CERN - DATA HANDLING DIVISION
CERN/DD/CO/69/5
K.S. Kölbig
J.A. Mignaco
E. Remiddi
July 1969

ON NIELSEN'S GENERALIZED POLYLOGARITHMS AND THEIR NUMERICAL CALCULATION

ON NIELSEN'S GENERALIZED POLYLOGARITHMS AND THEIR NUMERICAL CALCULATION

K.S. Kölbig
CERN, Geneva

J.A. Mignaco*)
University of Geneva

E. Remiddi
CERN, Geneva

ABSTRACT

The generalized polylogarithms of Nielsen are studied, in particular their functional relations. New integral expressions are obtained, and explicit relations for function values of particular arguments are given. An Algol procedure for calculating 10 functions of lowest order is presented. The numerical values of the Chebyshev coefficients used in this procedure are tabulated. A table of the real zeros of these functions is also given.

*) Fellow of Swiss National Research Fund.

1. INTRODUCTION

At the beginning of this century, Nielsen [1] published a monograph "Der Eulersche Dilogarithmus und seine Verallgemeinerungen", in which he discussed a family of functions defined by

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \frac{\log^{n-1} t \log^p(1-xt)}{t}\, dt \qquad (1.1)$$

for positive integers n and p and complex x. The log (1 - xt) is to be understood on its principal sheet. Consequently, we cut the x-plane from +1 to +∞. This formula embodies many particular cases treated separately by other authors, e.g., Euler's dilogarithm, Spence functions, Legendre's or Kummer's trilogarithm, polylogarithm, etc. For all these functions, however, the value $p = 1$ is characteristic, so that for $p > 1$ Eq. (1.1) defines functions not treated before the publication of Nielsen's monograph. We therefore call these functions $S_{n,p}(x)$ Nielsen's generalized polylogarithms.

Although Nielsen provided an extensive theory of the $S_{n,p}(x)$, his article (which unfortunately suffers from quite a number of misprints) does not seem to be sufficiently well-known to people working in areas where these functions appear. No reference could be traced in any of the relevant handbooks or integral tables.

Two of the authors (J.A.M. and E.R.) are engaged in calculations of higher order radiative corrections in quantum electrodynamics, where knowledge of the functions $S_{n,p}(x)$ seems to play an essential role [2]. In the past, special cases of $S_{n,p}(x)$ were discussed and some of their typical properties (functional relations) rediscovered by people working in the field mentioned [3],[4],[5]. Our interest in Nielsen's monograph was initiated by a reference given in the book of Lewin [6] who treats the special case $p = 1$ in detail.

The aim of the present article is twofold: to provide a useful tool for people interested in problems where $S_{n,p}(x)$ may appear, and to give an accurate method for its numerical evaluation, at least for some small values of n and p.

Nielsen, referring to the dilogarithm, wrote in his paper: "Man darf also mit Recht sagen, dass ein trübes Schicksal über dem Dilogarithmus und den ihn behandelnden Arbeiten geschwebt hat, so dass ein nicht uninteressanter Abschnitt der elementaren Integralrechnung beinahe ganz in Vergessenheit gesunken ist". It seems that, with these words, he unfortunately predicted the destiny of his own paper. It is therefore the hope of the authors that, by their small contribution to the problem, attention will again be drawn to Nielsen's important work on a useful family of functions.

2.1 Definitions and notation

The basic definition is given in Eq. (1.1). For $p = 1$, $n \geq 2$, one also writes

$$S_n(x) \equiv S_{n-1,1}(x) \quad . \tag{2.1}$$

These functions $S_n(x)$ are called polylogarithms. In particular, for $n = 2$ we have Euler's dilogarithm

$$S_2(x) = -\int_0^1 \frac{\log(1 - xt)}{t}\, dt = -\int_0^x \frac{\log(1 - t)}{t}\, dt \ , \tag{2.2}$$

for $n = 3$ the trilogarithm, etc. It is, then, appropriate to introduce the term "generalized polylogarithm" for the case $p > 1$.

The polylogarithms are closely related to the Spence functions [7]

$$L_n(x) = \int_1^x \frac{L_{n-1}(t)}{t - 1}\, dt \ ; \qquad L_o(t) = \frac{t - 1}{t} \ . \tag{2.3}$$

In fact, one has

$$S_n(x) = -L_n(1 - x) \ .$$

The terms used are not consistent throughout the literature. Some times the Spence function for $n = 2$ as defined in Eq. (2.3) is called the dilogarithm [8] and vice-versa [9]. The notation for the functions also varies, e.g. Lewin [6] and other authors [10],[11] use

$$Li_n(x) \equiv S_n(x) \ .$$

Integrating Eq. (1.1) for $p = 1$ by parts and using appropriate substitutions, one obtains

$$S_n(x) = \frac{x}{(n - 1)!} \int_0^\infty \frac{t^{n-1}}{e^t - x}\, dt = \frac{x}{(n - 1)!} \int_1^\infty \frac{\log^{n-1} t}{t(t - x)}\, dt \ . \tag{2.4}$$

For $n = 1$ it is easy to see from Eq. (1.1) that

$$S_{1,p}(x) = \frac{1}{p!} \int_0^{-\log(1-x)} \frac{t^p}{e^t - 1}\, dt = \frac{1}{p!}\, D_p(-\log(1-x)) \qquad (-\infty < x \leq 1) \,. \tag{2.5}$$

where

$$D_p(x) = \int_0^x \frac{t^p}{e^t - 1}\, dt \tag{2.6}$$

is known as the Debye function [8].

2.2 Some general properties

Again from the definition (1.1) of $S_{n,p}(x)$ we obtain by differentation and partial integration

$$\frac{d}{dx} S_{n,p}(\alpha x) = \frac{S_{n-1,p}(\alpha x)}{x} \qquad (n \geq 2) \,. \tag{2.7}$$

Hence

$$S_{n,p}(\alpha x) = \int_0^x \frac{S_{n-1,p}(\alpha t)}{t}\, dt \,. \tag{2.8}$$

With the usual definition of the logarithm on the branch cut $-\infty < x \leq 0$, namely

$$\log(x \pm i\epsilon) = \log|x| \pm i\pi\theta(-x) \,, \tag{2.9}$$

where $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ if $x < 0$, we find that $S_{n,p}(x)$ is real for $x \leq 1$. Across the cut from +1 to $+\infty$ the real part of $S_{n,p}(x)$ is continuous, whereas the imaginary part changes sign.

For our purposes it is sufficient to consider x as real, if not otherwise stated, although many of the formulas given remain valid for complex x, in particular the important relations (3.5) and (3.11).

The imaginary part of the boundary value of $S_{n,p}(x)$ at the cut is studied in detail in Section 4.

The values of $S_{n,p}(x)$ for the special arguments $x = 1$, $x = -1$, and $x = \frac{1}{2}$ are of particular interest in the theory of the generalized polylogarithms. Following Nielsen, we set

$$\begin{aligned} s_{n,p} &= S_{n,p}(1) \\ \sigma_{n,p} &= (-1)^p\, S_{n,p}(-1) \\ a_{n,p} &= S_{n,p}(\tfrac{1}{2}) \,. \end{aligned} \tag{2.10}$$

Relations between these values are studied in Section 3.4. As in Eq. (2.1) we write

$$s_n \equiv s_{n-1,1}\,, \qquad \sigma_n \equiv \sigma_{n-1,1}\,, \qquad a_n \equiv a_{n-1,1}\,.$$

We note already that for all n and p

$$s_{n,p} = s_{p,n}\,. \tag{2.11}$$

This can be deduced from Eq. (1.1) by the substitution $t' = 1 - t$ and integration by parts. No similar relation exists for $\sigma_{n,p}$ and $a_{n,p}$.

2.3 Relation to the hypergeometric function. Power series expansions.

The functions $S_{n,p}(x)$ are related to derivatives of the hypergeometric function with special arguments. From the integral representation

$${}_2F_1(\alpha,\beta;\,\gamma;\,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt \tag{2.12}$$

we obtain by comparison with Eq. (1.1)

$$S_{n,p}(x) = \frac{(-1)^{n-1}}{(n-1)!\,p!}\,\frac{\partial^{n+p-1}}{\partial\beta^{n-1}\partial\alpha^{p}}\left[\beta^{-1}\,{}_2F_1(\alpha,\beta;\,\beta+1;\,x)\right]_{\alpha=\beta=0} \tag{2.13}$$

Recalling the power series expansion

$${}_2F_1(\alpha,\beta;\,\gamma;\,z) = 1 + \frac{\alpha\beta}{\gamma}\,\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\,\frac{z^2}{2!} + \dots \tag{2.14}$$

we have

$$S_{n,p}(x) = \frac{(-1)^{n-1}}{(n-1)!\,p!}\,\frac{\partial^{n+p-1}}{\partial\beta^{n-1}\partial\alpha^{p}} \sum_{s=0}^{\infty} \frac{(\alpha)_s}{\beta+s}\,\frac{x^s}{s!}\,\Bigg|_{\alpha=\beta=0} \tag{2.15}$$

where

$$(\alpha)_s = \Gamma(\alpha+s)/\Gamma(\alpha) = \alpha(\alpha+1)\,\dots\,(\alpha+s-1)$$

$$(\alpha)_0 = 1$$

is the Pochhammer symbol. By differentiating with respect to β it follows that

$$S_{n,p}(x) = \frac{1}{p!}\frac{\partial^p}{\partial\alpha^p}\sum_{s=0}^{\infty}\frac{(\alpha)_s}{s^n}\frac{x^s}{s!}\Bigg|_{\alpha=0} . \tag{2.16}$$

Writing

$$(\alpha)_s = \sum_{\sigma=1}^{s}(-1)^{\sigma+s}\, S_s^{(\sigma)}\alpha^\sigma \tag{2.17}$$

where the $S_k^{(j)}$ are the Stirling numbers of the first kind, defined by [8] *)

$$\log^s(1+x) = s!\sum_{\sigma=s}^{\infty} S_\sigma^{(s)}\frac{x^\sigma}{\sigma!} \qquad (|x| < 1) \tag{2.18}$$

we see that

$$\frac{\partial^p}{\partial\alpha^p}(\alpha)_s\Bigg|_{\alpha=0} = \begin{cases} 0 & \text{for } s < p \\ (-1)^{p+s}\, p!\, S_s^{(p)} & \text{for } s \geq p . \end{cases} \tag{2.19}$$

Introducing this result into Eq. (2.16), we obtain

$$S_{n,p}(x) = \sum_{s=0}^{\infty}\frac{(-1)^s\, S_{p+s}^{(p)}}{(p+s)!(p+s)^n}\, x^{p+s} \tag{2.20}$$

which is the power series expansion of the generalized polylogarithm. Using the asymptotic relation [8]

$$\left|S_{p+s}^{(p)}\right| \sim \frac{(p+s-1)!(\gamma + \log(p+s))^{p-1}}{(p-1)!} \qquad (s\to\infty) \tag{2.21}$$

one can show that this expansion is valid for $|x| \leq 1$. Of course, Eq. (2.20) can also be obtained directly from the definition (1.1) with the help of Eq. (2.18).

