring{X}$ and using Corollary 3.4.6, we obtain

$$Q_r\mathcal{F}=\mathrm{Ker}(\partial^{\log})^{r+1}\quad(0\leqslant r\leqslant d).$$

PROPOSITION 3.4.16. *Let d be a positive integer. Let M be a module, let $\mathcal{N}\colon M\to M$ be a nilpotent endomorphism of M, and let Q_bM $(b\in\mathbb{Z})$ be an increasing filtration of M by submodules such that $Q_{-1}M=0$, $Q_dM=M$, and $\mathcal{N}(Q_bM)\subset Q_{b-1}M$. We define the submodules W_nM $(-d\leqslant n\leqslant d)$ by*

$$W_nM=\sum_{\substack{b-2a=n\\0\leqslant a\leqslant b\leqslant d}}\mathcal{N}^a(Q_bM).$$

We put $W_nM=0$ for $n\leqslant-d-1$ and $W_nM=M$ for $n\geqslant d+1$.

(1) *We have $W_{n-1}M\subset W_nM$ and $\mathcal{N}(W_nM)\subset W_{n-2}M$ for $n\in\mathbb{Z}$.*

(2) *Assume that, for $0\leqslant r\leqslant d$, $\mathcal{N}^r\colon W_rM\to W_{-r}M$ induces an isomorphism $\mathrm{gr}_r^W M\xrightarrow{\cong}\mathrm{gr}_{-r}^W M$. Then we have $\mathrm{Ker}\,\mathcal{N}^{r+1}=Q_rM$ for $0\leqslant r\leqslant d$.*

Proof. (1) The same as in Lemma 3.3.6.

(2) Note that $Q_rM\subset\mathrm{Ker}\,\mathcal{N}^{r+1}$ since $\mathcal{N}^{r+1}(Q_rM)\subset Q_{-1}M=0$. The claim is true for $r=d$ since $Q_dM=M$. Let r be an integer such that $0\leqslant r\leqslant d-1$ and assume that $\mathrm{Ker}\,\mathcal{N}^{r+2}=Q_{r+1}M$. Let x be an element of M such that $\mathcal{N}^{r+1}(x)=0$. By assumption, $x\in Q_{r+1}M\subset W_{r+1}M$. Since $\mathcal{N}^{r+1}$ induces an isomorphism $\mathrm{gr}_{r+1}^W M\xrightarrow{\cong}\mathrm{gr}_{-r-1}^W M$, we have $x\in W_rM$. By the definition of W_rM, x is written in the form $x=y+\mathcal{N}(z)$, where $y\in Q_rM$ and $z\in M$. Since $\mathcal{N}^{r+1}(y)=0$ by the remark in the beginning of the proof, we have $\mathcal{N}^{r+1}(\mathcal{N}(z))=0$.

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This implies that $z \in \operatorname{Ker} \mathcal{N}^{r+2} = Q_{r+1}M$ by assumption and $\mathcal{N}(z) \in \mathcal{N}(Q_{r+1}M) \subset Q_rM$. Hence, $x = y + \mathcal{N}(z) \in Q_rM$. □

3.5 Independence on Y

Let $\alpha' \colon X \to Y'$ be another morphism of fine log schemes over T satisfying the same conditions as α and let $g \colon Y' \to Y$ be a morphism over T such that $g \circ \alpha' = \alpha$. Let $\mathscr{E}$, $\mathscr{E}_X$, $\mathscr{E}_{X_I}$, α_I, $\mathcal{F}$, and $\mathcal{F}_I$ be as before Theorem 3.4.5. We define $\alpha'_I \colon X_I \to Y'$, $\mathcal{F}'$, and $\mathcal{F}'_I$ in the same way as α_I, $\mathcal{F}$, and $\mathcal{F}_I$ using α' instead of α. Then, by Proposition 2.6.5, we have

$$g_{+/S}\mathcal{F}' \cong \mathcal{F}, \quad g_{+/T}\mathcal{F}'_I \cong \mathcal{F}_I. \tag{3.5.1}$$

We obtain $\mathcal{H}^i(g_{+/S}\mathcal{F}') = 0$ and $\mathcal{H}^i(g_{+/T}\mathcal{F}'_I) = 0$ for $i \neq 0$. We regard $g_{+/S}\mathcal{F}'$ (respectively $g_{+/T}\mathcal{F}'_I$) as a left $\mathcal{D}_{Y/S}$-module (respectively $\mathcal{D}_{Y/T}$-module) in the following. Since $\mathcal{F}'$ is supported on X, the above vanishing implies that

$$g_{+/S}\mathcal{F}' \cong g_*(\mathcal{D}_{Y \leftarrow Y'/S} \otimes_{\mathcal{D}_{Y'/S}} \mathcal{F}'), \quad g_{+/T}\mathcal{F}'_I \cong g_*(\mathcal{D}_{Y \leftarrow Y'/T} \otimes_{\mathcal{D}_{Y'/T}} \mathcal{F}'_I). \tag{3.5.2}$$

Suppose that we have $\alpha'' \colon X \to Y''$ and $g' \colon Y'' \to Y'$ such that $\alpha' = g' \circ \alpha''$ and define $\mathcal{F}''$ and $\mathcal{F}''_I$ using α''. Then, by Proposition 2.6.6, the isomorphism $(g \circ g')_{+/S}\mathcal{F}'' \cong \mathcal{F}$ coincides with the composite of

$$(g \circ g')_{+/S}\mathcal{F}'' \cong g_{+/S}(g'_{+/S}\mathcal{F}'') \cong g_{+/S}\mathcal{F}' \cong \mathcal{F},$$

and similarly for $\mathcal{F}_I$, $\mathcal{F}'_I$, and $\mathcal{F}''_I$.

The isomorphisms (3.5.1) are compatible with the weight filtrations and the isomorphisms in Theorem 3.4.5(1) as follows.

Proposition 3.5.3. *Let the notation and assumptions be as above.*

(1) *The homomorphism $g_{+/S}\mathcal{F}' \to g_{+/S}\mathcal{F}'$ induced by the $\mathcal{D}_{Y'/S}$-linear homomorphism $\partial_t^{\log} \colon \mathcal{F}' \to \mathcal{F}'$ coincides with the action of $\partial_t^{\log} \in \mathcal{D}_{Y/S}$ on $g_{+/S}\mathcal{F}'$.*

(2) *For $n \in \mathbb{Z}$, we have $\mathcal{H}^i(g_{+/S}(\mathrm{gr}_n^W \mathcal{F}')) = 0$ and $\mathcal{H}^i(g_{+/S}W_n\mathcal{F}') = 0$ for $i \neq 0$.*

(3) *For $n \in \mathbb{Z}$, the image of $g_{+/S}W_n\mathcal{F}'$ (regarded as a left $\mathcal{D}_{Y/S}$-submodule of $g_{+/S}\mathcal{F}'$) under $g_{+/S}\mathcal{F}' \xrightarrow{\cong} \mathcal{F}$ coincides with $W_n\mathcal{F}$.*

(4) *For a non-empty set $I \subset \Lambda$ of cardinality $\leqslant d+1$, the following diagram is commutative in the category of $\mathcal{D}_{Y/T}$-modules.*

$$\begin{array}{ccccc}
g_{+/T}\mathcal{F}'_I & \xrightarrow{g_{+/T}(\kappa_I)} & g_{+/T}(\mathrm{gr}^W_{|I|-1}\mathcal{F}') & \xrightarrow[\text{Corollary 2.6.10}]{\cong} & g_{+/S}(\mathrm{gr}^W_{|I|-1}\mathcal{F}') \\
\downarrow \cong & & & & \downarrow \cong \\
\mathcal{F}_I & & \xrightarrow{\kappa_I} & & \mathrm{gr}^W_{|I|-1}\mathcal{F}
\end{array}$$

Here the right vertical isomorphism is induced by (2) and (3).

Proof. (1) By (3.5.2), it suffices to prove that $\partial_t^{\log} P = P \partial_t^{\log}$ for $P \in \mathcal{D}_{Y \leftarrow Y'/S}$. By Proposition 2.3.4, the actions of $\partial_t^{\log}$ on ω_Y and $\omega_{Y'}$ are 0. Since $g_* \colon T_{Y'/S} \to T_{Y/S}$ sends $\partial_t^{\log}$ to $\partial_t^{\log}$, this together with Propositions 2.4.5 and 2.1.11 implies the desired claim.

(2) By Theorem 3.4.5(1) and Corollary 2.6.10, $\mathcal{H}^i(g_{+/T}\mathcal{F}'_I) = 0$ implies that

$$\mathcal{H}^i(g_{+/S}(\mathrm{gr}_n^W \mathcal{F}')) = 0 \quad \text{for } i \neq 0.$$

Since $W_{-d-1}\mathcal{F}' = 0$, we obtain $\mathcal{H}^i(g_{+/S}W_n\mathcal{F}') = 0$ by induction on n.

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(3) By (1) and the definition of the weight filtration, it suffices to prove that the image of

$$\mathcal{H}^0(g_{+/S}Q_n\mathcal{F}')\longrightarrow g_{+/S}\mathcal{F}'\cong\mathcal{F}$$

coincides with $Q_n\mathcal{F}$. Note that, for any surjective homomorphism $\mathcal{F}_1\to\mathcal{F}_2$ of $\mathcal{D}_{Y'/S}$-modules supported in X, the homomorphism $\mathcal{H}^0(g_{+/S}\mathcal{F}_1)\to\mathcal{H}^0(g_{+/S}\mathcal{F}_2)$ is surjective. By the definition of $Q_n\mathcal{F}'$, the image in question coincides with the image of

$$g_*(\mathcal{D}_{Y\leftarrow Y'/S}\otimes_{\mathcal{D}_{Y'/S}}\alpha'_*(Q_n\mathcal{D}_{Y'\leftarrow X/S}\otimes_{\mathcal{D}_{X/S}}\mathscr{E}_X))\longrightarrow g_{+/S}\mathcal{F}'\cong\mathcal{F}.$$

Note that g_* is exact for sheaves supported in X. Hence, the claim follows from the fact that the natural homomorphism

$$\begin{aligned}\alpha'^{-1}(\mathcal{D}_{Y\leftarrow Y'/S})\otimes_{\alpha'^{-1}(\mathcal{D}_{Y'/S})}Q_n\mathcal{D}_{Y'\leftarrow X/S}\quad&\longrightarrow\quad\alpha'^{-1}(\mathcal{D}_{Y\leftarrow Y'/S})\otimes_{\alpha'^{-1}(\mathcal{D}_{Y'/S})}\mathcal{D}_{Y'\leftarrow X/S}\\&\xrightarrow[(2.6.3)]{\cong}\mathcal{D}_{Y\leftarrow X/S}\end{aligned}$$

is injective and its image is $Q_n\mathcal{D}_{Y\leftarrow X/S}$.

