ON NEARLY ABSOLUTELY ISOLATED
HYPERSURFACE SINGULARITIES OF DIMENSION 2

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ABSTRACT. A formula for the geometric genus of a nearly absolutely isolated hypersurface singularitiy of dimension 2 is found by using the canonical resolution. An upper bound for the fundamental cycle of such singularity is also given.

Introduction. Let $\pi: M \to V$ be a resolution of an isolated singularity of dimension 2. The number $\dim H^0(V, R^1\pi_*(\mathcal{O}_M))$ is defined to be the geometric genus of the singularity. We study the case in which V can be embeded in $\mathbb{C}^3$. Our major result is a formula (Theorem 2) for the geometric genus when the singularity is nearly absolutely isolated (see §2 for definition). Our proof is based on the canonical resolution of an m-tuple point, which is developed in the first section. The case $m = 2$ is well-known (cf. [2, pp. 47–48]). Finally we give a bound for the fundamental cycle of that kind of singularity as well as for its self-intersection number (Theorem 3).

The base field is the complex number field $\mathbb{C}$. A singular point p always means a hypersurface point of dimension 2. Sometimes we use the same notation for a line bundle and its corresponding divisor if it will not cause confusion.

1. m-tuple covering. Let $m \geqslant 2$. Let Y be a smooth surface covered by affine open sets $\{U_i\}_{i \in I}$. Let $C_0, \dots, C_{m-2}$ be effective divisors on Y locally defined in U_i by equations $c_{s,i} = 0$ ($0 \leqslant s \leqslant m - 2, i \in I$). Suppose there is a line bundle F over Y with transition function $\{f_{ij}\}$ over $\{U_i \cap U_j\}$ such that

$$(1) \qquad c_{s,i} = f_{ij}^{m-s} c_{s,j}$$

for all $0 \leqslant s \leqslant m - 2$. Let ϕ_i be the fibre coordinates over U_i. Then the equations

$$(2) \qquad \phi_i^m + c_{m-2,i}\phi_i^{m-2} + \cdots + c_{0,i} = 0$$

give rise to a surface X in F and the projection map from F to Y induces a finite morphism $f: X \to Y$ of degree m.

DEFINITION. The surface X constructed as above is called the *m-tuple cover of Y with branch locus data* $(C_0, \dots, C_{m-2})$.

Let D_i be the discriminant of the equation (2) for $i \in I$. Then $\{D_i\}_{i \in I}$ give rise to a divisor on Y, denoted by D. Obviously D is the branch locus of the map f. The map f is called *totally ramified* at a point $p \in Y$ if $f^{-1}(p)$ consists of one point.

Since the degree $(m-1)$ term in (2) has zero coefficient, the set of all totally ramified points in Y is $C_0 \cap \cdots \cap C_{m-2}$.

For any point $p \in Y$, let $\mu_p(C_s)$ be the multiplicity of C_s at p. Then a point $p \in C_0 \cap \cdots \cap C_{m-2}$ gives rise to an m-tuple point on X if and only if $\mu_p(C_s) \geq m - s$ for $0 \leq s \leq m - 2$.

Suppose $x = f^{-1}(p)$ is an isolated multiple point of X. We call x a *simple m-tuple point* if there is an open neighborhood U of p such that the support of the divisor $C_0 + C_1 + \cdots + C_{m-2}$ in U has two irreducible components which are nonsingular at the point p with normal crossing.

Suppose that X has only a finite number of m-tuple points. Let $p \in C_0 \cap \cdots \cap C_{m-2}$ be such that $f^{-1}(p)$ is an m-tuple point. Let $q_1: Y_1 \to Y$ be the blowing-up of Y at p. Let $E_1 = q_1^{-1}(p)$. Let

$$r_1 = \min\left(\left[\mu_p(C_{m-2})/2\right], \ldots, \left[\mu_p(C_0)/m\right]\right),$$

where $[a]$ is the integer part of a. We call r_1 the *multiplicity of the branch locus data* $(C_0, \ldots, C_{m-2})$ *at* p. Let the divisors $C_{1,s}$ $(0 \leq s \leq m - 2)$ be defined by

$$C_{1,s} = q_1^*(C_s) - r_1(m - s)E_1.$$

Let $F_1 = q_1^*(F) - r_1 E_1$. It is easy to check that the line bundle F_1 and the divisors $C_{1,0}, \ldots, C_{1,m-2}$ on Y_1 satisfy the condition (1). Let $f_1: X_1 \to Y_1$ be the m-tuple cover of Y_1 with branch locus data $(C_{1,0}, \ldots, C_{1,m-2})$. Obviously there is a map $\pi_1: X_1 \to X$ such that the diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\pi} & X \\
\downarrow f_1 & & \downarrow f \\
Y_1 & \xrightarrow{q_1} & Y
\end{array}
$$

commutes.