---

*) These numbers are connected with the generalized Bernoulli numbers, see [7]. For recurrence relations of $S_k^{(j)}$ see Eq. (6.6) below.

Nielsen writes

$$S_{n,p}(x) = \sum_{s=0}^{\infty} \frac{\omega_{p,p+s}}{(p+s)^{n+1}} x^{p+s} \tag{2.22}$$

so that in his notation

$$\omega_{p,p+s} = \frac{(-1)^s S_{p+s}^{(p)}}{(p+s-1)!} . \tag{2.23}$$

We note as special cases of Eq. (2.20) for the dilogarithm

$$S_{1,1}(x) \equiv S_2(x) = \sum_{s=1}^{\infty} \frac{x^s}{s^2} = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots, \tag{2.24}$$

and in general, for the polylogarithm*),

$$S_n(x) = \sum_{s=1}^{\infty} \frac{x^s}{s^n} = \frac{x}{1^n} + \frac{x^2}{2^n} + \frac{x^3}{3^n} + \ldots . \tag{2.25}$$

For p > 1, the series become more complicated, e.g. one can show that

$$S_{n-2,2}(x) = \sum_{s=2}^{\infty} \left(\sum_{r=1}^{s-1} \frac{1}{r} \right) \frac{x^s}{s^{n-1}} = \frac{x^2}{2^{n-1}} + \left(1 + \frac{1}{2}\right) \frac{x^3}{3^{n-1}} + \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{x^4}{4^{n-1}} + \ldots, \tag{2.26}$$

and

$$S_{n-3,3}(x) = \sum_{s=3}^{\infty} \left(\sum_{r=1}^{s-2} \frac{1}{r} \left(\sum_{q=r+1}^{s-1} \frac{1}{q} \right) \right) \frac{x^s}{s^{n-2}} = \frac{1}{2} \frac{x^3}{3^{n-2}} + \left[\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{2} \times \frac{1}{3} \right] \frac{x^4}{4^{n-2}} +$$

$$+ \left[\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right) + \frac{1}{3} \times \frac{1}{4} \right] \frac{x^5}{5^{n-2}} + \ldots . \tag{2.27}$$

---

*) This series is also a special case of Lerch's transcendent $\Phi(z,\zeta,\alpha) = \Sigma_{s=0}^{\infty} z^s/(\alpha+s)^\zeta$. In fact, one has $S_n(x) = x\Phi(x,n,1) \equiv F(x,n)$, sometimes called Jonquière's function [12],[13]. Mitchell [14] writes $S_n(x) = \zeta(1,n|x)$.

For $x \to 0$, we see from Eq. (2.20) that

$$S_{n,p}(x) = \frac{1}{p!p^n} x^p + 0(x^{p+1}) \tag{2.28}$$

3. TRANSFORMATIONS OF THE ARGUMENT OF $S_{n,p}(x)$

Following Nielsen, it is convenient to introduce the auxiliary functions

$$L_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^x \frac{\log^{n-1} t \log^p(1-t)}{t} dt \tag{3.1a}$$

$$M_{n,p}(x) = \frac{(-1)^{n-1}}{(n-1)!p!} \int_0^x \frac{\log^{n-1} t \log^p(1+t)}{t} dt \,. \tag{3.1b}$$

Their connection with $S_{n,p}(x)$ is easily obtained by performing the substitution $t' = tx$. One finds that

$$L_{n,p}(x) = \sum_{r=0}^{n-1} \frac{(-1)^r \log^r x}{r!} S_{n-r,p}(x) \tag{3.2a}$$

$$M_{n,p}(x) = (-1)^p \sum_{r=0}^{n-1} \frac{(-1)^r \log^r x}{r!} S_{n-r,p}(-x) \,. \tag{3.2b}$$

The corresponding inverse relations are

$$S_{n,p}(x) = \sum_{s=0}^{n-1} \frac{\log^s x}{s!} L_{n-s,p}(x) \tag{3.3a}$$

$$S_{n,p}(-x) = (-1)^p \sum_{s=0}^{n-1} \frac{\log^s x}{s!} M_{n-s,p}(x) \,. \tag{3.3b}$$

As is clear from their definitions, the functions $L_{n,p}(x)$ and $M_{n,p}(x)$ have a more complicated analytic structure than $S_{n,p}(x)$. They are, however, convenient for the study of some transformations of the argument of $S_{n,p}(x)$.

3.1 The reflection $x \to 1 - x$

By performing the substitution $t' = 1 - t$ in Eq. (3.1a) and then integrating by parts, we have*)

$$L_{n,p}(1 - x) + L_{p,n}(x) = s_{n,p} - \frac{(-1)^{n+p}}{n!p!} \log^p x \log^n(1 - x) . \qquad (3.4)$$

Expressing $L_{n,p}(x)$ in terms of $S_{n,p}(x)$ by means of Eq. (3.2a) we obtain with the help of

$$\sum_{s=0}^{n-1} (-1)^s \binom{n}{s} = (-1)^{n-1}$$

and

$$L_{n,p}(1) = S_{n,p}(1) = s_{n,p}$$

the relation

$$S_{n,p}(x) = \sum_{s=0}^{n-1} \frac{\log^s x}{s!} \left\{ s_{n-s,p} - \sum_{r=0}^{p-1} \frac{(-1)^r \log^r(1 - x)}{r!} S_{p-r,n-s}(1 - x) \right\}$$
$$+ \frac{(-1)^p}{n!p!} \log^n x \log^p (1 - x) . \qquad (3.5)$$

For $n = p = 1$, Eq. (3.5) reduces to the well-known formula for the dilogarithm

$$S_2(x) + S_2(1 - x) = s_2 - \log x \log (1 - x) . \qquad (3.6)$$

It follows from Eq. (3.5) that a similar relation, which would combine polylogarithms of arguments x or $1 - x$ with elementary functions only, does not exist for $S_n(x)$ if $n \geq 3$. Thus, for the trilogarithm one has

$$S_3(x) = s_3 - S_{1,2}(1 - x) + \log x \{s_2 - S_2(1 - x)\} - \frac{1}{2} \log^2 x \log (1 - x) . \qquad (3.7)$$

---

*) The corresponding formulae of Nielsen [1] § 12(8),(9) contain a sign error.

This relation contains a function $S_{1,2}(x)$ which is not a polylogarithm. However, if one considers the set of generalized polylogarithms $S_{n,p}(x)$, no functions other than these and elementary functions are required for arbitrary n and p in Eq. (3.5).

3.2 The inversion $x \to 1/x$

In this case, Nielsen proceeds as follows. By substituting $t' = 1/t$ in the integral of Eq. (3.1b) and using

$$M_{n,p}(1) = (-1)^p S_{n,p}(-1) = \sigma_{n,p} \tag{3.8}$$

he finds after some manipulation

$$M_{n,p}(x) = (-1)^n \left\{ \sum_{s=0}^{p-1} \binom{n+s-1}{s} M_{n+s,p-s}\left(\frac{1}{x}\right) - \frac{\log^{n+p} x}{(n-1)!p!(n+p)} \right\} + C_{n,p} \tag{3.9}$$

where

$$C_{n,p} = (1 - (-1)^n)\, \sigma_{n,p} - (-1)^n \sum_{r=1}^{p-1} \binom{n+r-1}{r} \sigma_{n+r,p-r} \,. \tag{3.10}$$

Applying Eq. (3.3b) and using the identity

$$\sum_{s=0}^{n-1} \frac{a^s}{s!} \binom{n+r-s-1}{r} y_{n+r-s} = \sum_{m=0}^{r} \frac{(-1)^m a^m}{m!} \binom{n+r-m-1}{r-m} x_{n+r-m}$$

where the quantities x_i and y_j are related by

$$x_n = \sum_{p=0}^{n-1} \frac{a^p}{p!} y_{n-p} \,, \qquad y_n = \sum_{p=0}^{n-1} \frac{(-1)^p a^p}{p!} x_{n-p} \,,$$

he obtains (after making the substitution $x \to -x$) the relation

$$S_{n,p}(x) = (-1)^n \sum_{s=0}^{p-1} (-1)^s \sum_{r=0}^{s} \frac{\log^r(-x)}{r!} \binom{n+s-r-1}{s-r} S_{n+s-r,p-s}\left(\frac{1}{x}\right) + (-1)^p \left\{ \sum_{r=0}^{n-1} \frac{\log^r(-x)}{r!} C_{n-r,p} + \frac{\log^{n+p}(-x)}{(n+p)!} \right\}. \tag{3.11}$$

For the dilogarithm, this formula becomes

$$S_2(x) = -S_2\left(\frac{1}{x}\right) - s_2 - \frac{1}{2}\log^2(-x) , \qquad (3.12)$$

and for the polylogarithm

$$S_n(x) = (-1)^{n-1} S_n\left(\frac{1}{x}\right) - \sum_{r=0}^{n-2} \frac{\log^r(-x)}{r!}\left(1 + (-1)^{n-r}\right)\sigma_{n-r} - \frac{\log^n(-x)}{n!} \qquad (3.13)$$

This relation was already known to Jonquière [15],[16]. Following [12],[13], this formula can also be written as

$$S_n(x) + (-1)^n S_n\left(\frac{1}{x}\right) = -\frac{(2\pi i)^n}{n!} B_n\left(\frac{\log x}{2\pi i}\right) \qquad (3.14)$$

where $B_n(x)$ is the Bernoulli polynomial of order n.