(4) Put $n=|I|-1$. The composite of the epimorphism

$$\mathcal{H}^0(g_{+/T}(\alpha'_{I*}(\mathcal{D}_{Y'\leftarrow X_I/T}\otimes_{\mathcal{O}_{X_I}}\mathscr{E}_{X_I})))\longrightarrow g_{+/T}\mathcal{F}'_I$$

with

$$g_{+/T}(\kappa_I)\colon g_{+/T}\mathcal{F}'_I\longrightarrow g_{+/T}(\mathrm{gr}_n^W\mathcal{F}')\cong\mathrm{gr}_n^W\mathcal{F}$$

is the same as the composite of

$$\begin{aligned}&\mathcal{H}^0(g_{+/T}(\alpha'_{I*}(\mathcal{D}_{Y'\leftarrow X_I/T}\otimes_{\mathcal{O}_{X_I}}\mathscr{E}_{X_I})))\longrightarrow\mathcal{H}^0(g_{+/T}(\alpha'_*(\mathrm{gr}_n^Q\mathcal{D}_{Y'\leftarrow X/S}\otimes_{\mathcal{O}_X}\mathscr{E}_X)))\\&\xrightarrow{\cong}\mathcal{H}^0(g_{+/S}(\alpha'_*(\mathrm{gr}_n^Q\mathcal{D}_{Y'\leftarrow X/S}\otimes_{\mathcal{O}_X}\mathscr{E}_X)))\longrightarrow\mathcal{H}^0(g_{+/S}(\mathrm{gr}_n^Q\mathcal{F}'))\longrightarrow\mathrm{gr}_n^Q\mathcal{F}\longrightarrow\mathrm{gr}_n^W\mathcal{F}.\end{aligned}$$

On the other hand, the composite of the epimorphism $\alpha_{I*}(\mathcal{D}_{Y\leftarrow X_I/T}\otimes_{\mathcal{O}_{X_I}}\mathscr{E}_{X_I})\to\mathcal{F}_I$ with $\kappa_I\colon\mathcal{F}_I\to\mathrm{gr}_n^W\mathcal{F}$ is the same as the composite of

$$\alpha_{I*}(\mathcal{D}_{Y\leftarrow X_I/T}\otimes_{\mathcal{O}_{X_I}}\mathscr{E}_{X_I})\longrightarrow\alpha_*(\mathrm{gr}_n^Q\mathcal{D}_{Y\leftarrow X/S}\otimes_{\mathcal{O}_X}\mathscr{E}_X)\longrightarrow\mathrm{gr}_n^Q\mathcal{F}\longrightarrow\mathrm{gr}_n^W\mathcal{F}.$$

Hence, the claim follows from the following two commutative diagrams.

$$\begin{array}{ccc}\alpha_I'^{-1}(\mathcal{D}_{Y\leftarrow Y'/T})\otimes_{\alpha_I'^{-1}(\mathcal{D}_{Y'/T})}\mathcal{D}_{Y'\leftarrow X_I/T} & \longrightarrow & \alpha'^{-1}(\mathcal{D}_{Y\leftarrow Y'/S})\otimes_{\alpha'^{-1}(\mathcal{D}_{Y'/S})}\mathrm{gr}_n^Q\mathcal{D}_{Y'\leftarrow X/S}\\ \cong\big\downarrow(2.6.3) & & \big\downarrow\cong\\ \mathcal{D}_{Y\leftarrow X_I/T} & \longrightarrow & \mathrm{gr}_n^Q\mathcal{D}_{Y\leftarrow X/S}\end{array}$$

$$\begin{array}{ccc}\mathcal{H}^0(g_{+/S}(\alpha'_*(Q_n\mathcal{D}_{Y'\leftarrow X/S}\otimes_{\mathcal{O}_X}\mathscr{E}_X))) & \longrightarrow & \mathcal{H}^0(g_{+/S}(Q_n\mathcal{F}'))\\ \cong\big\downarrow & & \big\downarrow\\ \alpha_*(Q_n\mathcal{D}_{Y\leftarrow X/S}\otimes_{\mathcal{O}_X}\mathscr{E}_X) & \longrightarrow & Q_n\mathcal{F}\end{array}$$

Here the right vertical isomorphism of the first diagram and the left vertical isomorphism of the second diagram are induced by the canonical isomorphism

$$\alpha'^{-1}(\mathcal{D}_{Y\leftarrow Y'/S})\otimes_{\alpha'^{-1}(\mathcal{D}_{Y'/S})}Q_n\mathcal{D}_{Y'\leftarrow X/S}\xrightarrow{\cong}Q_n\mathcal{D}_{Y\leftarrow X/S}$$

mentioned in the proof of (3). □

4. Nearby cycles: global case

Let X_0 be a fine and saturated log scheme smooth over T_0 such that $\mathring{X}_0$ is separated of finite type over S_0 and $\Omega^1_{X_0/T_0}$ has constant rank d. Throughout this section, we assume that X_0 satisfies the condition on X for the case $N = 1$ in the beginning of § 3. We also assume that we are given a morphism $\alpha_0\colon X_0 \to Y$ of fine log schemes over T such that the underlying morphism of schemes $\mathring{\alpha}_0\colon \mathring{X}_0 \to \mathring{Y}$ is a closed immersion, $\mathring{Y}$ is smooth and separated of finite type over S, the log structure M_Y is the inverse image of M_T, and $\Omega^1_{Y/T}$ has constant rank e. We assume that $p > 2$. In this section, we will define and study nearby cycles of a crystal $\mathcal{E}$ of $\mathcal{O}_{\mathring{X}_0/S}$-modules locally free of finite type on $(\mathring{X}_0/S)_{\mathrm{Ncrys}}$.

4.1 Preliminaries

Let X and X' denote fine and saturated log schemes over T satisfying the same conditions as in the beginning of § 2.6. Assume that we are given an isomorphism $g\colon X' \xrightarrow{\cong} X$ over T. Let B be S or T.

For a left $\mathcal{D}_{X/B}$-module $\mathcal{E}$, the natural isomorphism $g^{-1}(\mathcal{E}) \xrightarrow{\cong} g^*_{/B}(\mathcal{E})$ is compatible with the left actions of $g^{-1}(\mathcal{D}_{X/B}) \cong \mathcal{D}_{X'/B}$. Since the natural isomorphism $g^{-1}(\omega_X) \xrightarrow{\cong} \omega_{X'}$ is compatible with the right actions of $g^{-1}(\mathcal{D}_{X/B}) \cong \mathcal{D}_{X'/B}$, we see that the natural isomorphism

$$g^{-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \omega_X) \xrightarrow{\cong} g^*_{/B}(\mathcal{E}) \otimes_{\mathcal{O}_{X'}} \omega_{X'}$$

is compatible with the right actions of $g^{-1}(\mathcal{D}_{X/B}) \cong \mathcal{D}_{X'/B}$.

For a right $\mathcal{D}_{X/B}$-module $\mathcal{M}$, by applying the above argument to the left $\mathcal{D}_{X/B}$-module $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{M})$ and using Proposition 2.4.6, we see that the natural isomorphisms

$$g^{-1}(\mathcal{M}) \xrightarrow{\cong} g^{-1}(\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{M}) \otimes_{\mathcal{O}_X} \omega_X) \xrightarrow{\cong} g^*_{/B}(\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{M})) \otimes_{\mathcal{O}_{X'}} \omega_{X'} \qquad (4.1.1)$$

are compatible with the right actions of $g^{-1}(\mathcal{D}_{X/B}) \cong \mathcal{D}_{X'/B}$.

We have isomorphisms

$$g^{\natural}(\omega_X) \cong g^!(\omega_X) \cong g^! f^!(\mathcal{O}_S)[-d] \cong (f')^!(\mathcal{O}_S)[-d] \cong \omega_{X'} \qquad (4.1.2)$$

(see § 2.3). Here f (respectively f') denotes the structure morphism $X \to T$ (respectively $X' \to T$) and $d = \mathrm{rank}_{\mathcal{O}_X} \Omega^1_{X/T}$. By [Har66, Remark after Corollary 8.3 in ch. III] and an argument similar to the proof of Proposition 2.3.5, we see that the composite of the above isomorphisms is explicitly given by

$$g^{\natural}(\omega_X) \cong \mathcal{H}om_{g^{-1}(\mathcal{O}_X)}(\mathcal{O}_{X'}, g^{-1}(\omega_X)) \xrightarrow{\cong} \omega_{X'}; \quad \varphi \mapsto g^*(\varphi(1)). \qquad (4.1.3)$$

Using this isomorphism and Lemma 2.4.4(1), we see that the last term of (4.1.1) is canonically isomorphic to $g^{\natural}(\mathcal{M})$ as follows:

$$\begin{aligned} g^*(\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{M})) \otimes_{\mathcal{O}_{X'}} \omega_{X'} &\cong \mathcal{H}om_{\mathcal{O}_{X'}}(g^{\natural}(\omega_X), g^{\natural}(\mathcal{M})) \otimes_{\mathcal{O}_{X'}} \omega_{X'} \\ &\cong \mathcal{H}om_{\mathcal{O}_{X'}}(\omega_{X'}, g^{\natural}(\mathcal{M})) \otimes_{\mathcal{O}_{X'}} \omega_{X'} \\ &\cong g^{\natural}(\mathcal{M}). \end{aligned} \qquad (4.1.4)$$

Here we used Lemma 2.4.4(1) for the first isomorphism.

Lemma 4.1.5. *For a right $\mathcal{D}_{X/B}$-module $\mathcal{M}$, the homomorphism*

$$g^{-1}(\mathcal{M}) \xrightarrow{\cong} g^{\natural}(\mathcal{M}) = \mathcal{H}om_{g^{-1}(\mathcal{O}_X)}(\mathcal{O}_{X'}, g^{-1}(\mathcal{M}))$$

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obtained by composing (4.1.1) with (4.1.4) sends $m\in g^{-1}(\mathcal{M})$ *to* $\varphi_m\colon \mathcal{O}_{X'}\to g^{-1}(\mathcal{M})$ *defined by* $\varphi_m(a)=(g^*)^{-1}(a)\cdot m$. *Here* g^* *denotes the isomorphism* $g^{-1}(\mathcal{O}_X)\xrightarrow{\cong}\mathcal{O}_{X'}$.