By the choice of r_1, we see that X_1 has only a finite number of m-tuple points. We can repeat the same procedure for any m-tuple point on X_1.

LEMMA. *The above procedure will terminate after a finite number of steps.*

PROOF. It is easy to see that after a finite number of steps there are at most finitely many simple m-tuple points on the top surface. Thus we may assume that $f^{-1}(p)$ is a simple m-tuple point.

By definition, the simple m-tuple point $f^{-1}(p)$ is analytically isomorphic to an m-tuple point given by

$$x^m + y^{s_{m-2}}z^{t_{m-2}}x^{m-2} + \cdots + y^{s_1}z^{t_1}x + y^{s_0}z^{t_0} = 0$$

satisfying

(A) $s_i + t_i \geq m - i$ for $0 \leq i \leq m - 2$;

(B) $\min_{0 \leq i \leq m-2}(s_i + i) < m$;

(C) $\min_{0 \leq i \leq m-2}(t_i + i) < m$.

In particular, $\mu_p(C_i) = s_i + t_i$ for $0 \leq i \leq m - 2$.

Let $S = \{0 \leqslant i \leqslant m - 2: s_i/(m - i) = \min_{0 \leqslant k \leqslant m-2}(s_k/(m - k))\}$ and

$$T = \left\{0 \leqslant i \leqslant m - 2: t_i/(m - i) = \min_{0 \leqslant k \leqslant m-2}(t_k/(m - k))\right\}.$$

Let $\alpha = \min_{i \in T}\{s_i/(m - i)\} + \min_{i \in S}\{t_i/(m - i)\}$. We call this rational number α the "grade" of the m-tuple point $f^{-1}(p)$.

The exceptional divisor E_1 of the blowing-up q_1 intersects the branch locus of f_1 at two distinct points p_1 and p_1', which are the only points on Y_1 that might give rise to m-tuple points on X_1. Since the discussions of p_1 and p_1' are the same, it suffices to consider p_1. If $f_1^{-1}(p_1)$ is not an m-tuple point then there is nothing to prove. So we assume $f_1^{-1}(p_1)$ is an m-tuple point. Then it can be represented by

$$x^m + y^{s_{m-2}}z^{s_{m-2}+t_{m-2}-2r_1}x^{m-2} + \cdots + y^{s_1}z^{s_1+t_1-(m-1)r_1}x + y^{s_0}z^{s_0+t_0-mr_1} = 0$$

where $r_1 = \min([(s_{m-2} + t_{m-2})/2],\ldots,[(s_0 + t_0)/m])$. We want to show that the "grade" α_1 of $f_1^{-1}(p_1)$ is smaller than α. Assume that $j \in T$ and $k \in S$ are such that $s_j/(m - j) = \min_{i \in T}\{s_i/(m - i)\}$ and $t_k/(m - k) = \min_{i \in S}\{t_i/(m - i)\}$. We also assume that $(s_n + t_n)/(m - n) = \min_{0 \leqslant i \leqslant m-2}\{(s_i + t_i)/(m - i)\}$. By definition,

$$\alpha_1 = \min_{i \in T_1}\left\{s_i/(m - i)\right\} + \min_{i \in S_1}\left\{(s_i + t_i - (m - i)r_1)/(m - i)\right\}$$

where $S_1 = S$ and

$$T_1 = \left\{0 \leqslant i \leqslant m - 2: (s_i + t_i - (m - i)r_1)/(m - i)\right.$$

$$= \left.\min_{0 \leqslant k \leqslant m-2}((s_k + t_k - (m - k)r_1)/(m - k))\right\}.$$