Contrary to the case of the reflection $x \to 1 - x$, there is here a relation (3.13) which, for any n, combines polylogarithms of arguments x and $1/x$ with elementary functions only. Moreover, only polylogarithms with the same n appear in this relation. For $p \geq 2$, several generalized polylogarithms $S_{n',p'}(1/x)$ are needed in Eq. (3.11), but only elementary functions are used in addition to these.

We insist on the fact that the generalized polylogarithms $S_{n,p}(x)$ are a closed set, in the sense described above, under the transformations we have considered. $S_{n,p}(1 - x)$ and $S_{n,p}(1/x)$ can be expressed as linear combinations of $S_{n,p}(x)$ with coefficients constructed from rational numbers and the functions $\log(\pm x)$ and $\log(1 - x)$. In addition, there appear terms composed of $s_{n,p}$, $\sigma_{n,p}$, $\log(\pm x)$, $\log(1 - x)$. The Eqs. (3.5), (3.11) may be said to be homogeneous of degree $n + p$ in the sense that if $n + p$ is the degree of $S_{n,p}(x)$, $s_{n,p}$, $\sigma_{n,p}$, and if $\log^q u$ is of degree q, then each term in both equations is of degree $n + p$.

3.3 Other transformations

If we consider the reflection $P_1(x) = 1 - x$ and the inversion $P_2(x) = 1/x$ as belonging to the group of the bilinear transformations

$$P(x) \equiv \frac{\alpha x + \beta}{\gamma x + \delta}$$

with real coefficients and $\alpha\delta - \beta\gamma \neq 0$, we see that they generate a subgroup

$$
\begin{aligned}
P_0(x) &= P_1P_1(x) = P_2P_2(x) = x \\
P_1(x) &= 1 - x \\
P_2(x) &= \frac{1}{x} \\
P_3(x) &= P_2P_1(x) = \frac{1}{1-x} \\
P_4(x) &= P_1P_2(x) = \frac{x-1}{x} \\
P_5(x) &= P_2P_1P_2(x) = \frac{x}{x-1} \quad .
\end{aligned}
\tag{3.15}
$$

By repeated use of Eq. (3.5) and (3.11) it is therefore easy to see that

$$
S_{n,p}\left(\frac{1}{1-x}\right), \qquad S_{n,p}\left(\frac{x-1}{x}\right), \qquad S_{n,p}\left(\frac{x}{x-1}\right)
$$

can be expressed as homogeneous combinations (in the above sense) of $S_{n,p}(x)$ with the above-mentioned components and with additional terms involving log $(x - 1)$.

For reasons of simplicity we give explicitly only the known formulae for the dilogarithm:

$$
\begin{aligned}
S_2(1-x) &= s_2 - S_2(x) - \log x \log(1-x) \\
S_2\left(\frac{1}{x}\right) &= -S_2(x) - s_2 - \frac{1}{2}\log^2(-x) \\
S_2\left(\frac{1}{1-x}\right) &= S_2(x) - 2\,s_2 + \log x \log(1-x) - \frac{1}{2}\log^2(x-1) \\
S_2\left(\frac{x-1}{x}\right) &= S_2(x) + 2\,s_2 + \frac{1}{2}\log^2(-x) + \log x \log(1-x) - \log^2 x \\
S_2\left(\frac{x}{x-1}\right) &= -S_2(x) - \frac{1}{2}\log^2(1-x)
\end{aligned}
\tag{3.16}
$$

where we have used Eq. (2.9) and Eq. (3.23) below.

From Eq. (3.15), one obtains as solution of the equations

$$
P_j(x) = x \qquad (j = 1 , \ldots, 5)
$$

the six values

$$
x = +1 , \qquad -1 , \qquad \frac{1}{2} , \qquad 2 , \qquad \frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i \ .
$$

These numbers define the fixed points of the subgroup.

An interesting functional relation for $S_n(x)$ has been found by Rühl [17]. Using Eq. (2.4) one can prove by a partial fraction expansion that

$$S_n(x^m) = m^{n-1} \sum_{k=1}^{m} S_n(e_m^k x) \tag{3.17}$$

where $e_m^k = e^{2\pi ik/m}$ are the m^{th} roots of unity. In particular one has

$$S_n(x^2) = 2^{n-1} \left[S_n(x) + S_n(-x) \right] . \tag{3.18}$$

We add here that other higher order functional relations, which combine polylogarithms of more complicated arguments, have been discussed recently by Maier and Zahn [11] for the trilogarithm $S_3(x)$ and by Wechsung [18] for the pentalogarithm $S_5(x)$.

3.4 Relations between numerical values for special arguments

In this section we investigate the relations between the constants $s_{n,p}$, $\sigma_{n,p}$ and $a_{n,p}$. These numbers play a special role in the theory of the generalized polylogarithms, since they are values of $S_{n,p}(x)$ for arguments which are invariant under the bilinear transformations discussed in the previous section. In particular, explicit relations between these constants will be given for $n + p = 2, \ldots, 5$.

The s_n can be expressed in terms of the Riemann zeta function with integer arguments

$$s_n = \zeta(n) . \tag{3.19}$$

As is well-known from the theory of this function, we have for even $n = 2m$

$$s_{2m} = \zeta(2m) = \frac{2^{2m-1}\pi^{2m}|B_{2m}|}{(2m)!} \tag{3.20}$$

where the B_{2m} are the Bernoulli numbers. In particular,

$$s_2 = \frac{\pi^2}{6}$$

$$s_4 = \frac{2}{5} s_2^2 = \frac{\pi^4}{90} .$$

No similar relations are known for odd values $n = 2m + 1$.

We have noted already that for all n and p

$$s_{n,p} = s_{p,n} \, .$$

Furthermore, Nielsen proved that for all n and p, $s_{n,p}$ can be expressed as a polynomial in terms of $s_q \equiv s_{q-1,1}$ $(2 \le q \le n + p)$, with rational coefficients. This polynomial is homogeneous of degree n + p, if we say as in Section 3.2 that $s_{n,p}$ is of degree n + p. A closed formula for this polynomial has been found by Rühl [17], namely

$$\begin{aligned} s_{n,p} &= \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \frac{\log^{n-1} t \, \log^p(1-t)}{t} \, dt \\ &= \sum_{k=1}^{[(n+p)/2]} \frac{(-1)^{k+1}}{k!} \sum_{\substack{m_i \ge 2 \\ \Sigma_{i=1}^k m_i = n+p}} \frac{H_p(m_1, m_2, \ldots, m_k)}{m_1 m_2 \ldots m_k} \, s_{m_1} s_{m_2} \ldots s_{m_k} \end{aligned} \tag{3.21}$$

where

$$H_p(m_1, m_2, \ldots, m_k) = \sum_{\substack{\le m_i - 1 \\ p_i \ge 1 \\ \Sigma_{i=1}^k p_i = p}} \binom{m_1}{p_1} \binom{m_2}{p_2} \ldots \binom{m_k}{p_k} \tag{3.22}$$

The proof of this relation is given in Appendix 1. In particular, one finds that

$$s_{2,2} = \frac{3}{2} s_4 - \frac{1}{2} s_2^2 = \frac{1}{10} s_2^2 = \frac{1}{4} s_4$$

$$s_{2,3} = 2\, s_5 - s_2 s_3$$

$$s_{3,3} = \frac{10}{3} s_6 - \frac{3}{2} s_2 s_4 - s_3^2 + \frac{1}{6} s_2^3 \quad ^{*)}$$

$$s_{4,2} = \frac{5}{2} s_6 - s_2 s_4 - \frac{1}{2} s_3^2$$

$$s_{4,3} = 5\, s_7 - 2\, s_2 s_5 - \frac{5}{2} s_3 s_4 + \frac{1}{2} s_2^2 s_3 \quad ^{*)}$$

$$s_{4,4} = \frac{35}{4} s_8 - \frac{10}{3} s_2 s_6 - 4\, s_3 s_5 - \frac{17}{8} s_4^2 + \frac{3}{4} s_2^2 s_4 + s_2 s_3^2 - \frac{1}{24} s_2^4 \, .$$

Another general result is known for σ_n, namely

$$\sigma_n = (1 - 2^{1-n}) \, s_n \, , \tag{3.23}$$

---

*) Note that [1] § 18(19) does not give the last term correctly.

which follows from Eq. (3.18). No similar relations seem to exist for the $\sigma_{n,p}$ in the case $p \geq 2$ or for the $a_{n,p}$.

By exploiting the information contained in Eq. (3.5), (3.11) for the special arguments -1, 0, ½, 1, 2, the following results are found:

n + p = 2

$$s_2 = -\int_0^1 t^{-1} \log(1-t)\, dt = \frac{\pi^2}{6}$$

$$\sigma_2 = \int_0^1 t^{-1} \log(1+t)\, dt = \frac{1}{2} s_2$$

$$a_2 = -\int_0^1 t^{-1} \log\left(1 - \frac{1}{2} t\right) dt = \frac{1}{2} s_2 - \frac{1}{2} \log^2 2$$

n + p = 3

$$\sigma_{1,2} = \frac{1}{2} \int_0^1 t^{-1} \log^2 (1+t)\, dt = \frac{1}{8} s_3 \tag{3.24}$$

$$\sigma_3 = -\int_0^1 t^{-1} \log t \log(1+t)\, dt = \frac{3}{4} s_3$$

$$a_{1,2} = \frac{1}{2} \int_0^1 t^{-1} \log^2 \left(1 - \frac{1}{2} t\right) dt = \frac{1}{8} s_3 - \frac{1}{6} \log^3 2$$

$$a_3 = \int_0^1 t^{-1} \log t \log\left(1 - \frac{1}{2} t\right) dt = \frac{7}{8} s_3 - \frac{1}{2} s_2 \log 2 + \frac{1}{6} \log^3 2 \quad ^{*)}$$

n + p = 4

$$s_4 = -\frac{1}{2} \int_0^1 t^{-1} \log^2 t \log(1-t)\, dt = \frac{2}{5} s_2^2 \quad ^{**)}$$

$$s_{2,2} = -\frac{1}{2} \int_0^1 t^{-1} \log t \log^2 (1-t)\, dt = \frac{1}{10} s_2^2$$

*) Note the misprint in [1] § 20(3).