Proof. Straightforward computation using the explicit description of the homomorphism (2.4.3) in the proof of Lemma 2.4.4(1). □

If we apply (4.1.1) to $\mathcal{D}_{X/B}$, we obtain an isomorphism

$$g^{-1}(\mathcal{D}_{X/B})\xrightarrow{\cong}\mathcal{D}_{X\leftarrow X'/B} \tag{4.1.6}$$

compatible with the left actions of $g^{-1}(\mathcal{D}_{X/B})$ and the right actions of $g^{-1}(\mathcal{D}_{X/B})\cong\mathcal{D}_{X'/B}$. Hence, for a left $\mathcal{D}_{X/B}$-module $\mathcal{E}$, by taking the tensor product of the isomorphism $g^{-1}\mathcal{E}\xrightarrow{\cong}g^*_{/B}\mathcal{E}$ with (4.1.6) over $g^{-1}(\mathcal{D}_{X/B})\cong\mathcal{D}_{X'/B}$ and applying g_*, we obtain a $\mathcal{D}_{X/B}$-linear isomorphism

$$\mathcal{E}\xrightarrow{\cong}g_{+/B}g^*_{/B}(\mathcal{E}). \tag{4.1.7}$$

LEMMA 4.1.8. *For a left* $\mathcal{D}_{X/S}$*-module* $\mathcal{E}$*, the following diagram is commutative.*

$$\begin{array}{ccc}
r_X(\mathcal{E}) & \xrightarrow[\cong]{(4.1.7)} & r_X(g_{+/S}g^*_{/S}(\mathcal{E})) \\
{\scriptstyle (4.1.7)}\downarrow\cong & & {\scriptstyle \text{Corollary 2.6.10}}\downarrow\cong \\
g_{+/T}g^*_{+/T}r_X(\mathcal{E}) & \xrightarrow[\cong]{(2.5.1)} & g_{+/T}r_{X'}g^*_{+/S}(\mathcal{E})
\end{array}$$

Here r_X *and* $r_{X'}$ *denote the functors in (2.5.1).*

Proof. This follows from the following commutative diagram.

$$\begin{array}{ccc}
g^{-1}(\mathcal{D}_{X/S}) & \xrightarrow[\cong]{(4.1.6)} & \mathcal{D}_{X\leftarrow X'/S} \\
{\scriptstyle (2.1.8)}\uparrow & & {\scriptstyle (2.6.8)}\uparrow \\
g^{-1}(\mathcal{D}_{X/T}) & \xrightarrow[\cong]{(4.1.6)} & \mathcal{D}_{X\leftarrow X'/T}
\end{array}$$

□

We will need the following compatibility of $c^*_{g,g'/B}$ and $c_{g,g',+/B}$ defined in Propositions 2.5.2 and 2.6.7 for two isomorphisms g, $g'\colon X'\xrightarrow{\cong}X$.

PROPOSITION 4.1.9. *For any two isomorphisms* $g,g'\colon X'\xrightarrow{\cong}X$ *over* T *which coincide modulo* p*, the following diagram is commutative.*

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow[(4.1.7)]{\cong} & g'_{+/B}g'^*_{/B}(\mathcal{E}) \\
{\scriptstyle (4.1.7)}\downarrow\cong & & \cong\downarrow{\scriptstyle c_{g,g',+/B}(g'^*_{/B}(\mathcal{E}))} \\
g_{+/B}g^*_{/B}(\mathcal{E}) & \xrightarrow[g_{+/B}\circ c^*_{g,g'/B}(\mathcal{E})]{\cong} & g_{+/B}g'^*_{/B}(\mathcal{E})
\end{array}$$

This proposition is obtained by applying the following more general proposition to $\mathcal{M}=\mathcal{D}_{X/B}$ and taking g_*.

PROPOSITION 4.1.10. *Let* $g,g'\colon X'\xrightarrow{\cong}X$ *be two isomorphisms over* T *which coincide modulo* p*. Then, for a right* $\mathcal{D}_{X/B}$*-module* $\mathcal{M}$ *and a left* $\mathcal{D}_{X/B}$*-module* $\mathcal{E}$*, the following diagram*

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is commutative.

$$\begin{array}{ccc} g'^{-1}(\mathcal{M}\otimes_{\mathcal{D}_{X/B}}\mathcal{E}) & \xrightarrow{\cong} & (g'^{*}_{/B}(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X,\mathcal{M}))\otimes_{\mathcal{O}_{X'}}\omega_{X'})\otimes_{\mathcal{D}_{X'/B}} g'^{*}_{/B}(\mathcal{E}) \\ \| & & \downarrow\cong \\ g^{-1}(\mathcal{M}\otimes_{\mathcal{D}_{X/B}}\mathcal{E}) & \xrightarrow{\cong} & (g^{*}_{/B}(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X,\mathcal{M}))\otimes_{\mathcal{O}_{X'}}\omega_{X'})\otimes_{\mathcal{D}_{X'/B}} g^{*}_{/B}(\mathcal{E}) \end{array}$$

Here the lower horizontal homomorphism is defined by taking the tensor product of (4.1.1) and $g^{-1}(\mathcal{E})\xrightarrow{\cong} g^*_{/B}(\mathcal{E})$ *over* $g^{-1}(\mathcal{D}_{X/B})\cong\mathcal{D}_{X'/B}$*, and similarly for the upper one. The right vertical one is defined by applying* $c^*_{g,g'/B}$ *(Proposition 2.5.2) to the left* $\mathcal{D}_{X/B}$*-modules* $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X,\mathcal{M})$ *and* $\mathcal{E}$*.*

In the rest of this subsection, we will prove Proposition 4.1.10. To prove it, we give another description of the right vertical isomorphism via (4.1.4). Let g, g' be as in Proposition 4.1.10. Since the canonical PD structure on $p\mathcal{O}_{X'}$ is nilpotent by the assumption that $p>2$, the morphism $(g,g')\colon X'\to X\times_B X$ induces a PD morphism $h\colon X'\to P^n_{X/B}$ for a sufficiently large n. Hence, for a right $\mathcal{D}_{X/B}$-module $\mathcal{M}$, we have an isomorphism

$$g^\natural(\mathcal{M})\cong h^\natural p_1^{n\natural}(\mathcal{M})\xrightarrow[h^\natural(\varepsilon_n)]{\cong} h^\natural p_2^{n\natural}(\mathcal{M})\cong g'^\natural(\mathcal{M}). \tag{4.1.11}$$

Here

$$\varepsilon_n\colon p_1^{n\natural}(\mathcal{M})\xrightarrow{\cong} p_2^{n\natural}(\mathcal{M})$$

is the isomorphism associated to the right action of $\mathcal{D}_{X/B}$ on $\mathcal{M}$ by Theorem 2.2.5. For the right $\mathcal{D}_{X/B}$-module ω_X, the definition of the right action in § 2.3 implies that the above isomorphism coincides with the composite of

$$g^\natural(\omega_X)\xrightarrow{\cong}\omega_{X'}\xleftarrow{\cong} g'^\natural(\omega_X)$$

obtained from (4.1.2) and (4.1.3).

Proposition 4.1.12. *Let* g, g' *be the same as in Proposition 4.1.10. Then, for a right* $\mathcal{D}_{X/B}$*-module* $\mathcal{M}$*, the following diagram is commutative.*

$$\begin{array}{ccc} g'^{*}_{/B}(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X,\mathcal{M}))\otimes_{\mathcal{O}_{X'}}\omega_{X'} & \xrightarrow{\cong} & g^{*}_{/B}(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X,\mathcal{M}))\otimes_{\mathcal{O}_{X'}}\omega_{X'} \\ {\scriptstyle(4.1.4)}\downarrow\cong & & \cong\downarrow{\scriptstyle(4.1.4)} \\ g'^\natural(\mathcal{M}) & \xleftarrow[(4.1.11)]{\cong} & g^\natural(\mathcal{M}) \end{array}$$

Here the upper horizontal isomorphism is defined by applying $c^*_{g,g'/B}$ *(Proposition 2.5.2) to* $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X,\mathcal{M})$*.*

Proof. By the proof of Proposition 2.4.1, the left action of $\mathcal{D}_{X/B}$ on $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X,\mathcal{M})$ corresponds to the system of isomorphisms defined in the following way:

$$\begin{aligned} p_2^{n*}(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X,\mathcal{M})) &\overset{(2.4.3)}{\cong} \mathcal{H}\mathrm{om}_{\mathcal{P}^n_{X/B}}(p_2^{n\natural}(\omega_X),p_2^{n\natural}(\mathcal{M})) \\ &\cong \mathcal{H}\mathrm{om}_{\mathcal{P}^n_{X/B}}(p_1^{n\natural}(\omega_X),p_1^{n\natural}(\mathcal{M})) \overset{(2.4.3)}{\cong} p_1^{n*}(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X,\mathcal{M})). \end{aligned}$$

Let h be as before (4.1.11). Then, by taking h^* and using Lemma 2.4.4(2) for (2.4.3), we see that $c^*_{g,g'/B}(\mathcal{H}om_{\mathcal{O}_X}(\omega_X,\mathcal{M}))$ is the composite of

$$g'^*(\mathcal{H}om_{\mathcal{O}_X}(\omega_X,\mathcal{M})) \overset{(2.4.3)}{\cong} \mathcal{H}om_{\mathcal{O}_{X'}}(g'^{\natural}(\omega_X), g'^{\natural}(\mathcal{M}))$$
$$\cong \mathcal{H}om_{\mathcal{O}_{X'}}(g^{\natural}(\omega_X), g^{\natural}(\mathcal{M})) \overset{(2.4.3)}{\cong} g^*(\mathcal{H}om_{\mathcal{O}_X}(\omega_X,\mathcal{M})),$$

where the second isomorphism is induced by (4.1.11) for ω_X and $\mathcal{M}$. By the remark after (4.1.11), we see that the diagram in the proposition is commutative. □

Proof of Proposition 4.1.10. Let h be as before (4.1.11) and let $\theta_n\colon \mathcal{E}\to\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{P}^n_{X/B}$ be the homomorphism corresponding to the left action of $\mathcal{D}_{X/B}$ on $\mathcal{E}$ (Proposition 2.2.3). Then, for $e\in\mathcal{E}$, the image of $g'^{-1}(e)$ under the composite of

$$g'^{-1}(\mathcal{E}) \xrightarrow{\cong} g'^*_{/B}(\mathcal{E}) \xrightarrow[c^*_{g,g'/B}(\mathcal{E})]{\cong} g^*_{/B}(\mathcal{E}) \xleftarrow{\cong} g^{-1}(\mathcal{E})$$

is the image of $h^{-1}(\theta_n(e))$ by

$$h^{-1}(p_1^{n*}(\mathcal{E})) \xrightarrow{h^*} g^*(\mathcal{E}) \xrightarrow{\cong} g^{-1}\mathcal{E},$$

which is $g^{-1}(P\cdot e)$, where P denotes the section $\mathcal{P}^n_{X/B}\xrightarrow{h^*}\mathcal{O}_{X'}\underset{\cong}{\xleftarrow{g^*}}\mathcal{O}_X$ of $\mathcal{D}_{X/B,n}$. On the other hand, by the construction of the isomorphism (4.1.11) and Lemma 4.1.5, we see that the image of $g^{-1}(m)$ ($m\in\mathcal{M}$) under the composite of

$$g^{-1}(\mathcal{M}) \xrightarrow{\cong} g^{\natural}(\mathcal{M}) \xrightarrow[(4.1.11)]{\cong} g'^{\natural}(\mathcal{M}) \xleftarrow{\cong} g'^{-1}(\mathcal{M})$$

is the image of the section $g^{-1}(\{b\mapsto (g^*)^{-1}\circ h^*(b)\cdot m\})$ of $g^{-1}(\mathcal{H}om_{\mathcal{O}_X}(p^n_{1*}\mathcal{P}^n_{X/B},\mathcal{M}))$ by

$$g^{-1}(\mathcal{H}om_{\mathcal{O}_X}(p^n_{1*}\mathcal{P}^n_{X/B},\mathcal{M})) \xrightarrow[h^{-1}(\varepsilon_n)]{\cong} g'^{-1}(\mathcal{H}om_{\mathcal{O}_X}(p^n_{2*}\mathcal{P}^n_{X/B},\mathcal{M})) \longrightarrow g'^{-1}(\mathcal{M}),$$

where the second map is the evaluation at 1. The latter image is $g'^{-1}(m\cdot P)$ for $P\in\mathcal{D}_{X/B,n}$ as above. By Proposition 4.1.12, we see that the diagram of the proposition is commutative. □

4.2 Affine case

Let $\alpha_0\colon X_0\to Y$ be as in the beginning of § 4 and assume that the underlying scheme of X_0 is affine in this subsection. We consider both T and S as a base and apply the result for T to the intersections of smooth components of X_0 (cf. (4.2.5)).