Clearly $n \in T_1$. Hence $\alpha_1 \leqslant s_n/(m - n) + (s_k + t_k - (m - k)r_1)/(m - k)$. Since $t_n/(m - n) \geqslant t_j/(m - j)$ and $(s_n + t_n)/(m - n) \leqslant (s_j + t_j)/(m - j)$ we have $s_n/(m - n) \leqslant s_j/(m - j)$. On the other hand, $s_k/(m - k) < 1$ (by property (B)) and $r_1 \geqslant 1$ imply that $(s_k + t_k - (m - k)r_1)/(m - k) < t_k/(m - k)$. Hence

$$\alpha_1 \leqslant s_j/(m - j) + (s_k + t_k - (m - k)r_1)/(m - k)$$

$$< s_j/(m - j) + t_k/(m - k) = \alpha.$$

Since all "grades" during the process of the resolution are nonnegative rational numbers with denominators less than a fixed number $m!$, the procedure must terminate after a finite number of steps. $\square$

If X has only a finite number of m-tuple points then, by the lemma, there will be a commutative diagram

$$\begin{array}{ccccccccccc}
X_n & \xrightarrow{\pi_n} & X_{n-1} & \rightarrow \cdots \rightarrow & X_2 & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & X \\
f_n \downarrow & & f_{n-1} \downarrow & & f_2 \downarrow & & f_1 \downarrow & & f \downarrow \\
Y_n & \xrightarrow{q_n} & Y_{n+1} & \rightarrow \cdots \rightarrow & Y_2 & \xrightarrow{q_2} & Y_1 & \xrightarrow{q_1} & Y
\end{array}$$

such that X_n has no m-tuple points. But X_n is not necessarily normal.

DEFINITION. The surface X_n is called the *canonical resolution of m-tuple points* in X. If X_n is normal, then the above resolution is called a *normal* canonical resolution.

Here the term "resolution" is not in the usual sense. Note that during the canonical resolution each map q_i is related to a number r_i, which is the multiplicity of the branch locus data of the m-tuple covering map f_{i-1}. The sequence $r_1, \ldots, r_n$ can be used to decide some geometric invariants of the surface X_n.

THEOREM 1. *The holomorphic Euler characteristic of the canonical resolution X_n of the m-tuple point p in X is given by*

$$(3) \quad \chi(X_n, \mathcal{O}_{X_n}) = m\chi(Y, \mathcal{O}_Y) - \sum_{i=1}^{n} \left(\frac{(m-1)m(2m-1)}{12} r_i^2 - \frac{(m-1)m}{4} r_i \right).$$

PROOF. Let $F_i^\#$ be the completion of the line bundle F_i. Let $[Y_i]$ be the zero section of $F_i^\#$. There are exact sequences

$$0 \to \mathcal{O}_{F_i^\#}(-X_i) \to \mathcal{O}_{F_i^\#} \to \mathcal{O}_{X_i} \to 0,$$
$$0 \to \mathcal{O}_{F_i^\#}(-[Y_i]) \to \mathcal{O}_{F_i^\#} \to \mathcal{O}_{Y_i} \to 0,$$
$$0 \to \mathcal{O}_{F_i^\#}(-2[Y_i]) \to \mathcal{O}_{F_i^\#}(-[Y_i]) \to \mathcal{O}_{Y_i}(-F_i) \to 0,$$
$$\vdots$$
$$0 \to \mathcal{O}_{F_i^\#}(-X_i) \to \mathcal{O}_{F_i^\#}(-(m-1)[Y_i]) \to \mathcal{O}_{Y_i}(-(m-1)F_i) \to 0$$

due to the facts $m[Y_i] \sim X_i$ in $F_i^\#$ and $Y_i|_{Y_i} \sim F_i$.

Using these exact sequences and the Riemann-Roch Theorem, we have

$$\chi(X_i, \mathcal{O}_{X_i}) = \chi(F_i^\#, \mathcal{O}_{F_i^\#}) - \chi(F_i^\#, \mathcal{O}_{F_i^\#}(-X_i))$$
$$= \chi(F_i^\#, \mathcal{O}_{F_i^\#}) - \chi(F_i^\#, \mathcal{O}_{F_i^\#}(-(m-1)[Y_i])) + \chi(Y_i, \mathcal{O}_{Y_i}(-(m-1)F_i))$$
$$= \cdots$$
$$= \chi(Y_i, \mathcal{O}_{Y_i}) + \chi(Y_i, \mathcal{O}_{Y_i}(-F_i)) + \cdots + \chi(Y_i, \mathcal{O}_{Y_i}(-(m-1)F_i))$$
$$= m\chi(Y_i, \mathcal{O}_{Y_i}) + (\tfrac{1}{2})F_i(F_i + k_i) + \cdots + (\tfrac{1}{2})(m-1)F_i((m-1)F_i + k_i)$$
$$= m\chi(Y_i, \mathcal{O}_{Y_i}) + (\tfrac{1}{2})\left(\sum_{r=1}^{m-1} r^2 \right) F_i^2 + ((m-1)m/4)F_i k_i$$
$$= m\chi(Y, \mathcal{O}_Y) + ((m-1)m(2m-1)/12)F_i^2 + ((m-1)m/4)F_i k_i$$