**) Found by other methods, given here for completeness.

$$\sigma_{1,3} = \frac{1}{6}\int_0^1 t^{-1}\log^3(1+t)\,dt$$

$$= \frac{2}{5}s_2^2 - a_4 - \frac{7}{8}s_3\log 2 + \frac{1}{4}s_2\log^2 2 - \frac{1}{24}\log^4 2$$

$$\sigma_{2,2} = -\frac{1}{2}\int_0^1 t^{-1}\log t\log^2(1+t)\,dt$$

$$= -\frac{3}{4}s_2^2 + 2\,a_4 + \frac{7}{4}s_3\log 2 - \frac{1}{2}s_2\log^2 2 + \frac{1}{12}\log^4 2$$

$$\sigma_4 = \frac{1}{2}\int_0^1 t^{-1}\log^2 t\log(1+t)\,dt = \frac{7}{20}s_2^2$$

$$a_{1,3} = -\frac{1}{6}\int_0^1 t^{-1}\log^3\left(1-\frac{1}{2}t\right)dt$$

$$= \frac{2}{5}s_2^2 - a_4 - \frac{7}{8}s_3\log 2 + \frac{1}{4}s_2\log^2 2 - \frac{1}{12}\log^4 2 \qquad (3{,}24)$$

$$a_{2,2} = -\frac{1}{2}\int_0^1 t^{-1}\log t\log^2\left(1-\frac{1}{2}t\right)dt$$

$$= \frac{1}{20}s_2^2 - \frac{1}{8}s_3\log 2 + \frac{1}{24}\log^4 2$$

<u>$n + p = 5$</u>

$$s_{2,3} = \frac{1}{6}\int_0^1 t^{-1}\log t\log^3(1-t)\,dt = 2\,s_5 - s_2 s_3 \;^{*)}$$

$$\sigma_{1,4} = \frac{1}{24}\int_0^1 t^{-1}\log^4(1+t)\,dt$$

$$= s_5 - a_5 - a_4\log 2 - \frac{7}{16}s_3\log^2 2 + \frac{1}{6}s_2\log^3 2 - \frac{1}{30}\log^5 2$$

---

*) Found by other methods, given here for completeness.

$$\sigma_{2,3} = -\frac{1}{6}\int_0^1 t^{-1}\log t \log^3(1+t)\,dt$$

$$= -\frac{33}{32}s_5 - \frac{1}{2}s_2s_3 + 2a_5 + 2a_4\log 2 + \frac{7}{8}s_3\log^2 2$$

$$-\frac{1}{3}s_2\log^3 2 + \frac{1}{15}\log^5 2$$

$$\sigma_{3,2} = \frac{1}{4}\int_0^1 t^{-1}\log^2 t\log^2(1+t)\,dt = -\frac{29}{32}s_5 + \frac{1}{2}s_2s_3$$

$$\sigma_5 = -\frac{1}{6}\int_0^1 t^{-1}\log^3 t\log(1+t)\,dt = \frac{15}{16}s_5$$

$$a_{1,4} = \frac{1}{24}\int_0^1 t^{-1}\log^4\left(1-\frac{1}{2}t\right)dt$$

$$= s_5 - a_5 - a_4\log 2 - \frac{7}{16}s_3\log^2 2 + \frac{1}{6}s_2\log^3 2 - \frac{1}{24}\log^5 2 \qquad (3.24)$$

$$a_{2,3} = \frac{1}{6}\int_0^1 t^{-1}\log t\log^3\left(1-\frac{1}{2}t\right)dt$$

$$= \frac{63}{32}s_5 - \frac{1}{2}s_2s_3 - a_5 + \frac{7}{16}s_3\log^2 2 - \frac{2}{5}s_2^2\log 2 - \frac{1}{12}s_2\log^3 2$$

$$+\frac{1}{60}\log^5 2$$

$$a_{3,2} = \frac{1}{4}\int_0^1 t^{-1}\log^2 t\log^2\left(1-\frac{1}{2}t\right)dt$$

$$= \frac{1}{32}s_5 - \frac{1}{2}s_2s_3 + a_5 + a_4\log 2 - \frac{1}{20}s_2^2\log 2 + \frac{1}{2}s_3\log^2 2 -$$

$$-\frac{1}{6}s_2\log^3 2 + \frac{1}{40}\log^5 2$$

We can summarize these results as follows. For n + p = 2 and 3 these constants can all be expressed in terms of s_2, s_3, and log 2. For n + p = 4 and 5, however, not only s_4 and s_5 are needed, but also a_4 and a_5. No expression involving only s_4 and s_5 is known for a_4 and a_5.

We remark that, according to Eq. (2.25), for $n \geq 2$

$$a_n = \sum_{r=1}^{\infty} \frac{1}{2^r r^n} \qquad (3.25)$$

so that we could write here for $n = 1$

$$\log 2 = a_1 = \sum_{r=1}^{\infty} \frac{1}{2^r r} . \qquad (3.26)$$

It should be noted that the general structure of the relations (3.24) for higher $n + p$ remains unknown.

Many of the above relations are contained in Nielsen's paper, in most cases implicitly. Furthermore, the following general relations, which can be deduced by using properties of the function $L_{n,p}(x)$, are satisfied by the above expressions

$$a_{1,p} = \sigma_{1,p} - \frac{\log^{p+1} 2}{(p+1)!} \qquad (3.27)$$

and

$$s_{n,p} = \sum_{q=1}^{n} \binom{n+p-q-1}{p-1} \sigma_{q,n+p-q} + \sum_{q=1}^{p} \binom{n+p-q-1}{n-1} \sigma_{q,n+p-q} . \qquad (3.28)$$

This last formula corrects Eq.§ 19 (12) of Nielsen*). Nielsen actually claims in this connection that for all odd $n + p$, $\sigma_{n,p}$ can be expressed as a homogeneous polynomial (in the sense of Section 3.2) in terms of s_q $(2 \leq q \leq n+p)$ alone, with rational coefficients, e.g.

$$\left.\begin{matrix}\sigma_{1,4} \\ \sigma_{2,3}\end{matrix}\right\} = r' s_2 s_3 + r'' s_5 \qquad (3.29)$$

However, using Eq. (3.28) for $n + p = 5$, one finds

$$\begin{aligned} 2\,\sigma_{1,4} + \sigma_{2,3} &= s_{1,4} - \sigma_{3,2} - \sigma_{4,1} \\ 6\,\sigma_{1,4} + 3\,\sigma_{2,3} &= s_{2,3} - \sigma_{3,2} \end{aligned} \qquad (3.30)$$

*) Note that [1], Eqs. § 19(11) and (13) are also wrong, as can be easily verified numerically for small n,p.

The appearance of the combination $2\,\sigma_{1,4} + \sigma_{2,3}$ prevents us from finding independent separate expressions for $\sigma_{1,4}$ and $\sigma_{2,3}$. Because of this, and in view of the fact that Nielsen's proof is erroneous, we doubt whether his statement is correct.

4. DISPERSION RELATIONS

It was noted above that $S_{n,p}(x)$ has a branch cut along the real axis from 1 to ∞. In this section we want to exploit the information contained in the integral representation of $S_{n,p}(x)$ in terms of its discontinuity across the branch cut. Relations obtained through this procedure are usually called dispersion relations by theoretical physicists. In this way one can express a number of integrals involving $S_{n,p}(x)$ in terms of $S_{n,p}(x)$ of higher order and logarithms.

From the definition (1.1) or from Eq. (3.11), which are both valid for complex arguments $z = x$, we see that for $z \to \infty$

$$S_{n,p}(z) = O(\log^{n+p} z)$$

or

$$\lim_{z\to\infty} \frac{S_{n,p}(z)}{z} = 0$$

in the whole of the cut z-plane. We note further from Eq. (2.28) that $S_{n,p}(z)/z$ is regular at $z = 0$, because of $p \geq 1$. Since $S_{n,p}(z)/z$ has no singularities other than the branch cut along $1 < x < \infty$, it is possible to apply the Cauchy theorem to this function, giving

$$\frac{1}{z}\, S_{n,p}(z) = \frac{1}{2\pi i} \int_C \frac{S_{n,p}(\zeta)}{\zeta(\zeta - z)}\, d\zeta \tag{4.1}$$

for any contour not crossing the branch cut. If the contour is as shown in Fig. 1, the integral over the circle vanishes for $\zeta \to \infty$ and one obtains the "dispersion relation"

$$\begin{aligned} \frac{1}{z}\, S_{n,p}(z) &= \frac{1}{2\pi i} \int_1^\infty \frac{S_{n,p}(\zeta + i\epsilon) - S_{n,p}(\zeta - i\epsilon)}{\zeta(\zeta - z)}\, d\zeta \\ &= \frac{1}{\pi} \int_1^\infty \frac{\operatorname{Im} S_{n,p}(\zeta)}{\zeta(\zeta - z)}\, d\zeta\,. \end{aligned} \tag{4.2}$$

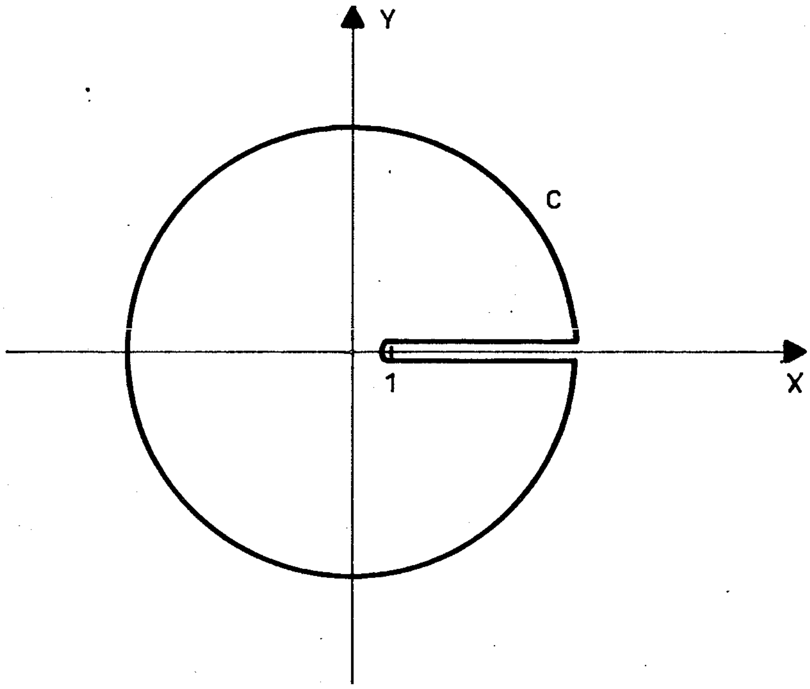

Fig. 1

The boundary value along the cut is obtained from formula (4.2) if we set z real plus a small positive imaginary part $i\epsilon$ ($\epsilon \to 0+$), which means

$$\int \frac{f(\zeta)d\zeta}{\zeta - z - i\epsilon} = P \int \frac{f(\zeta)d\zeta}{\zeta - z} + i\pi f(z) \ .$$