Let B denote S or T as in § 2, and let $\mathscr{G}$ be a crystal of $\mathcal{O}_{X_0/B}$-modules locally free of finite type on the nilpotent crystalline site $(X_0/B)_{\mathrm{Ncrys}}$. By assumption, there exists a smooth lifting $X\to T$ of $X_0\to T_0$, which is unique up to (non-canonical) isomorphisms [Kat89, Proposition 3.14(1)]. By the uniqueness and the assumption on X_0, we see that X/T satisfies the conditions in the beginning of § 3. For such an X, the PD thickenings $X_0\to P^n_{X/B}(r)$ and the projections among them are objects and morphisms of the site $(X_0/B)_{\mathrm{Ncrys}}$. Hence, the evaluation $\mathscr{G}_X$ of $\mathscr{G}$ on $X_0\to X$ is regarded as a left $\mathcal{D}_{X/B}$-module by Theorem 2.2.1. For an X as above, there exists also a morphism $\alpha\colon X\to Y$ whose composite with $X_0\to X$ is α_0 [Kat89, Corollary 3.11]. The underlying morphism of schemes of such an α is always a closed immersion. By Theorem 3.1.2(1), we have $\mathcal{H}^i(\alpha_{+/B}(\mathscr{G}_X))=0$ for $i\neq 0$. We regard $\alpha_{+/B}(\mathscr{G}_X)$ as a left $\mathcal{D}_{Y/B}$-module in the following.

Now suppose that we are given two such pairs (X, α) and (X', α'). If we choose an isomorphism $\iota\colon X' \xrightarrow{\cong} X$ over T inducing the identity on X_0, then we have the following isomorphisms:

$$\begin{aligned} &\alpha_{+/B}\mathscr{G}_X \xrightarrow[\text{(4.1.7)}]{\cong} \alpha_{+/B}\iota_{+/B}\iota^*_{/B}\mathscr{G}_X \xrightarrow{\cong} \alpha_{+/B}\iota_{+/B}\mathscr{G}_{X'} \\ &\xrightarrow[\text{Proposition 2.6.5}]{\cong} (\alpha\circ\iota)_{+/B}\mathscr{G}_{X'} \xrightarrow[\text{Proposition 2.6.7}]{\cong} \alpha'_{+/B}\mathscr{G}_{X'}. \end{aligned} \tag{4.2.1}$$

Proposition 4.2.2.

(1) *The composite of the isomorphism (4.2.1) is independent of the choice of the isomorphism* ι. *Let* $c_{(X',\alpha'),(X,\alpha)/B}$ *denote the composite.*

(2) *If* $X = X'$, *the isomorphism* $c_{(X',\alpha'),(X,\alpha)/B}$ *coincides with* $c_{\alpha',\alpha,+/B}$ *in Proposition 2.6.7.*

(3) *For three pairs* (X, α), (X', α'), *and* (X'', α''), *we have*

$$c_{(X'',\alpha''),(X,\alpha)/B} = c_{(X'',\alpha''),(X',\alpha')/B} \circ c_{(X',\alpha'),(X,\alpha)/B}.$$

(4) *Let* $\mathscr{G}$ *be a crystal of* $\mathcal{O}_{X_0/S}$*-modules locally free of finite type on* $(X_0/S)_{\mathrm{Ncrys}}$ *and let* $\mathscr{G}'$ *denote the inverse image of* $\mathscr{G}$ *on* $(X_0/T)_{\mathrm{Ncrys}}$. *Let* r_X *denote the functor* $\mathcal{D}_{X/S}$-**Mod** $\to$ $\mathcal{D}_{X/T}$-**Mod** *induced by (2.1.8), and define* $r_{X'}$ *and* r_Y *similarly. Then the following diagram is commutative.*

$$\begin{array}{ccccc} r_Y\circ\alpha_{+/S}(\mathscr{G}_X) & \xrightarrow[\cong]{\text{Corollary 2.6.10}} & \alpha_{+/T}\circ r_X(\mathscr{G}_X) & \xrightarrow[\cong]{} & \alpha_{+/T}(\mathscr{G}'_X) \\ {\scriptstyle r_Y\circ c_{(X',\alpha'),(X,\alpha)/S}}\downarrow{\scriptstyle\cong} & & & & {\scriptstyle\cong}\downarrow{\scriptstyle c_{(X',\alpha'),(X,\alpha)/T}} \\ r_Y\circ\alpha'_{+/S}(\mathscr{G}_{X'}) & \xrightarrow[\cong]{\text{Corollary 2.6.10}} & \alpha'_{+/T}\circ r_{X'}(\mathscr{G}_{X'}) & \xrightarrow[\cong]{} & \alpha'_{+/T}(\mathscr{G}'_{X'}) \end{array}$$

Proof. (1) Let $\iota'\colon X' \xrightarrow{\cong} X$ be another isomorphism. The claim follows from the following commutative diagram.

$$\begin{array}{ccccccc} \alpha_{+/B}\mathscr{G}_X & \xrightarrow{\cong} & \alpha_{+/B}\iota_{+/B}\iota^*_{/B}\mathscr{G}_X & \xrightarrow{\cong} & \alpha_{+/B}\iota_{+/B}\mathscr{G}_{X'} & \xrightarrow{\cong} & (\alpha\circ\iota)_{+/B}\mathscr{G}_{X'} & \searrow^{c_{\alpha',\alpha\circ\iota,+/B}}_{\cong} & \\ & \searrow_{\cong} & & & {\scriptstyle\alpha_{+/B}(c_{\iota,\iota',+/B})}\uparrow{\scriptstyle\cong} & & {\scriptstyle c_{\alpha\circ\iota,\alpha\circ\iota',+/B}}\uparrow{\scriptstyle\cong} & & \\ & & \alpha_{+/B}\iota'_{+/B}\iota'^*_{/B}\mathscr{G}_X & \xrightarrow{\cong} & \alpha_{+/B}\iota'_{+/B}\mathscr{G}_{X'} & \xrightarrow{\cong} & (\alpha\circ\iota')_{+/B}\mathscr{G}_{X'} & \xrightarrow[c_{\alpha',\alpha\circ\iota',+/B}]{\cong} & \alpha'_{+/B}\mathscr{G}_{X'} \end{array}$$

The left-hand square is (respectively the middle square and the right-hand triangle are) commutative by Proposition 4.1.9 (respectively Proposition 2.6.7).

(2) We are reduced to the fact that the composite of

$$\mathcal{D}_{Y\leftarrow X/B} \xrightarrow[\mathcal{D}_{Y\leftarrow X/B}\otimes(4.1.6)]{\cong} \mathcal{D}_{Y\leftarrow X/B} \otimes_{\mathcal{D}_{X/B}} \mathcal{D}_{X \overset{\mathrm{id}}{\leftarrow} X/B} \xrightarrow[(2.6.3)]{\cong} \mathcal{D}_{Y\leftarrow X/B}$$

is the identity map, which is verified by a straightforward computation.

(3) Choose isomorphisms $\iota\colon X' \xrightarrow{\cong} X$ and $\iota'\colon X'' \xrightarrow{\cong} X'$. By using Propositions 2.6.7 and 2.6.6, we are reduced to showing that the composite of the isomorphisms

$$\mathscr{G}_X \xrightarrow{\cong} \iota_{+/B}\iota^*_{/B}\mathscr{G}_X \xrightarrow{\cong} \iota_{+/B}\iota'_{+/B}\iota'^*_{/B}\iota^*_{/B}\mathscr{G}_X$$

coincides with the isomorphism

$$\mathscr{G}_X \xrightarrow{\cong} (\iota\circ\iota')_{+/B}(\iota\circ\iota')^*_{/B}\mathscr{G}_X \cong \iota_{+/B}\iota'_{+/B}\iota'^*_{/B}\iota^*_{/B}\mathscr{G}_X.$$

This follows from the following commutative diagram

$$\begin{array}{ccc}
\mathcal{D}_{X/B} & \xrightarrow[\text{(4.1.6)}]{\cong} & \mathcal{D}_{X\leftarrow X'/B} \\
{\scriptstyle (4.1.6)}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle \mathcal{D}_{X\leftarrow X'/B}\otimes(4.1.6)} \\
\mathcal{D}_{X\leftarrow X''/B} & \xleftarrow[\text{(2.6.3)}]{\cong} & \mathcal{D}_{X\leftarrow X'/B}\otimes_{\mathcal{D}_{X'/B}}\mathcal{D}_{X'\leftarrow X''/B}
\end{array}$$

which is verified by a direct computation. Here we regard sheaves on X'_{Zar} (respectively X''_{Zar}) as sheaves on X_{Zar} via ι (respectively $\iota \circ \iota'$).

(4) The claim follows from Lemmas 2.6.11, 2.6.12, and 4.1.8. □

DEFINITION 4.2.3. Let B be S or T, and let $\mathscr{G}$ be a crystal of $\mathcal{O}_{X_0/B}$-modules locally free of finite type on $(X_0/B)_{\mathrm{Ncrys}}$. We define the left $\mathcal{D}_{Y/B}$-module $\alpha_{0,+/B}(\mathscr{G})$ to be $\alpha_{+/B}(\mathscr{G}_X)$, which is independent of the choice of (X, α) up to canonical isomorphisms by Proposition 4.2.2. When $B = S$, we call $\alpha_{0,+/S}(\mathscr{G})$ *the nearby cycles of* $\mathscr{G}$ *realized on* Y.

For a crystal $\mathscr{G}$ of $\mathcal{O}_{X_0/S}$-modules locally free of finite type on $(X_0/S)_{\mathrm{Ncrys}}$, $\alpha_{0,+/S}(\mathscr{G})$ regarded as a left $\mathcal{D}_{Y/T}$-module via (2.1.8) is canonically isomorphic to $\alpha_{0,+/T}(\mathscr{G}')$ by Proposition 4.2.2(4), where $\mathscr{G}'$ denotes the inverse image of $\mathscr{G}$ on $(X_0/T)_{\mathrm{Ncrys}}$.