where k_i is the canonical divisor of Y_i. Since $F_i = q_i^*(F_{i-1}) - r_i E_i$ and $k_i = q_i^*(k_{i-1}) + E_i$, the theorem follows immediately. $\square$

2. Nearly absolutely isolated m-tuple points. Let p be an isolated m-tuple surface singularity which can be embedded in $\mathbf{C}^3$. In other words, $p \in V \subset \mathbf{C}^3$ where V is a surface. We may assume that p is the only singularity on V. Let $\pi \colon M \to V$ be a resolution of p. The set $A = \pi^{-1}(p)$ is called the *exceptional set*. The number $h(p) = \dim H^0(V, R^1\pi_*(\mathcal{O}_M))$ is called the *geometric genus* of p.

In this section we use the construction of m-tuple covering and Theorem 1 to find a formula for $h(p)$ for a certain type of p.

Denote by x, y, z the coordinates of $\mathbf{C}^3$. Assume that $p = (0, 0, 0)$. Since $V \subset \mathbf{C}^3$, V is defined by a single equation. Without loss of generality, we may assume the equation is given by

$$z^m = a_{m-2}(x, y)z^{m-2} + a_{m-3}(x, y)z^{m-3} + \cdots + a_1(x, y)z + a_0(x, y)$$

where $a_0(x, y), \ldots, a_{m-2}(x, y)$ are polynomials in x, y. For any polynomial $g(x, y)$, define the *order* of $g(x, y)$ to be the lowest degree of all its terms, denoted by $\operatorname{ord}(g)$. Since p has multiplicity m, $\operatorname{ord}(a_i) \geqslant m - i$. Let x, y, w be the homogeneous coordinates in $\mathbf{P}^2$. Let

$$N = \max_{0 \leqslant i \leqslant m-2} \left(\left[\frac{\deg(a_i) + m - i - 1}{m - i} \right] \right)$$

and let $A_i(x, y, w) = w^{N(m-i)} a_i(x/w, y/w)$ for each $0 \leqslant i \leqslant m - 2$. Each $A_i(x, y, w)$ is a homogeneous polynomial of degree $(m - i)N$. Let F be the line bundle $\mathcal{O}_{\mathbf{P}^2}(N)$. Let C_i be the zero locus of $A_i(x, y, w)$ on $\mathbf{P}^2$. Then it is easy to see that the fibre coordinates of F satisfy (1) for each affine open subset of $\mathbf{P}^2$. Let X be the m-tuple cover of $\mathbf{P}^2$ with branch locus data $(C_0, \ldots, C_{m-2})$ as defined before. Then V is merely a subscheme of X. Hence we may think of X as our original V and $p \in X$ sits over $0 \in \mathbf{P}^2$. Let $\pi: X_n \to X$ be the canonical resolution of the m-tuple point p as defined in the previous section. Suppose X_n is normal. Then there are finite number of multiple points of X_n on $\pi^{-1}(p)$ with multiplicities smaller than m. These are called *infinitely near* multiple points of p. Suppose that their canonical resolutions are normal. Then we can consider their infinitely near multiple points and repeat the same process. If all canonical resolutions involved in the above process are normal then the m-tuple point p is called *nearly absolutely isolated*. (Remember that an isolated singularity is absolutely isolated if it can be resolved by a sequence of blowing-ups.) Obviously all isolated double points are nearly absolutely isolated.

THEOREM 2. *Let p be a nearly absolutely isolated m-tuple point of dimension 2 which can be embedded in $\mathbf{C}^3$. Then the geometric genus $h(p)$ of p is given by the formula*

$$(4) \qquad h(p) = \sum_{\lambda=2}^{m} \sum_{q \in I_\lambda} \sum_{i=1}^{n_{\lambda,q}} \left(\frac{(\lambda-1)\lambda(2\lambda-1)}{12} r_{i,q}^2 - \frac{(\lambda-1)\lambda}{4} r_{i,q} \right)$$

where I_λ is the set of all infinitely near λ-tuple points of p and $r_{1,q}, \ldots, r_{n_{\lambda,q},q}$ is the sequence of multiplicities of the branch locus data of q.