Changing the variable of integration to $t = 1/\zeta$ and replacing z by x, we finally have

$$S_{n,p}(x) = -\frac{1}{\pi} \int_0^1 \frac{\text{Im } S_{n,p}\left(\frac{1}{t}\right)}{t - \frac{1}{x}} dt \ . \tag{4.3}$$

The value of Im $S_{n,p}(1/t)$ can be found from relation (3.11), noting that on the right-hand side the imaginary parts come from log (-x) only. In general, we have

$$
\begin{aligned}
&-\frac{1}{\pi}\,\mathrm{Im}\,S_{n,p}\left(\frac{1}{t}\right) \\
&= -(-1)^n \sum_{s=1}^{p-1} (-1)^s \sum_{m=1}^{s} (-1)^m \binom{n+s-m-1}{s-m} S_{n-s-m,p-s}(t)\,\times \\
&\qquad \times \sum_{k=0}^{[(m-1)/2]} (-1)^k \frac{\pi^{2k}\log^{m-2k-1}|t|}{(2k+1)!\,(m-2k-1)!} - \\
&\quad - (-1)^p \sum_{r=1}^{n-1} (-1)^r C_{n-r,p} \sum_{k=0}^{[(r-1)/2]} (-1)^k \frac{\pi^{2k}\log^{r-2k-1}|t|}{(2k+1)!\,(r-2k-1)!} - \\
&\quad - (-1)^n \sum_{k=0}^{[(n+p-1)/2]} (-1)^k \frac{\pi^{2k}\log^{n+p-2k-1}|t|}{(2k+1)!\,(n+p-2k-1)!}\,.
\end{aligned}
\tag{4.4}
$$

where the sum over s or r is replaced by zero if $p = 1$ or $n = 1$, respectively. In particular, we obtain for $p = 1$ either from Eq. (4.4) or directly from the definition (1.1)

$$
-\frac{1}{\pi}\,\mathrm{Im}\,S_n\left(\frac{1}{t}\right) = \frac{(-1)^n}{(n-1)!}\log^{n-1} t \;^{*)} \tag{4.5}
$$

so that

$$
S_n(x) = \frac{(-1)^n}{(n-1)!}\int_0^1 \frac{\log^{n-1} t}{t-\frac{1}{x}} \;^{**)}\,, \tag{4.6}
$$

which can be found also from Eq. (2.4).

For $p = 2$, one can start from Eq. (4.4) with $n = 1$, and by applying the operation $\int_1^x dt/t$ repeatedly one obtains, with the help of Eq. (2.7), the expression

*) Lewin [6] page 2 and following uses a different convention for the imaginary parts.

**) This relation is also true for $n = 1$, if we define $S_1(x) = -\log(1 - x)$.

$$-\frac{1}{\pi}\,\mathrm{Im}\,S_{n-2,2}\left(\frac{1}{t}\right) = (-1)^{n-1}\left\{S_{n-1}(t) - \sum_{r=0}^{n-3} s_{n-r-1}\frac{\log^r t}{r!} + \frac{\log^{n-1} t}{(n-1)!}\right\}. \quad (4.7)$$

Inserting this into Eq. (4.3) and using Eq. (4.6) gives

$$\int_0^1 \frac{S_{n-1}(t)}{t-\frac{1}{x}}\,dt$$

$$= (-1)^{n-1}\left(S_{n-2,2}(x) + S_n(x)\right) + \sum_{r=2}^{n-2}(-1)^r\,s_{n-r}S_r(x) + s_{n-1}\log(1-x)\,. \quad (4.8)$$

Hence in particular

$$\int_0^1 \frac{S_2(t)}{t-\frac{1}{x}}\,dt = S_{1,2}(x) + S_3(x) + s_2\log(1-x) \quad (4.9)$$

$$\int_0^1 \frac{S_3(t)}{t-\frac{1}{x}}\,dt = -S_{2,2}(x) - S_4(x) + s_2S_2(x) + s_3\log(1-x) \quad (4.10)$$

$$\int_0^1 \frac{S_4(t)}{t-\frac{1}{x}}\,dt = S_{3,2}(x) + S_5(x) + s_3S_2(x) - s_2S_3(x) + s_4\log(1-x)\,. \quad (4.11)$$

With the help of the derivative relation (2.7), and after several integrations by parts, the integral on the left-hand side of Eq. (4.8) can be written as

$$\int_0^1 \frac{S_{n-1}(t)}{t-\frac{1}{x}}\,dt = s_{n-1}\log(1-x) + \sum_{r=2}^{m}(-1)^r\,s_{n-r}S_r(x) +$$

$$+ (-1)^{m+1}\int_0^1 \frac{S_{n-m-1}(t)\,S_m(xt)}{t}\,dt \quad (4.12)$$

which yields, by comparison with Eq. (4.8), an expression for an integral containing a product of two polylogarithms, namely

$$\int_0^1 \frac{S_{n-m-1}(t)\, S_m(xt)}{t}\, dt = (-1)^m \left\{(-1)^n \left[S_{n-2,2}(x) + S_n(x)\right] - \right.$$

$$\left. - \sum_{r=m+1}^{n-2} (-1)^r s_{n-r} S_r(x)\right\} \qquad (m \geq 2, \quad n - m \geq 3) \tag{4.13}$$

This formula was found by Nielsen in a different way*).

The procedure used for deriving formula (4.3) can also be applied to the function

$$\log (1 - x)\, S_n(x) \; .$$

This permits us to write the representation

$$\log (1 - x)\, S_n(x) = -\frac{1}{\pi} \int_0^1 \frac{\mathrm{Im} \left\{\log \left(1 - \frac{1}{t}\right) S_n\left(\frac{1}{t}\right)\right\}}{t - \frac{1}{x}}\, dt \; . \tag{4.14}$$

Using

$$\mathrm{Im}\, \{f(z)g(z)\} = \mathrm{Im}\, f(z)\, \mathrm{Re}\, g(z) + \mathrm{Re}\, f(z)\, \mathrm{Im}\, g(z)$$

we obtain, using Eq. (4.5),

$$\log (1 - x)\, S_n(x)$$

$$= \int_0^1 \frac{dt}{t - \frac{1}{x}} \left[\mathrm{Re}\, S_n\left(\frac{1}{t}\right) + \frac{(-1)^n}{(n-1)!} \left(\log^{n-1} t \log (1 - t) - \log^n t\right)\right] . \tag{4.15}$$

From Eq. (3.13) we find

$$(-1)^{n-1}\, \mathrm{Re}\, S_n\left(\frac{1}{t}\right) = S_n(t) - \sum_{r=2}^{n} \left(1 + (-1)^r\right) s_r \frac{\log^{n-r} t}{(n - r)!} + \frac{\log^n t}{n!} \; . \tag{4.16}$$

*) Note the misprint in [1] §16(4). The right-most term should read $S_{n+q-1,2}(x)$.

Inserting this result into Eq. (4.15) and using Eqs. (4.6), (4.8), we get

$$\frac{(-1)^n}{(n-1)!} \int_0^1 \frac{dt}{t - \frac{1}{x}} \log^{n-1} t \log(1-t)$$

$$= \log(1-x)\left(S_n(x) - s_n\right) + S_{n-1,2}(x) - n\, S_{n+1}(x) + \sum_{r=2}^{n-1} s_r S_{n-r+1}(x) , \tag{4.17}$$

This formula is valid for $n \geq 2$, the sum on the right-hand side being equal to zero for $n = 2$. In particular, we have

$$\int_0^1 \frac{dt}{t - \frac{1}{x}} \log t \log(1-t)$$

$$= \log(1-x)\left(S_2(x) - s_2\right) + S_{1,2}(x) - 2\, S_3(x)$$

$$-\frac{1}{2!} \int_0^1 \frac{dt}{t - \frac{1}{x}} \log^2 t \log(1-t)$$

$$= \log(1-x)\left(S_3(x) - s_3\right) + S_{2,2}(x) - 3\, S_4(x) + s_2 S_2(x) \tag{4.18}$$

$$\frac{1}{3!} \int_0^1 \frac{dt}{t - \frac{1}{x}} \log^3 t \log(1-t)$$

$$= \log(1-x)\left(S_4(x) - s_4\right) + S_{3,2}(x) - 4\, S_5(x) + s_2 S_3(x) + s_3 S_2(x) .$$

For $n = 1$, the relation degenerates into

$$-\int_0^1 \frac{dt}{t - \frac{1}{x}} \log(1-t) = \frac{1}{2} \log^2(1-x) - S_2(x) . \tag{4.19}$$

We note that from the last two equations (4.18) one has again

$$s_{2,2} = \frac{1}{10} s_2^2 \; ; \qquad s_{3,2} = 2\, s_5 - s_2 s_3 \, .$$

It is also possible to derive from Eq. (4.17) the relation

$$s_{n-1,2} = \frac{n}{2} s_{n+1} - \frac{1}{2}\left(s_2 s_{n-1} + s_3 s_{n-2} + \dots + s_{n-1} s_2\right) \qquad (4.20)$$

which was found by Nielsen in a different way.

Integral representations similar to Eq. (4.3) and (4.14) can be written not only for $S_{n,p}(x)$ but also for any product of these functions. These relations, however, become more and more complicated and we shall not enter into a systematic exposition here.