Let $\mathscr{E}$ be a crystal of $\mathcal{O}_{\mathring{X}_0/S}$-modules locally free of finite type on $(\mathring{X}_0/S)_{\mathrm{Ncrys}}$ and let $\mathscr{G}$ be the inverse image of $\mathscr{E}$ on $(X_0/S)_{\mathrm{Ncrys}}$. Then $\mathscr{G}_X$ has trivial monodromy (Definition 3.3.2) and we may apply the construction in § 3.3 and obtain the weight filtration $W_n(\alpha_{+/S}(\mathscr{G}_X))$ $(n \in \mathbb{Z})$ of $\alpha_{+/S}(\mathscr{G}_X)$. See before Theorem 3.4.5.

PROPOSITION 4.2.4. *For* $\mathscr{E}$ *and* $\mathscr{G}$ *as above, the isomorphism*

$$c_{(X',\alpha'),(X,\alpha)/S}\colon \alpha_{+/S}(\mathscr{G}_X) \xrightarrow{\cong} \alpha'_{+/S}(\mathscr{G}_{X'})$$

induces an isomorphism between the weight filtrations:

$$W_n(\alpha_{+/S}(\mathscr{G}_X)) \xrightarrow{\cong} W_n(\alpha'_{+/S}(\mathscr{G}_{X'})) \quad (n \in \mathbb{Z}).$$

Proof. Since $c_{(X',\alpha'),(X,\alpha)/S}$ is an isomorphism of $\mathcal{D}_{X/S}$-modules, it suffices to prove that $c_{(X',\alpha'),(X,\alpha)/S}$ induces an isomorphism between the filtrations $Q_\bullet$ (cf. § 3.3). Choose an isomorphism $\iota\colon X' \xrightarrow{\cong} X$. By Proposition 4.2.2, we may assume that $\alpha' = \alpha \circ \iota$ or $X = X'$. In the first (respectively the second) case, the claim follows from the fact that the composite of

$$\iota^{-1}(\mathcal{D}_{Y\leftarrow X/S}) \xrightarrow[\iota^{-1}(\mathcal{D}_{Y\leftarrow X/S})\otimes(4.1.6)]{\cong} \iota^{-1}(\mathcal{D}_{Y\leftarrow X/S}) \otimes_{\iota^{-1}(\mathcal{D}_{X/S})} \mathcal{D}_{X\leftarrow X'/S}$$
$$\xrightarrow[(2.6.3)]{\cong} \mathcal{D}_{Y\leftarrow X'/S}$$

(respectively the isomorphism $\mathcal{D}_{Y \overset{\alpha}{\leftarrow} X/S} \cong \mathcal{D}_{Y \overset{\alpha'}{\leftarrow} X/S}$ used in the construction of $\alpha_{+/S} \cong \alpha'_{+/S}$) is a filtered isomorphism for the filtrations $Q_\bullet$. □

By Proposition 4.2.4, the nearby cycles $\alpha_{0,+/S}(\mathscr{G})$ of $\mathscr{G}$ realized on Y are canonically endowed with the weight filtration $W_n(\alpha_{0,+/S}(\mathscr{G}))$ $(n \in \mathbb{Z})$.

In the following, we assume that all irreducible components of $\mathring{X}_0$ are smooth over S_0 and define $X_{0,\lambda}$ $(\lambda \in \Lambda)$ and $X_{0,I}$, X_I, and $\alpha_I\colon X_I \to Y$ $(I \subset \Lambda)$ as in § 3.4. Let $\alpha_{0,I}$ denote the exact closed immersion $X_{0,I} \to Y$ over T induced by $\mathring{X}_{0,I} \to \mathring{X}_0 \xrightarrow{\mathring{\alpha}_0} \mathring{Y}$. Let $\mathscr{G}_I$ denote the inverse

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image of $\mathscr{E}$ on $(X_{0,I}/T)_{\mathrm{Ncrys}}$ and let $\mathscr{G}_{I,X_I}$ denote the $\mathcal{D}_{X_I/T}$-module obtained by evaluating $\mathscr{G}_I$ on $X_{0,I} \to X_I$. By applying Proposition 4.2.2 to $B = T$, $(X_{0,I}, \alpha_{0,I})$, and $\mathscr{G}_I$, we obtain an isomorphism

$$c_{(X'_I,\alpha'_I),(X_I,\alpha_I)/T} \colon \alpha_{I,+/T}(\mathscr{G}_{I,X_I}) \xrightarrow{\cong} \alpha'_{I,+/T}(\mathscr{G}_{I,X'_I}) \tag{4.2.5}$$

for two pairs (X,α) and (X',α'), which satisfies the cocycle condition as in Proposition 4.2.2(3).

Proposition 4.2.6. *For two pairs (X,α), (X',α') and a non-empty subset I of Λ of cardinality $\leqslant d+1$, the following diagram is commutative.*

$$\begin{array}{ccc} \alpha_{I,+/T}(\mathscr{G}_{I,X_I}) & \xrightarrow{\kappa_I} & \mathrm{gr}^W_{|I|-1}(\alpha_{+/S}(\mathscr{G}_X)) \\ {\scriptstyle c_{(X'_I,\alpha'_I),(X_I,\alpha_I)/T}}\downarrow\cong & & \cong\downarrow{\scriptstyle c_{(X',\alpha'),(X,\alpha)/S}} \\ \alpha'_{I,+/T}(\mathscr{G}_{I,X'_I}) & \xrightarrow{\kappa_I} & \mathrm{gr}^W_{|I|-1}(\alpha'_{+/S}(\mathscr{G}_{X'})) \end{array}$$

Proof. Since the two vertical isomorphisms satisfy the cocycle condition, it suffices to prove the claim when $X = X'$ or $\alpha' = \alpha \circ \iota$ for an isomorphism $\iota \colon X' \xrightarrow{\cong} X$ of liftings of X_0. In the case $X = X'$, the claim is reduced to the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{D}_{Y \overset{\alpha_I}{\leftarrow} X_I/T} & \longrightarrow & \mathrm{gr}^Q_{|I|-1}\,\mathcal{D}_{Y \overset{\alpha}{\leftarrow} X/S} \\ \cong\downarrow & & \cong\downarrow \\ \mathcal{D}_{Y \overset{\alpha'_I}{\leftarrow} X_I/T} & \longrightarrow & \mathrm{gr}^Q_{|I|-1}\,\mathcal{D}_{Y \overset{\alpha'}{\leftarrow} X/S} \end{array}$$

where the left (respectively right) vertical isomorphism is the isomorphism (respectively is induced by the isomorphism $\mathcal{D}_{Y \overset{\alpha}{\leftarrow} X/S} \cong \mathcal{D}_{Y \overset{\alpha'}{\leftarrow} X/S}$) used in the construction of $\alpha_{I,+/T} \cong \alpha'_{I,+/T}$ (respectively $\alpha_{+/S} \cong \alpha'_{+/S}$). Let $\underline{X}$ be $\mathring{X}$ with the inverse image of M_T. Then the morphisms α and α_I factor through the same morphism $\underline{\alpha} \colon \underline{X} \to Y$, and the upper horizontal homomorphism of the above diagram is written as follows:

$$\underline{\alpha}^*(\mathcal{D}_{Y/T} \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_X} \omega_{X_I} \longrightarrow \underline{\alpha}^*(\mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_X} \mathrm{gr}^P_{|I|-1}\,\omega_X.$$

Here we take $\underline{\alpha}^*$ with respect to the action of $\mathcal{O}_Y$ through ω_Y^{-1}. Hence, the commutativity follows from (2.5.1) and Lemma 2.4.7 applied to $\mathcal{M} = \mathcal{D}_{Y/S}$ and $\mathcal{N} = \omega_Y$. In the case $\alpha' = \alpha \circ \iota$, ι induces an isomorphism $\iota_I \colon X'_I \xrightarrow{\cong} X_I$ and we have $\alpha'_I = \alpha_I \circ \iota_I$. The right vertical isomorphism is induced by

$$\mathrm{id} \otimes \iota^* \colon (\mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_Y} \omega_X \xrightarrow{\cong} (\mathcal{D}_{Y/S} \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_Y} \omega_{X'},$$

and the left one is induced by the isomorphism obtained by replacing $\mathcal{D}_{Y/S}$, ω_X, $\omega_{X'}$, and ι with $\mathcal{D}_{Y/T}$, ω_{X_I}, $\omega_{X'_I}$, and ι_I. Hence, the claim follows from the compatibility of the homomorphisms $\omega_{X_I^{(\prime)}} \to \mathrm{gr}^P_{|I|-1}\,\omega_{X^{(\prime)}}$ with ι_I^* and ι^*. □

Let $\mathscr{E}_{X_0/S}$ (respectively $\mathscr{E}_{X_{0,I}/T}$) denote the inverse image $\mathscr{G}$ (respectively $\mathscr{G}_I$) of $\mathscr{E}$ on $(X_0/S)_{\mathrm{Ncrys}}$ (respectively $(X_{0,I}/T)_{\mathrm{Ncrys}}$). Then, by Proposition 4.2.6, we have a canonical morphism of $\mathcal{D}_{Y/T}$-modules:

$$\kappa_I \colon \alpha_{0,I,+/T}(\mathscr{E}_{X_{0,I}/T}) \longrightarrow \mathrm{gr}^W_{|I|-1}(\alpha_{0,+/S}(\mathscr{E}_{X_0/S})) \tag{4.2.7}$$

for each non-empty subset I of Λ of cardinality $\leqslant d+1$.

4.3 General case

Let $\alpha_0\colon X_0 \to Y$ be as in the beginning of § 4. Let B be S or T, and let $\mathscr{G}$ be a crystal of $\mathcal{O}_{X_0/B}$-modules locally free of finite type on $(X_0/B)_{\mathrm{Ncrys}}$. Let V be an open log subscheme of Y (i.e. an open subscheme of $\mathring{Y}$ endowed with the inverse image of M_Y) such that the underlying scheme of $U_0 := X_0 \times_Y V$ is affine. Choose a smooth lifting U of U_0 over T, and a morphism $\alpha_U\colon U \to V$ over T whose composite with $U_0 \to U$ is $\alpha_0|_{U_0}$. Let V' be an open log subscheme of V such that the underlying scheme of $U_0' := X_0 \times_Y V'$ is affine, and put $U' = U \times_V V'$ and $\alpha_{U'} = \alpha_U|_{U'}$. Then we have a natural isomorphism $(\alpha_{U+/B}\mathscr{G}_U)|_{V'} \cong \alpha_{U'+/B}\mathscr{G}_{U'}$ of $\mathcal{D}_{V'/B}$-modules, where $\mathscr{G}_U$ denotes the $\mathcal{D}_{U/B}$-module obtained by evaluating $\mathscr{G}$ on $U_0 \hookrightarrow U$ and similarly for $\mathscr{G}_{U'}$ (cf. the beginning of § 4.2). If we are given another pair $(\widetilde{U}, \alpha_{\widetilde{U}})$, then the above isomorphisms for (U, α_U) and $(\widetilde{U}, \alpha_{\widetilde{U}})$ are compatible with the isomorphisms $c_{(U,\alpha_U),(\widetilde{U},\alpha_{\widetilde{U}})/B}$ and $c_{(U',\alpha_{U'}),(\widetilde{U}',\alpha_{\widetilde{U}'})/B}$ (cf. Proposition 4.2.2), since the isomorphisms in Propositions 2.6.5 and 2.6.7 and the isomorphism (4.1.7) are compatible with the restrictions to Zariski open subschemes. Hence, we have a canonical isomorphism of $\mathcal{D}_{V'/B}$-modules

$$\rho_{V',V}\colon (\alpha_{U_0,+/B}(\mathscr{G}_{U_0/B}))|_{V'} \xrightarrow{\cong} \alpha_{U_0',+/B}(\mathscr{G}_{U_0'/B}).$$