PROOF. Let $\tilde{X}$ be the resolution of p in the sense that there is a morphism $\xi: \tilde{X} \to X$ such that $\xi|_{\tilde{X} - \xi^{-1}(p)}$ is an isomorphism and $\tilde{X}$ is nonsingular on $\xi^{-1}(p)$. Theorem 1 implies that $\chi(X, \mathcal{O}_X) - \chi(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is equal to the right-hand side of (4). It is well known that $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \chi(X, \mathcal{O}_X) - h(p)$. Therefore (4) is true. $\square$

Since p has multiplicity m, $\operatorname{ord}(a_i(x, y)) \geqslant m - i$. Let T_i be the tangent cone of C_i at 0 for $2 \leqslant i \leqslant m - 2$. We say that p is *generic* when 0 is an ordinary $(m - i)$-tuple point of C_i for each i and $T_i \cap T_j = 0$ for any $i \neq j$.

COROLLARY. *If p is a generic m-tuple point, then $h(p) = \binom{m}{3}$.*

PROOF. Being generic implies that p is resolved by blowing up once. Therefore in (4) there is only one term

$$\frac{(m-1)m(2m-1)}{12} - \frac{(m-1)m}{4} = \binom{m}{3}. \qquad \square$$

3. Fundamental cycles. On the exceptional set A there is a unique positive cycle Z satisfying

(i) $A_i Z \leq 0$ for any irreducible component A_i of Z;

(ii) Z is minimal with respect to the above property.

The cycle Z is called the *fundamental cycle*. For details, see [1].

Let p be a nearly absolutely isolated m-tuple point as in the last section. There is a commutative diagram

$$
\begin{array}{ccccccccccccc}
\tilde{X} & \xrightarrow{\psi} & X_n & \xrightarrow{\pi_n} & X_{n-1} & \xrightarrow{\pi_{n-1}} & \cdots & \rightarrow & X_2 & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & X \\
 & & f_n \downarrow & & f_{n-1} \downarrow & & & & \downarrow & & f_1 \downarrow & & f \downarrow \\
 & & Y_n & \xrightarrow{q_n} & Y_{n-1} & \xrightarrow{q_{n-1}} & \cdots & \rightarrow & Y_2 & \xrightarrow{q_2} & Y_1 & \xrightarrow{q_1} & Y
\end{array}
$$

where ψ is the minimal resolution of X_n. Let $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n \circ \psi$ and let $\sigma = f_1 \circ \pi_2 \circ \cdots \circ \pi_n \circ \psi$. Then $\pi \colon \tilde{X} \to X$ is a resolution of p, though not necessarily minimal.

THEOREM 3. *Let Z be the fundamental cycle of p on $\tilde{X}$. Then $Z \leq \sigma^*(E_1)$, the pull-back of the exceptional divisor $E_1 = q_1^{-1}(0)$ on Y_1. Moreover, $Z^2 \geq -m$ and the equality holds if and only if $Z = \sigma^*(E_1)$.*

PROOF. Let $\overline{Z} = \sigma^*(E_1)$. Then $\overline{Z}^2 = mE_1^2 = -m$. For any irreducible component A_i of A, we have

$$
A_i \overline{Z} = (\sigma_* A_i) E_1 = \begin{cases} 0, & \text{if } \sigma(A_i) \text{ is a point;} \\ -1, & \text{if } \sigma(A_i) = E_1. \end{cases}
$$

Hence $\overline{Z}$ satisfies the first condition of the definition of fundamental cycle. Therefore $Z \leq \overline{Z}$.

Write $\overline{Z} = Z + Z'$. Then $Z^2 + 2ZZ' + Z'^2 = -m$. By the definition of Z, $ZZ' \leq 0$. Since the intersection matrix of A is negative definite, $Z'^2 \leq 0$ and $Z'^2 = 0$ if and only if $Z' = 0$. This finishes the proof. $\square$

COROLLARY. *If p is generic, then $Z = \sigma^*(E_1)$ and $Z^2 = -m$.*

PROOF. In this case $\tilde{X} = X_1$, and $\sigma^*(E_1)$ has no multiple components. Hence $Z = \sigma^*(E_1)$. $\square$

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