5. ZEROS OF $S_{n,p}(x)$

A numerical tabulation of $S_{n,p}(x)$ for real x and given values n, p indicates that Re $S_{n,p}(x)$ has p zeros on the positive real axis, whereas Im $S_{n,p}(x)$ has p - 1 zeros on this axis. It is likely that this behaviour is true for all integer values $n > 0$, $p > 0$.

The zeros are listed in the following table. The power of 10 is given in brackets.

	Zeros of Re $S_{n,p}(x)$	Zeros of Im $S_{n,p}(x)$
n = 1, p = 1	1.25951 70369 845 (1)	-
n = 1, p = 2	1.52632 71090 716 2.02726 44791 049 (2)	5.50374 16162 127
n = 2, p = 1	8.51716 73342 884 (1)	-
n = 1, p = 3	1.13731 31682 930 4.05240 89793 124 1.90643 92968 251 (3)	1.72742 72723 880 2.18159 46206 047 (1)
n = 2, p = 2	2.54017 89405 685 1.59132 98541 959 (3)	1.50703 76165 000 (1)
n = 3, p = 1	5.92142 82381 245 (2)	-
n = 1, p = 4	1.04904 29488 273 1.82294 50380 456 1.00395 03309 198 (1) 1.56965 39058 200 (4)	1.25797 98074 030 3.44695 79596 227 7.34010 94853 004 (1)
n = 2, p = 3	1.42483 20595 226 8.10590 99054 472 1.49730 20568 994 (4)	2.69029 17059 732 6.35446 47062 776 (1)
n = 3, p = 2	4.43896 57602 040 1.21889 20295 292 (4)	4.11373 61548 842 (1)
n = 4, p = 1	4.18245 60820 696 (3)	-

6. THE NUMERICAL CALCULATION OF $S_{n,p}(x)$

6.1 Existing tables and computational procedures

A review of tables of polylogarithms and related functions is given by Fletcher et al. [7]. No tables seem to exist for the generalized polylogarithms $S_{n,p}(x)$ with $p > 1$.

An Algol procedure [19] is available for the dilogarithm $S_2(x)$. This procedure calculates the function for real x to about 13 significant digits.

6.2 A method for the calculation of $S_{n,p}(x)$

We discuss a method for the numerical evaluation of the generalized polylogarithm function $S_{n,p}(x)$ for some small positive integers n, p and arbitrary real x.

In the interval $-1 \leq x \leq \frac{1}{2}$, we define the function

$$\tilde{S}_{n,p}(x) = p!\; p^n\, x^{-p}\, S_{n,p}(x) \tag{6.1}$$

which has the property

$$\tilde{S}_{n,p}(0) = 1 \; . \tag{6.2}$$

For a fixed pair of integers n, p we approximate this normalized function $\tilde{S}_{n,p}(x)$ in the given interval by a Chebyshev series

$$\tilde{S}_{n,p}(x) \approx \sum_r{}' \, c_r^{n,p} \, T_r\left(\frac{4x+1}{3}\right) \tag{6.3}$$

where

$$T_r(x) = \cos\,(r \arccos x) \tag{6.4}$$

is the Chebyshev polynomial of degree r.

For $-1 \leq x \leq \frac{1}{2}$, $S_{n,p}(x)$ is then obtained directly from Eq. (6.1) by means of this approximation. In the case $\frac{1}{2} < x \leq 2$, we use in addition the reflection formula (3.5). For $-\infty < x < -1$ and $2 < x < \infty$ the inversion formula (3.11) is used.

We have restricted ourselves to the calculation of the $c_r^{n,p}$ for the integers $1 \leq n \leq 4$, $1 \leq p \leq 4$, $n + p \leq 5$. We present in this paper Chebyshev coefficients for the 10 functions $S_{1,1}$; $S_{1,2}$; $S_{2,1}$; $S_{1,3}$; $S_{2,2}$; $S_{3,1}$; $S_{1,4}$; $S_{2,3}$; $S_{3,2}$; $S_{4,1}$. The method may, however, be used in principle for the calculation of $S_{n,p}(x)$ for other positive integer values of n and p.

Because of the fact that $\tilde{S}_{n,p}(x)$ has a branch point at $x = 1$ and all its derivatives higher than the $(n - 1)$st become infinite at $x = 1$, it is clear that the function cannot be approximated by a polynomial near this point. We have therefore chosen $x = \frac{1}{2}$ empirically as the upper limit of the basic interval for the approximation.

6.2.1 The calculation of the Chebyshev coefficients

For the calculation of the $c_r^{n,p}$ we used the method developed by Håvie [20], which is available as a Fortran program written in double precision mode.

In order to calculate the required input values $\tilde{S}_{n,p}(x)$ in $-1 \leq x \leq \frac{1}{2}$, we proceeded as follows. In the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$, we used the power series expansion (2.20), which gives

$$\tilde{S}_{n,p}(x) = \sum_{s=0}^{\infty} (-1)^s \frac{p!}{(p+s)!} \left(\frac{p}{p+s}\right)^n S_{p+s}^{(p)} x^s \; . \tag{6.5}$$

This sum was calculated in double precision mode until an accuracy of $\sim 10^{-20}$ was reached. The Stirling numbers $S_\sigma^{(p)}$ were generated beforehand by their stable recurrence relation [8]

$$\begin{aligned} S_{\sigma+1}^{(p)} &= S_\sigma^{(p-1)} - \sigma S_\sigma^{(p)} \\ S_1^{(0)} &= 0 \; , \quad S_p^{(p)} = 1 \end{aligned} \tag{6.6}$$

and the quotients $S_{\sigma+1}^{(p)}/S_\sigma^{(p)}$ were stored for $p = 1(1)4$, $\sigma = 1(1)100$.

In the interval $-1 \leq x < -\frac{1}{2}$, the evaluation of the power series becomes difficult when x approaches -1. A direct numerical integration of the definition (1.1) is not advisable for $n > 1$, since the integrand behaves like

$$t^{-1} \log^{n-1} t \log^p (1 - xt) = O(t^{p-1} \log^{n-1} t) \tag{6.7}$$

for $t \to 0$. For $n = 1$, however, the logarithmic factor in the right-hand side disappears and in this case we have integrated

$$\tilde{S}_{1,p}(x) = (-1)^p\, p\, x^{-p} \int_0^1 t^{-1} \log^p (1 - xt)\, dt \tag{6.8}$$

numerically, using an accurate double precision Gaussian integration routine [24]. Special care has been taken in the computation of the logarithm near $t = 0$, and a relative error of $\sim 10^{-20}$ was reached in the result $\tilde{S}_{n,p}(x)$.

In the case $n > 1$, we write Eq. (1.1) as

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \left(\int_0^\delta + \int_\delta^1 \right) \frac{\log^{n-1} t \log^p (1 - xt)}{t}\, dt \tag{6.9}$$

Substituting $t = \delta t'$ in the first integral, we obtain the relation

$$\frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^\delta \frac{\log^{n-1} t \log^p (1 - xt)}{t}\, dt = \sum_{s=0}^{n-1} \frac{(-1)^s \log^s \delta}{s!} S_{n-s,p}(\delta x)\ . \tag{6.10}$$

For $\delta = 1/2$ this leads to

$$\begin{aligned} \tilde{S}_{n,p}(x) &= 2^{-p} \sum_{s=0}^{n-1} \frac{(p \log 2)^s}{s!} \tilde{S}_{n-s,p}\left(\frac{x}{2}\right) + \\ &+ \frac{(-1)^{n+p-1}}{(n-1)!} p^n x^{-p} \int_{1/2}^1 \frac{\log^{n-1} t \log^p (1 - xt)}{t}\, dt\ . \end{aligned} \tag{6.11}$$

Because we now have $-1/2 \leq x/2 \leq -1/4$, the functions $\tilde{S}_{n-s,p}(x/2)$ may now be calculated from the power series (6.5). The integral on the right-hand-side was integrated numerically in the same way as in Eq. (6.8), since the integrand has no singularities in the region of integration.

6.3 The Algol procedure

We give an Algol procedure for the computation of the real and imaginary parts of $S_{n,p}(x)$ for arbitrary real x and the values of n and p given above. The procedure makes use of an auxiliary procedure Cheby already described by Clenshaw et al. [22]

```
procedure Snp(n,p,x,ReSnp,ImSnp);
 value n,p,x; integer n,p; real x,ReSnp,ImSnp;
comment This procedure calculates the real part ReSnp and the imaginary part ImSnp
of Nielsen's generalized polylogarithm function
```

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\log^{n-1} t \log^p (1-xt)}{t} dt$$

for given integers n and p satisfying $1 \le n \le 4$, $1 \le p \le 4$, $n + p \le 5$. Thus, any of the 10 functions $S_{1,1}$, $S_{1,2}$, $S_{2,1}$, $S_{1,3}$, $S_{2,2}$, $S_{3,1}$, $S_{1,4}$, $S_{2,3}$, $S_{3,2}$, $S_{4,1}$ can be evaluated. The argument x can be any real number. For $x \le 1$, $S_{n,p}(x)$ is real valued and ImSnp will be set to zero. In most cases, 13 to 14 significant digits are correct, for exceptions see Section 6.5. An error exit La is provided for illegal values of n or p;

```
begin integer r,s,n1,p1,j,k;
 real x1,y,srr,sri,sr1r,sr1i,h,q,chsum;
 array Sat1,C[1:4,1:4],fact,vr,vi[0:5],w[0:4],pi[1:5];
 switch Chebysw := L11,L12,L13,L14,La,La,La,La,La,La,L21,L22,
                   L23,La,La,La,La,La,La,La,L31,L32,La,La,La,
                   La,La,La,La,La,L41;
 switch retsw := R1,R2,R3,R4,R5;
 switch vsw := V1,V2,V3,V4,V5;
 if  n < 1 ∨ n > 4 ∨ p < 1 ∨ p > 4 ∨ n + p > 5 then go to La;
```

comment All constants needed in the different branches of the procedure are given in the following statements, although for certain values n,p,x it may happen that only some of them are used in the relevant branch. The following notations hold: fact[r] = r!, pi[r]= π^r, Sat1[r,s] = $S_{r,s}(1) = s_{r,s}$, C[r,s] = $C_{r,s}$;

```
 fact[0] := fact[1] := 1; pi[1] := 3.14159 26535 89793;
 for r := 2 step 1 until 5 do
  begin
   fact[r]:= r × fact[r-1]; pi[r] := pi[1] × pi[r-1]
  end;
```