Here $\mathscr{G}_{U_0/B}$ denotes the inverse image of $\mathscr{G}$ on $(U_0/B)_{\mathrm{Ncrys}}$, $\alpha_{U_0} = \alpha_0|_{U_0}$, and similarly for $\mathscr{G}_{U_0'/B}$, $\alpha_{U_0'}$. See also Definition 4.2.3. For an open log subscheme V'' of V' such that the underlying scheme of $U_0'' := X_0 \times_Y V''$ is affine, we have

$$\rho_{V'',V} = \rho_{V'',V'} \circ \rho_{V',V}|_{V''}.$$

Lemma 4.3.1. *Let the notation and assumptions be as above. There exists a left $\mathcal{D}_{Y/B}$-module $\mathcal{F}$ with an isomorphism of $\mathcal{D}_{V/B}$-modules $\rho_V\colon \mathcal{F}|_V \xrightarrow{\cong} \alpha_{U_0,+/B}(\mathscr{G}_{U_0/B})$ for each open log subscheme V of Y with affine $U_0 = X_0 \times_Y V$ such that the following diagram is commutative for every two open log subschemes $V' \subset V$ of Y with affine inverse images U_0' and U_0 on X_0.*

$$\begin{array}{ccc} \mathcal{F}|_{V'} & \xrightarrow[\rho_V|_{V'}]{\cong} & (\alpha_{U_0,+/B}(\mathscr{G}_{U_0/B}))|_{V'} \\ & {\scriptstyle \rho_{V'}}\searrow^{\cong} & \downarrow{\scriptstyle \rho_{V',V}}\ \cong \\ & & \alpha_{U_0',+/B}(\mathscr{G}_{U_0'/B}) \end{array}$$

Furthermore, $(\mathcal{F}, \{\rho_V\})$ is unique up to unique isomorphisms.

Proof. Put $\mathcal{F}_V := \alpha_{U_0,+/B}(\mathscr{G}_{U_0/B})$ for simplicity. For two open log subschemes V, V' of Y with affine inverse images on X_0, define the isomorphism $\rho_{V',V}$ to be the composite

$$\rho_{V\cap V',V'}^{-1} \circ \rho_{V\cap V',V}\colon \mathcal{F}_V|_{V\cap V'} \xrightarrow{\cong} \mathcal{F}_{V\cap V'} \xleftarrow{\cong} \mathcal{F}_{V'}|_{V\cap V'}.$$

Then $\rho_{V',V}$ satisfies the cocycle condition for three V, V', and V'', and there exists a unique $\mathcal{F}$ with $\rho_V\colon \mathcal{F}|_V \xrightarrow{\cong} \mathcal{F}_V$ compatible with the isomorphism $\rho_{V',V}$. Since the compatibility with $\rho_{V',V}$ for every V, V' is equivalent to that for every V, V' satisfying $V' \subset V$, this implies the claim. □

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Definition 4.3.2. Let the notation and assumptions be as above. We define the left $\mathcal{D}_{Y/B}$-module $\alpha_{0,+/B}(\mathscr{G})$ to be $\mathcal{F}$ considered in Lemma 4.3.1. When $B = S$, we call $\alpha_{0,+/S}(\mathscr{G})$ *the nearby cycles of* $\mathscr{G}$ *realized on* Y.

Let $\mathscr{G}$ be a crystal of $\mathcal{O}_{X_0/S}$-modules locally free of finite type on $(X_0/S)_{\mathrm{Ncrys}}$ and let $\mathscr{G}'$ denote the inverse image of $\mathscr{G}$ on $(X_0/T)_{\mathrm{Ncrys}}$. Then we have a canonical isomorphism as $\mathcal{D}_{Y/T}$-modules

$$\alpha_{0,+/S}(\mathscr{G}) \cong \alpha_{0,+/T}(\mathscr{G}') \tag{4.3.3}$$

such that the following diagram is commutative for every open log subscheme of Y with affine $U_0 := X_0 \times_Y V$.

$$\begin{array}{ccc} \alpha_{0,+/S}(\mathscr{G})|_V & \xrightarrow[\cong]{\rho_V} & \alpha_{U_0,+/S}(\mathscr{G}_{U_0/S}) \\ \cong\downarrow & & \cong\downarrow \\ \alpha_{0,+/T}(\mathscr{G}')|_V & \xrightarrow[\cong]{\rho_V} & \alpha_{U_0,+/T}(\mathscr{G}'_{U_0/T}) \end{array}$$

Here the right vertical isomorphism is the one mentioned after Definition 4.2.3.

Let $\mathscr{E}$ be a crystal of $\mathcal{O}_{\mathring{X}_0/S}$-modules locally free of finite type on $(\mathring{X}_0/S)_{\mathrm{Ncrys}}$, and let $\mathscr{G}$ be the inverse image of $\mathscr{E}$ on $(X_0/S)_{\mathrm{Ncrys}}$. Then the isomorphism $\rho_{V',V}$ is a filtered isomorphism with respect to the weight filtrations. Hence, the weight filtration on $\alpha_{U_0,+/S}(\mathscr{G}_{U_0/S})$ glues and gives the weight filtration $W_n(\alpha_{0,+/S}(\mathscr{G}))$ $(n \in \mathbb{Z})$ on the nearby cycles $\alpha_{0,+/S}(\mathscr{G})$ of $\mathscr{G}$ realized on Y.

As in the beginning of § 3.4, let $X_{0,\lambda}$ $(\lambda \in \Lambda)$ be the irreducible components of $\mathring{X}_0$ and assume that $X_{0,\lambda}$ is smooth over S_0 for every $\lambda \in \Lambda$. We choose and fix a total order of the finite set Λ. For a non-empty subset I of Λ, we define $X_{0,I}$ to be the fiber product of $X_{0,\lambda}$ $(\lambda \in I)$ over $\mathring{X}_0$ endowed with the inverse image of M_T, and $\alpha_{0,I}$ to be the T-morphism $X_{0,I} \to Y$ induced by $\mathring{X}_{0,I} \to \mathring{X}_0 \xrightarrow{\mathring{\alpha}_0} \mathring{Y}$.

For two open log subschemes $V' \subset V$ with affine $U_0 := X_0 \times_Y V$ and $U_0' := X_0 \times_Y V'$, the following diagram is commutative.

$$\begin{array}{ccc} (\alpha_{U_0,I,+/T}(\mathscr{E}_{U_0,I/T}))|_{V'} & \xrightarrow[(4.2.7)]{\kappa_I|_{V'}} & (\mathrm{gr}^W_{|I|-1}(\alpha_{U_0,+/S}(\mathscr{E}_{U_0/S})))|_{V'} \\ \rho_{V',V}\downarrow\cong & & \rho_{V',V}\downarrow\cong \\ \alpha_{U'_0,I,+/T}(\mathscr{E}_{U'_{0,I}/T}) & \xrightarrow[(4.2.7)]{\kappa_I} & \mathrm{gr}^W_{|I|-1}(\alpha_{U'_0,+/S}(\mathscr{E}_{U'_0/S})) \end{array}$$

Here $U_{0,I} = X_{0,I} \times_Y V$, $\alpha_{U_0,I} = \alpha_{0,I}|_{U_{0,I}}$, and $\mathscr{E}_{U_{0,I}/T}$ (respectively $\mathscr{E}_{U_0/S}$) denotes the inverse image of $\mathscr{E}$ on $(U_{0,I}/T)_{\mathrm{Ncrys}}$ (respectively $(U_0/S)_{\mathrm{Ncrys}}$). Similarly for $U'_{0,I}$, etc.

Let $\mathscr{E}_{X_0/S}$ (respectively $\mathscr{E}_{X_{0,I}/T}$) denote the inverse image of $\mathscr{E}$ on $(X_0/S)_{\mathrm{Ncrys}}$ (respectively $(X_{0,I}/T)_{\mathrm{Ncrys}}$). Then we can glue κ_I and obtain a canonical morphism of $\mathcal{D}_{Y/T}$-modules:

$$\kappa_I \colon \alpha_{0,I,+/T}(\mathscr{E}_{X_{0,I}/T}) \longrightarrow \mathrm{gr}^W_{|I|-1}(\alpha_{0,+/S}(\mathscr{E}_{X_0/S})). \tag{4.3.4}$$

From Theorem 3.4.5 and Corollary 3.4.6, we obtain the isomorphisms

$$\bigoplus_{\substack{b-2a=n \\ 0\leqslant a\leqslant b\leqslant d}} \bigoplus_{\substack{I\subset\Lambda \\ |I|=b+1}} \alpha_{0,I,+/T}(\mathscr{E}_{X_{0,I}/T}) \xrightarrow{\cong} \mathrm{gr}^W_n(\alpha_{0,+/S}(\mathscr{E}_{X_0/S}));\ (x_I)_{a,b,I} \mapsto \sum_{a,b,I} (\partial_t^{\log})^a(\kappa_I(x_I)) \tag{4.3.5}$$

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for $-d \leqslant n \leqslant d$ and the isomorphisms

$$(\partial_t^{\log})^n\colon\ \mathrm{gr}_n^W(\alpha_{0,+/S}(\mathscr{E}_{X_0/S})) \xrightarrow{\cong} \mathrm{gr}_{-n}^W(\alpha_{0,+/S}(\mathscr{E}_{X_0/S})) \tag{4.3.6}$$

for $0 \leqslant n \leqslant d$.

4.4 Independence on Y

Let $\alpha_0\colon X_0 \to Y$ be as in the beginning of § 4. Let $\alpha'_0\colon X_0 \to Y'$ be another morphism satisfying the same conditions as α_0 and let $g\colon Y' \to Y$ be a morphism over T such that $g \circ \alpha'_0 = \alpha_0$.

We first assume that $\mathring{X}_0$ is affine. Let B be S or T, and let $\mathscr{G}$ be a crystal of $\mathcal{O}_{X_0/B}$-modules locally free of finite type on $(X_0/B)_{\mathrm{Ncrys}}$. Let $X \to T$ be a smooth lifting of X_0 over T and let $\alpha'\colon X \to Y'$ be a morphism over T such that the composite with $X_0 \to X$ is α'_0. Put $\alpha = g \circ \alpha'$. Let $\mathscr{G}_X$ be the $\mathcal{D}_{X/B}$-module associated to $\mathscr{G}$ as in the beginning of § 4.2. Then, by Proposition 2.6.5, we have a canonical isomorphism $g_{+/B}\alpha'_{+/B}\mathscr{G}_X \cong \alpha_{+/B}\mathscr{G}_X$. For another pair $(\widetilde{X}, \widetilde{\alpha}')$ and $\widetilde{\alpha} = g \circ \widetilde{\alpha}'$, we see that the above isomorphisms for (X, α') and $(\widetilde{X}, \widetilde{\alpha}')$ are compatible with the isomorphisms $c_{(X,\alpha'),(\widetilde{X},\widetilde{\alpha}')/B}$ and $c_{(X,\alpha),(\widetilde{X},\widetilde{\alpha})/B}$ (cf. Proposition 4.2.2) by Propositions 2.6.6 and 2.6.7(2). Hence, we have a canonical isomorphism of $\mathcal{D}_{Y/B}$-modules:

$$g_{+/B}\alpha'_{0,+/B}\mathscr{G} \cong \alpha_{0,+/B}\mathscr{G}. \tag{4.4.1}$$

By Lemma 2.6.11, we see that the following diagram is commutative for a crystal $\mathscr{G}$ of $\mathcal{O}_{X_0/S}$-modules locally free of finite type on $(X_0/S)_{\mathrm{Ncrys}}$.

$$\begin{array}{ccccc} r_Y g_{+/S}\alpha'_{0,+/S}(\mathscr{G}) & \xrightarrow[\cong]{\text{Lemma 2.6.12}} & g_{+/T} r_{Y'}\alpha'_{0,+/S}(\mathscr{G}) & \xrightarrow{\cong} & g_{+/T}\alpha'_{0,+/T}(\mathscr{G}') \\ {\scriptstyle \cong}\downarrow{\scriptstyle (4.4.1)} & & & & {\scriptstyle \cong}\downarrow{\scriptstyle (4.4.1)} \\ r_Y\alpha_{0,+/S}(\mathscr{G}) & & \xrightarrow{\qquad\qquad\cong\qquad\qquad} & & \alpha_{0,+/T}(\mathscr{G}') \end{array} \tag{4.4.2}$$

where r_Y and $r_{Y'}$ are defined as in (2.5.1) and $\mathscr{G}'$ denotes the inverse image of $\mathscr{G}$ on $(X_0/T)_{\mathrm{Ncrys}}$.

Suppose that we have $\alpha''_0\colon X \to Y''$ and $g'\colon Y'' \to Y'$ such that $\alpha'_0 = g' \circ \alpha''_0$. Then, by Proposition 2.6.6, we see that the isomorphism

$$(g \circ g')_{+/B}\alpha''_{0,+/B}\mathscr{G} \cong \alpha_{0,+/B}\mathscr{G}$$

coincides with

$$(g \circ g')_{+/B}\alpha''_{0,+/B}\mathscr{G} \cong g_{+/B}(g'_{+/B}\alpha''_{0,+/B}\mathscr{G}) \cong g_{+/B}\alpha'_{0,+/B}\mathscr{G} \cong \alpha_{0,+/B}\mathscr{G}.$$

In the following, we consider a general X_0 and do not assume that $\mathring{X}$ is affine. Let $\widetilde{V} \subset V$ be two open log subschemes of Y such that the underlying schemes of $U_0 := X_0 \times_Y V$ and $\widetilde{U}_0 := X_0 \times_Y \widetilde{V}$ are affine. Put $V' = V \times_Y Y'$ and $\widetilde{V}' = \widetilde{V} \times_Y Y'$. Then, since the isomorphism in Proposition 2.6.5 is compatible with the restrictions to Zariski open subschemes, we see that the following diagram is commutative.

$$\begin{array}{ccc} (g_{V',+/B}\alpha'_{U_0,+/B}\mathscr{G}_{U_0/B})|_{\widetilde{V}} & \xrightarrow[(4.4.1)]{\cong} & (\alpha_{U_0,+/B}\mathscr{G}_{U_0/B})|_{\widetilde{V}} \\ {\scriptstyle g_{\widetilde{V}',+/B}(\rho_{\widetilde{V}',V'})}\downarrow{\scriptstyle \cong} & & {\scriptstyle \rho_{\widetilde{V},V}}\downarrow{\scriptstyle \cong} \\ g_{\widetilde{V}',+/B}\alpha'_{\widetilde{U}_0,+/B}\mathscr{G}_{\widetilde{U}_0/B} & \xrightarrow[(4.4.1)]{\cong} & \alpha_{\widetilde{U}_0,+/B}\mathscr{G}_{\widetilde{U}_0/B} \end{array}$$

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Here

$$g_{V'} = g|_{V'}, \quad \alpha'_{U_0} = \alpha'_0|_{U_0}, \quad \alpha_{U_0} = \alpha_0|_{U_0}, \quad \mathscr{G}_{U_0/B} = \mathscr{G}|_{(U_0/B)_{\mathrm{Ncrys}}}$$

and similarly for $\widetilde{V}'$ and $\widetilde{U}_0$. Hence, by gluing the isomorphism (4.4.1) for α_{U_0}, α'_{U_0}, $g_{V'}$, and $\mathscr{G}_{U_0/B}$, we obtain a canonical isomorphism of $\mathcal{D}_{Y/B}$-modules:

$$g_{+/B}\alpha'_{0,+/B}\mathscr{G} \cong \alpha_{0,+/B}\mathscr{G}. \tag{4.4.3}$$

The diagram (4.4.2) is still commutative for a general X_0, since the question is Zariski local on $\mathring{Y}$. We also have the compatibility with respect to compositions of the g.

Next let us consider a crystal $\mathscr{E}$ of $\mathcal{O}_{\mathring{X}/S}$-modules locally free of finite type on $(\mathring{X}/S)_{\mathrm{Ncrys}}$. We assume that every irreducible component of X_0 is smooth and define $X_{0,\lambda}$ $(\lambda \in \Lambda)$, $X_{0,I}$ $(I \subset \Lambda)$, $\alpha_{0,I} \colon X_{0,I} \to Y$, $\mathscr{E}_{X_0/S}$, and $\mathscr{E}_{X_{0,I}/T}$ in the same way as in § 4.3. We define $\alpha'_{0,I} \colon X_{0,I} \to Y'$ in the same way as $\alpha_{0,I}$ using α'_0 instead of α_0. Then, from Proposition 3.5.3, we obtain the following proposition.

Proposition 4.4.4.

(1) *The homomorphism* $g_{+/S}\alpha'_{0,+/S}\mathscr{E}_{X_0/S} \to g_{+/S}\alpha'_{0,+/S}\mathscr{E}_{X_0/S}$ *induced by the* $\mathcal{D}_{Y'/S}$*-linear homomorphism* $\partial_t^{\log} \colon \alpha'_{0,+/S}\mathscr{E}_{X_0/S} \to \alpha'_{0,+/S}\mathscr{E}_{X_0/S}$ *coincides with the action of* $\partial_t^{\log} \in \mathcal{D}_{Y/S}$ *on* $g_{+/S}\alpha'_{0,+/S}\mathscr{E}_{X_0/S}$.

(2) *For* $n \in \mathbb{Z}$, *we have*

$$\mathcal{H}^i(g_{+/S}(\mathrm{gr}^W_n(\alpha'_{0,+/S}\mathscr{E}_{X_0/S}))) = 0$$

and

$$\mathcal{H}^i(g_{+/S}(W_n(\alpha'_{0,+/S}\mathscr{E}_{X_0/S}))) = 0 \quad \text{for } i \neq 0.$$

(3) *For* $n \in \mathbb{Z}$, *the image of* $g_{+/S}(W_n(\alpha'_{0,+/S}\mathscr{E}_{X_0/S}))$ *under*

$$g_{+/S}\alpha'_{0,+/S}\mathscr{E}_{X_0/S} \xrightarrow{\cong} \alpha_{0,+/S}\mathscr{E}_{X_0/S}$$

coincides with $W_n(\alpha_{0,+/S}\mathscr{E}_{X_0/S})$.

(4) *For a non-empty set* $I \subset \Lambda$ *of cardinality* $\leqslant d+1$, *the following diagram is commutative in the category of* $\mathcal{D}_{Y/T}$*-modules, where* r_Y *and* $r_{Y'}$ *are defined as in (2.5.1).*

$$\begin{array}{ccccc}
g_{+/T}\alpha'_{0,I,+/T}\mathscr{E}_{X_{0,I}/T} & \xrightarrow[(4.3.4)]{g_{+/T}(\kappa_I)} & g_{+/T}r_{Y'}(\mathrm{gr}^W_{|I|-1}(\alpha'_{0,+/S}\mathscr{E}_{X_0/S})) & \xrightarrow[\cong]{\text{Corollary 2.6.10}} & r_Y g_{+/S}(\mathrm{gr}^W_{|I|-1}(\alpha'_{0,+/S}\mathscr{E}_{X_0/S})) \\
\downarrow{\cong} & & & & \downarrow{\cong} \\
\alpha_{0,I,+/T}\mathscr{E}_{X_{0,I}/T} & & \xrightarrow[(4.3.4)]{\kappa_I} & & r_Y(\mathrm{gr}^W_{|I|-1}(\alpha_{0,+/S}\mathscr{E}_{X_0/S})).
\end{array}$$

4.5 Cohomology and weight spectral sequence

Let $\alpha_0 \colon X_0 \to Y$ be as in the beginning of § 4 and let $h \colon Y \to T$ be the structure morphism. Let B be S or T and let $\mathscr{G}$ be a crystal of $\mathcal{O}_{X_0/B}$-modules locally free of finite type on $(X_0/B)_{\mathrm{Ncrys}}$. Then we see that the object $h_{+/B}\alpha_{0,+/B}\mathscr{G}$ of $D^-(\mathcal{D}_{T/B}\text{-}\mathbf{Mod})$ is independent of the choice of α_0 as follows. Let $\alpha'_0 \colon X_0 \to Y'$ be another morphism satisfying the same conditions as α_0. Put $Z = Y \times_T Y'$ and let β_0 be the morphism $X_0 \to Z$ induced by α_0 and α'_0, which also satisfies the same conditions as α_0. Let h' (respectively h'') denote the structure morphism $Y' \to T$ (respectively $Z \to T$). Then, by using (4.4.3) for two projections $p_Y \colon Z \to Y$ and $p_{Y'} \colon Z \to Y'$

and Proposition 2.6.5, we obtain

$$c_{\alpha_0',\alpha_0/B}\colon h_{+/B}\alpha_{0,+/B}\mathscr{G}\cong h_{+/B}p_{Y+/B}\beta_{0,+/B}\mathscr{G}\cong h''_{+/B}\beta_{0,+/B}\mathscr{G}$$
$$\cong h'_{+/B}p_{Y'+/B}\beta_{0,+/B}\mathscr{G}\cong h'_{+/B}\alpha'_{0,+/B}\mathscr{G}.$$