comment The numerical values of $s_{r,s}$ are given by

$$(s_{r,s}) = \begin{pmatrix} \zeta(2) & \zeta(3) & \zeta(4) & \zeta(5) \\ \zeta(3) & \tfrac{1}{4}\zeta(4) & 2\zeta(5)-\zeta(2)\zeta(3) & * \\ \zeta(4) & 2\zeta(5)-\zeta(2)\zeta(3) & * & * \\ \zeta(5) & * & * & * \end{pmatrix}$$

and of $C_{r,s}$ by

$$(C_{r,s}) = \begin{pmatrix} \zeta(2) & \zeta(3) & \zeta(4) & \zeta(5) \\ 0 & -\tfrac{7}{4}\zeta(4) & -\zeta(5)-\zeta(2)\zeta(3) & * \\ \tfrac{7}{4}\zeta(4) & \zeta(5)+\zeta(2)\zeta(3) & * & * \\ 0 & * & * & * \end{pmatrix}$$

where $\zeta(m)$ is the Riemann zeta function;

```
 Sat1[1,1] := C[1,1] := 1.64493 40668 48226;
 Sat1[1,2] := Sat1[2,1] := C[1,2] := 1.20205 69031 59594;
 Sat1[1,3] := Sat1[3,1] := C[1,3] := 1.08232 32337 11138;
 Sat1[1,4] := Sat1[4,1] := C[1,4] := 1.03692 77551 43370;
 Sat1[2,2] := 0.27058 08084 27785;
 Sat1[2,3] := Sat1[3,2] := 0.09655 11599 89444;
 C[2,1] := C[4,1] := 0;
 C[3,1] := 1.89406 56589 94492; C[2,2] := - C[3,1];
 C[3,2] := 3.01423 21054 40666; C[2,3] := - C[3,2];
 if x = 1 then
  begin
   ReSnp := Sat1[n,p]; ImSnp := 0; go to Lo
  end
 else if x ≥ -1 ∧ x ≤ 0.5 then
  begin
   n1 := n; p1 := p; x1 := x;
   y := (4 × x1 + 1)/3; j := 1; go to Lch;
R1: ReSnp := chsum; ImSnp := 0; go to Lo
  end
 else if x < -1 ∨ x > 2 then
  begin comment The inversion formula (3.11) is used;
   k := n + p; x1 := x; j := 4;
V: vr[0] := 1; vr[1] := ln(abs(x1));
   vi[0] := vi[1] := 0;
   for r := 2 step 1 until k do
    begin
     vr[r] := vr[1] × vr[r-1]; vi[r] := 0
    end;
  if x > 1 then
   begin comment If ξ < 0, the real and imaginary parts of log^r ξ are computed
         in the following statements. The definition log (-ξ) = log |ξ| - iπ
         is used for ξ > 0;
```

```
        go to vsw[k];
V5:     vr[5] := vr[5] - 10 × pi[2] × vr[3] + 5 × pi[4] × vr[1];
        vi[5] := 10 × pi[3] × vr[2] - 5 × pi[1] × vr[4] - pi[5];
V4:     vr[4] := vr[4] - 6 × pi[2] × vr[2] + pi[4];
        vi[4] := 4 × (pi[3] × vr[1] - pi[1] × vr[3]);
V3:     vr[3] := vr[3] - 3 × pi[2] × vr[1];
        vi[3] := -3 × pi[1] × vr[2] + pi[3];
V2:     vr[2] := vr[2] - pi[2];
        vi[2] := -2 × pi[1] × vr[1];
V1:     vi[1] := -pi[1]
       end;
      for r := 2 step 1 until k do
       begin
        vr[r] := vr[r]/fact[r];
        vi[r] := vi[r]/fact[r]
       end;
      go to retsw[j];
R4: x1 := 1/x; y := (4 × x1 + 1)/3 ; j := 2 ;
      sr1r := vr[k]; sr1i := vi[k];
      for r := 0 step 1 until n - 1 do
       begin
        h := C[n-r,p];
        sr1r := sr1r + vr[r] × h; sr1i := sr1i + vi[r] × h
      end;
     ReSnp := ImSnp := 0; q := 1;
     for s := 0 step 1 until p - 1 do
      begin
       srr := sri := 0; p1 := p - s;
       for r := 0 step 1 until s do
        begin
         n1 := n + s - r; go to Lch;
R2:      h := (fact[n1-1]/(fact[s-r] × fact[n-1])) × chsum;
         srr := srr + vr[r] × h; sri := sri + vi[r] × h
        end;
       ReSnp := ReSnp + q × srr; ImSnp := ImSnp + q × sri;
       q := -q
      end;
    h := (-1)↑n; q := (-1)↑p;
     ReSnp := h × ReSnp + q × sr1r;
     ImSnp := h × ImSnp + q × sr1i;
```

```
    go to Lo
   end
  else
   begin comment The reflection formula (3.5) is used;
    w[0] := 1; w[1] := ln(x);
    for r := 2 step 1 until n do
     w[r] := w[1] × w[r-1]/r;
    k := p; x1 := 1 - x; j := 5; go to V;
R5: y := (4 × x1 + 1)/3; j := 3;
    h := (-1)↑p × w[n];
    ReSnp := vr[p] × h; ImSnp := vi[p] × h;
    for s := 0 step 1 until n - 1 do
     begin
      srr := sri := 0; q := 1; p1 := n - s;
      for r := 0 step 1 until p - 1 do
       begin
        n1 := p - r; go to Lch;
R3:     h := q × chsum;
        srr := srr + vr[r] × h; sri := sri + vi[r] × h;
        q := -q
       end;
      ReSnp := ReSnp + w[s] × (Sat1[p1,p] - srr);
      ImSnp := ImSnp - w[s] × sri
     end;
    go to Lo
   end;
Lch: begin comment The appropriate Chebyshev approximation is calculated;
      go to Chebysw[10 × n1-10+p1];
L11: chsum := Cheby(y,cS11,nS11); go to L;
L12: chsum := Cheby(y,cS12,nS12); go to L;
L21: chsum := Cheby(y,cS21,nS21); go to L;
L13: chsum := Cheby(y,cS13,nS13); go to L;
L22: chsum := Cheby(y,cS22,nS22); go to L;
L31: chsum := Cheby(y,cS31,nS31); go to L;
L14: chsum := Cheby(y,cS14,nS14); go to L;
L23: chsum := Cheby(y,cS23,nS23); go to L;
L32: chsum := Cheby(y,cS32,nS32); go to L;
L41: chsum := Cheby(y,cS41,nS41);
L:   chsum := chsum × x1↑p1/(fact[p1] × p1↑n1);
      go to retsw[j]
     end;
Lo: end Snp
```

6.4 Numerical values of the Chebyshev coefficients

The coefficients $c_r^{n,p}$ are presented in the same format as given by Clenshaw et al. [21], where further explanations can be found. The numbers were calculated in double precision mode on a CDC 6600 computer, and then rounded to 15 decimal digits. All the expansions are given for the interval $-1 \leq x \leq \frac{1}{2}$

cS11

$$S_{1,1}(x) = x \sum{}' c_r T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	26.0
0	1.93506 43008 69969
1	0.16607 30329 27855
2	0.02487 93229 24228
3	0.00468 63619 59447
4	0.00100 16274 96164
5	0.00023 20021 96094
6	0.00005 68178 22718
7	0.00001 44963 00557
8	38163 29463
9	10299 04264
10	2835 75385
11	793 87055
12	225 36705
13	64 74338
14	18 79117
15	5 50291
16	1 62421
17	48274
18	14437
19	4342
20	1312
21	398
22	121
23	37
24	11
25	4
26	1

$\Sigma' c_r =$ 1.16448 10529 30025

$\Sigma' (-1)^r c_r =$ .82246 70334 24113

cS12

$$S_{1,2}(x) = \frac{x^2}{4} \sum{}' c_r T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	28.0
0	1.90361 77825 56648
1	0.43131 13184 65320
2	0.10002 25071 49049
3	0.02442 41559 52197
4	0.00622 51246 37235
5	0.00164 07883 12350
6	0.00044 40792 02645
7	0.00012 27749 41678
8	0.00003 45398 12838
9	98586 95646
10	28485 69950
11	8317 08473
12	2450 39499
13	727 64962
14	217 58023
15	65 46158
16	19 80328
17	6 02044
18	1 83845
19	56368
20	17346
21	5356
22	1659
23	515
24	160
25	50
26	16
27	5
28	2

$\Sigma' c_r =$ 1.51604 80676 82043

$\Sigma' (-1)^r c_r =$ .60102 84515 79798

cS21

$$S_{2,1}(x) = x \sum{}' c_r T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	23.0
0	1.95841 72133 83495
1	0.08518 81314 86831
2	0.00855 98522 20133
3	0.00121 17721 44129
4	0.00020 72276 85308
5	0.00003 99695 86914
6	83806 40658
7	18684 89447
8	4366 60867
9	1059 17334
10	264 78920
11	67 87000
12	17 76536
13	4 73417
14	1 28121
15	35143
16	9754
17	2736
18	775
19	221
20	64
21	18
22	5
23	2

$\Sigma' c_r =$ 1.07442 63872 16080

$\Sigma' (-1)^r c_r =$ .90154 26773 69695

cS13

$$S_{1,3}(x) = \frac{x^3}{18} \sum{}' c_r T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	30.0
0	1.96322 05598 27298
1	0.72926 80632 07265
2	0.22774 71490 93207
3	0.06809 08329 61969
4	0.02013 70118 30638
5	0.00595 47848 01966
6	0.00176 76901 39589
7	0.00052 74821 85025
8	0.00015 82746 14600
9	0.00004 77492 20758
10	0.00001 44792 04084
11	44115 48863
12	13500 38702
13	4148 17790
14	1279 33069
15	395 90696
16	122 90546
17	38 26584
18	11 94589
19	3 73858
20	1 17273
21	36866
22	11613
23	3665
24	1159
25	367
26	116
27	37
28	12
29	4
30	1