Let α_0'' be a morphism $X_0\to Y''$ satisfying the same conditions as α_0. Put $\widetilde{Z}=Y\times_T Y'\times_T Y''$, let $\widetilde{\beta}_0\colon X_0\to\widetilde{Z}$ be the morphism induced by α_0, α_0', and α_0'', let $q_Y\colon\widetilde{Z}\to Y$ be the projection to Y, and define $q_{Y'}$ and $q_{Y''}$ similarly. Then, by using (4.4.3) for $\widetilde{Z}\to Y\times_T Y'$ and Proposition 2.6.6, we see that $c_{\alpha_0',\alpha_0/B}$ is also given as the composite of

$$h_{+/B}\alpha_{0,+/B}\mathscr{G}\cong h_{+/B}q_{Y+/B}\widetilde{\beta}_{0,+/B}\mathscr{G}\cong(h\circ q_Y)_{+/B}\widetilde{\beta}_{0,+/B}\mathscr{G}$$
$$=(h'\circ q_{Y'})_{+/B}\widetilde{\beta}_{0,+/B}\mathscr{G}\cong h'_{+/B}q_{Y'+/B}\widetilde{\beta}_{0,+/B}\mathscr{G}\cong h'_{+/B}\alpha'_{0,+/B}\mathscr{G}.$$

Here we also used the compatibility of (4.4.3) with compositions of the g. We have similar descriptions for $c_{\alpha_0'',\alpha_0/B}$, and $c_{\alpha_0'',\alpha_0'/B}$. Hence, we have $c_{\alpha_0'',\alpha_0/B}=c_{\alpha_0'',\alpha_0'/B}\circ c_{\alpha_0',\alpha_0/B}$. Thus, we see that $h_{+/B}\alpha_{0,+/B}\mathscr{G}$ is independent of the choice of α_0 up to canonical isomorphisms.

DEFINITION 4.5.1. Let X_0 be as in the beginning of § 4 and assume that there exists $\alpha_0\colon X_0\to Y$. (For example, such an α_0 exists when $\mathring{X}_0$ is projective.) Let B be S or T, and let f_0 denote the morphism $X_0\to T$. For a crystal $\mathscr{G}$ of $\mathcal{O}_{X_0/B}$-modules locally free of finite type on $(X_0/B)_{\mathrm{Ncrys}}$, we define the object $f_{0,+/B}\mathscr{G}$ of $D^-(\mathcal{D}_{B/S}\text{-}\mathbf{Mod})$ to be $h_{+/B}\alpha_{0,+/B}\mathscr{G}$.

LEMMA 4.5.2. *Let $\mathscr{G}$ be a crystal of $\mathcal{O}_{X_0/S}$-modules locally free of finite type on $(X_0/S)_{\mathrm{Ncrys}}$. Then we have a canonical isomorphism $r_T f_{0,+/S}\mathscr{G}\cong f_{0,+/T}\mathscr{G}'$, where r_T denotes the functor $D^-(\mathcal{D}_{T/S}\text{-}\mathbf{Mod})\to D^-(\mathcal{O}_T\text{-}\mathbf{Mod})$ and $\mathscr{G}'$ denotes the inverse image of $\mathscr{G}$ on $(X_0/T)_{\mathrm{Ncrys}}$.*

Proof. It suffices to show that the following diagram is commutative.

$$\begin{array}{ccccc} r_T h_{+/S}\alpha_{0,+/S}(\mathscr{G}) & \xrightarrow[\cong]{\text{Corollary 2.6.10}} & h_{+/T}r_Y\alpha_{0,+/S}(\mathscr{G}) & \xrightarrow[\cong]{(4.3.3)} & h_{+/T}\alpha_{0,+/T}(\mathscr{G}') \\ \cong\downarrow c_{\alpha_0',\alpha_0/S} & & & & \cong\downarrow c_{\alpha_0',\alpha_0/T} \\ r_T h'_{+/S}\alpha'_{0,+/S}(\mathscr{G}) & \xrightarrow[\cong]{\text{Corollary 2.6.10}} & h'_{+/T}r_{Y'}\alpha'_{0,+/S}(\mathscr{G}) & \xrightarrow[\cong]{(4.3.3)} & h'_{+/T}\alpha'_{0,+/T}(\mathscr{G}') \end{array}$$

This follows from Lemma 2.6.11 and the commutative diagram (4.4.2), which holds for general X_0. □

If $N=1$ (i.e. $T=T_0$), then, by Proposition 2.6.5 and Corollary 2.6.2, we see that $h_{+/B}\alpha_{0,+/B}\mathscr{G}$ is canonically isomorphic to $\mathbb{R}f_{0,*}(\mathscr{G}_{X_0}\otimes_{\mathcal{O}_{X_0}}\Omega^{\bullet}_{X_0/T_0})[d]$ as an object of $D^-(k\text{-}\mathbf{Vect})$. If $\mathring{X}_0$ is proper over S_0, then we can prove that $H^i(f_{0,+/B}\mathscr{G})$ is a finitely generated W_N-module by reducing to the case $N=1$ and using the above comparison with the de Rham cohomology. With these observations, it is natural to ask the following question.

Question 4.5.3. Let X_0, B, and f_0 be as in Definition 4.5.1. Then, for a crystal $\mathscr{G}$ of $\mathcal{O}_{X_0/B}$-modules locally free of finite type on $(X_0/B)_{\mathrm{crys}}$, is there a canonical isomorphism

$$H^i(f_{0,+/B}\mathscr{G})\cong H^{i+d}((X_0/T)_{\mathrm{crys}},\mathscr{G})$$

of W_N-modules?

When $B=S$, it is also natural to ask whether the endomorphism $\partial_t^{\log}$ on the left-hand side corresponds to the monodromy operator N on the right-hand side defined in the same way as in [HK94, 3.5].

Now let us consider a crystal $\mathscr{E}$ of $\mathcal{O}_{\mathring{X}_0/S}$-modules locally free of finite type on $(\mathring{X}_0/S)_{\mathrm{Ncrys}}$. We assume that every irreducible component of X_0 is smooth and define $X_{0,\lambda}$ $(\lambda \in \Lambda)$, $X_{0,I}$ $(I \subset \Lambda)$, $\alpha_{0,I}\colon X_{0,I} \to Y$, $\mathscr{E}_{X_0/S}$, and $\mathscr{E}_{X_{0,I}/T}$ in the same way as in § 4.3. Let $f_{0,I}$ denote the morphism $X_{0,I} \to T$. Then, from the isomorphism (4.3.5), Proposition 4.4.4, and Corollary 2.6.10, we obtain the following theorem.

Theorem 4.5.4. *Let the notation and assumptions be as above. Then there exists a canonical spectral sequence:*

$$E_1^{i,j} = \bigoplus_{\substack{b-2a=-i \\ 0\leqslant a\leqslant b\leqslant d}} \bigoplus_{\substack{I\subset\Lambda \\ |I|=b+1}} H^{i+j}(f_{0,I,+/T}\mathscr{E}_{X_{0,I}/T}) \Longrightarrow E_\infty^{i+j} = H^{i+j}(f_{0,+/S}\mathscr{E}_{X_0/S}).$$

Furthermore, there exists a morphism of spectral sequences of degree $(2, -2)$*:*

$$\begin{array}{ccc} E_1^{i,j} = \bigoplus_{\substack{b-2a=-i \\ 0\leqslant a\leqslant b\leqslant d}} \bigoplus_{\substack{I\subset\Lambda \\ |I|=b+1}} H^{i+j}(f_{0,I,+/T}\mathscr{E}_{X_{0,I}/T}) & \Longrightarrow & E_\infty^{i+j} = H^{i+j}(f_{0,+/S}\mathscr{E}_{X_0/S}) \\ \downarrow & & \downarrow \partial_t^{\log} \\ E_1^{i+2,j-2} = \bigoplus_{\substack{b-2a=-i-2 \\ 0\leqslant a\leqslant b\leqslant d}} \bigoplus_{\substack{I\subset\Lambda \\ |I|=b+1}} H^{i+j}(f_{0,I,+/T}\mathscr{E}_{X_{0,I}/T}) & \Longrightarrow & E_\infty^{i+j} = H^{i+j}(f_{0,+/S}\mathscr{E}_{X_0/S}) \end{array}$$

The homomorphism $E_1^{i,j} \to E_1^{i+2,j-2}$ of the E_1 term is defined by 0 on the component of $E_1^{i,j}$ for $a = b$ and the identity map from the component of $E_1^{i,j}$ for (a, b, I) with $a < b$ to that of $E_1^{i+2,j-2}$ for $(a + 1, b, I)$.

Proof. We will prove the existence of the morphism of spectral sequences. Let I be a subset of Λ of cardinality $\leqslant d + 1$ and put $b = |I| - 1$. By the construction of κ_I in Lemma 3.4.9 and Proposition 3.3.3(1), we see that the composite of

$$\alpha_{0,I,+/T}\mathscr{E}_{X_{0,I}/T} \xrightarrow[(4.3.4)]{\kappa_I} \mathrm{gr}_b^W(\alpha_{0,+/S}\mathscr{E}_{X_0/S}) \xrightarrow{(\partial_t^{\log})^{b+1}} \mathrm{gr}_{-b-2}^W(\alpha_{0,+/S}\mathscr{E}_{X_0/S})$$

is 0. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{\substack{b-2a=n \\ 0\leqslant a\leqslant b\leqslant d}} \bigoplus_{\substack{I\subset\Lambda \\ |I|=b+1}} \alpha_{0,I,+/T}\mathscr{E}_{X_{0,I}/T} & \xrightarrow{\cong} & \mathrm{gr}_n^W(\alpha_{0,+/S}\mathscr{E}_{X_0/S}) \\ \downarrow & & \downarrow \partial_t^{\log} \\ \bigoplus_{\substack{b-2a=n-2 \\ 0\leqslant a\leqslant b\leqslant d}} \bigoplus_{\substack{I\subset\Lambda \\ |I|=b+1}} \alpha_{0,I,+/T}\mathscr{E}_{X_{0,I}/T} & \xrightarrow{\cong} & \mathrm{gr}_{n-2}^W(\alpha_{0,+/S}\mathscr{E}_{X_0/S}) \end{array}$$

for $-d + 2 \leqslant n \leqslant d$. Here the left vertical homomorphism is defined by 0 on the component of the source for $a = b$ and the identity map from the component of the source for (a, b, I) with $a < b$ to that of the target for $(a + 1, b, I)$. On the other hand, by the same argument as in the proof of Proposition 3.5.3(1), we see that the endomorphism of $h_{+/S}\alpha_{0,+/S}\mathscr{E}_{X_0/S}$ induced by the action of $\partial_t^{\log} \in \mathcal{D}_{Y/S}$ on $\alpha_{0,+/S}\mathscr{E}_{X_0/S}$ coincides with the action of $\partial_t^{\log} \in \mathcal{D}_{T/S}$ on $h_{+/S}\alpha_{0,+/S}\mathscr{E}_{X_0/S}$. Hence, by taking $h_{+/T}$ of the above commutative diagram and using Corollary 2.6.10, we obtain the desired morphism of spectral sequences. □

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