$\Sigma' c_r =$ 2.03533 01589 58561

$\Sigma' (-1)^r c_r =$ .42754 25938 07133

cS22

$$S_{2,2}(x) = \frac{x^2}{8} \sum{}' c_r T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	26.0
0	1.90043 70392 79041
1	0.29052 52916 14329
2	0.05081 77406 17156
3	0.00995 54376 72804
4	0.00211 73389 50305
5	0.00047 85947 05496
6	0.00011 33432 13084
7	0.00002 78473 31042
8	70478 81078
9	18278 87396
10	4838 74921
11	1303 38422
12	356 37694
13	98 71736
14	27 65856
15	7 82787
16	2 23540
17	64352
18	18660
19	5446
20	1599
21	472
22	140
23	42
24	13
25	4
26	1

$\Sigma' c_r =$ 1.30426 36530 97896

$\Sigma' (-1)^r c_r =$ .70228 53725 49243

cS31

$$S_{3,1}(x) = x \sum{}' c_r T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	21.0
0	1.97600 02334 44585
1	0.04364 06760 96007
2	0.00295 09117 82781
3	0.00031 47780 97198
4	0.00004 31484 60293
5	69381 82301
6	12464 03504
7	2429 36280
8	504 08271
9	109 90749
10	24 94666
11	5 85396
12	1 41269
13	34915
14	8809
15	2263
16	591
17	156
18	42
19	11
20	3
21	1

$\Sigma' c_r =$ 1.03495 81233 47798

$\Sigma' (-1)^r c_r =$ .94703 28294 97246

cS14

$$S_{1,4}(x) = \frac{x^4}{96} \sum\nolimits' c_r \, T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	32.0
0	2.12810 42369 22831
1	1.06917 20744 98122
2	0.41527 19325 17684
3	0.14610 33293 62224
4	0.04904 73264 87838
5	0.01606 34086 03956
6	0.00518 88935 07895
7	0.00166 29871 73242
8	0.00053 05827 99692
9	0.00016 88702 92506
10	0.00005 36832 80590
11	0.00001 70592 33125
12	54217 43743
13	17239 40821
14	5485 32751
15	1746 77950
16	556 75505
17	177 62342
18	56 72244
19	18 13132
20	5 80123
21	1 85789
22	59555
23	19108
24	6136
25	1972
26	634
27	204
28	66
29	21
30	7
31	2
32	1

$\Sigma' c_r =$ 2.76734 02168 90396

$\Sigma' (-1)^r c_r =$ .30096 09217 61363

cS23

$$S_{2,3}(x) = \frac{x^3}{54} \sum\nolimits' c_r \, T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	28.0
0	1.90128 06437 35539
1	0.54138 28546 51707
2	0.13649 97959 03205
3	0.03417 94232 82074
4	0.00869 02788 35829
5	0.00225 28408 41555
6	0.00059 51608 98065
7	0.00015 99561 77662
8	0.00004 36521 30961
9	0.00001 20747 46884
10	33801 81755
11	9563 24763
12	2731 31288
13	786 69684
14	228 31955
15	66 72049
16	19 61910
17	5 80178
18	1 72463
19	51511
20	15453
21	4654
22	1407
23	427
24	130
25	40
26	12
27	4
28	1

$\Sigma' c_r =$ 1.67446 10799 35393

$\Sigma' (-1)^r c_r =$ .51848 46959 29011

cS32

$$S_{3,2}(x) = \frac{x^2}{16} \sum\nolimits' c_r \, T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	24.0
0	1.91537 01309 26997
1	0.19725 24967 95338
2	0.02603 37031 39182
3	0.00409 38216 82611
4	0.00072 68170 71098
5	0.00014 09187 92608
6	0.00002 92045 89136
7	63763 11440
8	14516 78496
9	3420 52806
10	829 43024
11	206 07841
12	52 28234
13	13 50664
14	3 54513
15	94364
16	25433
17	6931
18	1908
19	530
20	148
21	42
22	12
23	3
24	1

$\Sigma' c_r =$ 1.18597 03081 89863

$\Sigma' (-1)^r c_r =$ .78298 23527 99505

S cS41

$$S_{4,1}(x) = x \sum\nolimits' c_r \, T_r\left(\frac{4x+1}{3}\right)$$

r	c_r
-1	18.0
0	1.98687 30334 26943
1	0.02225 77012 68255
2	0.00101 47557 47032
3	0.00008 17515 62503
4	89997 35466
5	12082 39865
6	1861 69135
7	317 47228
8	58 52146
9	11 47394
10	2 36517
11	50820
12	11308
13	2592
14	610
15	147
16	36
17	9
18	2

$\Sigma' c_r =$ 1.01680 11584 84537

$\Sigma' (-1)^r c_r =$ .97211 97704 46910

6.5 Numerical tests and accuracy

The Algol procedure has been tested on a CDC 3800 computer at the University of Geneva. A FORTRAN version has been written using complex arithmetic and tested for the following cases on a CDC 6600 computer at CERN.

i) Evaluation of $\tilde{S}_{1,p}(x) = p!\, p\, x^{-p}\, S_{1,p}(x)$ for $p = 1(1)4$, $x = -100(10)-10(1)-1(0.1)0.9(0.01)0.99$, $x = 1 - 10^{-m}$ $[m = 3(1)13]$. The results have been compared with those obtained from

$$\tilde{S}_{1,p}(x) = p\, x^{-p} \int_0^{-\log(1-x)} \frac{t^p}{e^t - 1}\, dt$$

by numerical integration. Agreement was found in most cases to 13 or 14 significant digits, except in the case $p > 2$ for $m \simeq 8$, where in the worst case, $p = 4$, at least 10 significant digits agreed.

ii) Evaluation of $\tilde{S}_{n,1}(x) = x^{-1}\, S_{n,1}(x)$ for $n = 1(1)4$, $x = -0.7(0.1)0.9(0.01)0.99$. The results have been compared with those found from

$$\tilde{S}_{n,1}(x) = \sum_{s=0}^{\infty} \frac{x^s}{(s+1)^{n+1}}$$

by direct summation. Agreement was found to 13 or 14 significant digits in all cases.

iii) Evaluation of

$$S_{n,p}(x) = \int_0^x \frac{S_{n-1,p}(t)}{t}\, dt$$

for $x < 1$ and

$$S_{n,p}(x) = s_{n,p} + \int_1^x \frac{S_{n-1,p}(t)}{t}\, dt$$

for $x > 1$ by numerical integration for $n = 2(1)4$, $p = 1(1)4$, $n + p \le 5$, $x = \pm 20000, \pm 10000, \pm 1000, \pm 100, \pm 10, \pm 5, -2, -1, 0.5, 1.5, 1 \pm 10^{-m}$ with $m = 6,11$ for $\mathrm{Re}\, S_{n,p}(x)$ and $x = 1.01, 1.1, 1.5, 3.5, 10, 100, 1000, 10000, 20000$ for $\mathrm{Im}\, S_{n,p}(x)$. Agreement to 13 or 14 significant digits was found in practically all cases.

Of course, accuracy is lost for values of x in the neighbourhood of a zero. One or two digits of Re $S_{n,p}(x)$ may also be lost for $p > 3$ in the regions $-2 \lesssim x \lesssim -1$ and $0.5 \lesssim x \lesssim 0.8$. In the case of Im $S_{n,p}(x)$, accuracy will be lost for $1 < x < 1 + \delta$ $(\delta > 0)$, because of Im $S_{n,p}(1) = 0$. δ varies between < 0.01 for $n = 1$, p arbitrary, and approximately 0.19 for $n = 3$, $p = 2$. Nevertheless, an absolute accuracy of around 10^{-12} is likely to be obtained in all these cases.

The results obtained for the polylogarithms $S_n(x)$ were also checked against the 10-figure table of the Amsterdam Mathematisch Centrum [23]. Occasional discrepancies of one or two units in the tenth figure were found.

The authors had no access to the 20 figure table of Shafer [24].

Acknowledgement

We thank Dr. W. Rühl for reading the manuscript and for helpful comments, in particular for giving the proofs of Eq. (3.17) and (3.21).

APPENDIX 1

The proof of Eq. (3.21) uses some of the ideas of Nielsen's proof and proceeds as follows. Using Eq. (2.12) with $-\alpha$ instead of α, one obtains, analogously to Eq. (2.13),

$$s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial\beta^{n-1}\partial\alpha^{p}} \left[\beta^{-1} {}_2F_1(-\alpha,\beta;\beta+1;1)\right]_{\alpha=\beta=0}$$

which can be written as [25]

$$s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial\beta^{n-1}\partial\alpha^{p}} \left[\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\beta\,\Gamma(1+\alpha+\beta)}\right]_{\alpha=\beta=0} .$$

Using [25]

$$\log\Gamma(1+z) = -\gamma z + \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m} z^m \qquad (|z| \leq 1)$$

we can write

$$s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial\beta^{n-1}\partial\alpha^{p}} \left[\beta^{-1}\, e^{\varphi(\alpha,\beta)}\right]_{\alpha=\beta=0}$$

where

$$\varphi(\alpha,\beta) = -\sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m} \left[(\alpha+\beta)^m - \alpha^m - \beta^m\right] .$$

Nielsen's theorem follows by noting that any $\zeta(q)$ in this expression is multiplied by a homogeneous polynomial of degree q in α and β. Expanding the exponential function yields

$$s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial\beta^{n-1}\partial\alpha^{p}}$$

$$\left\{\beta^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \prod_{i=1}^{k} \sum_{m_i=2}^{\infty} (-1)^{m_i} \frac{\zeta(m_i)}{m_i} \left[(\alpha+\beta)^{m_i} - \alpha^{m_i} - \beta^{m_i}\right]\right\}_{\alpha=\beta=0} .$$

In order to carry out the differentiations with respect to α, we apply the Leibniz formula

$$\frac{d^p}{dx^p}\left\{\prod_{i=1}^{k} f_i(x)\right\} = \left\{\sum_{i=1}^{k} f_i(x)\right\}^{(p)}$$

$$= \sum \frac{p!}{p_1!p_2! \ldots p_k!} f_1^{(p_1)}(x) f_2^{(p_2)}(x) \ldots f_k^{(p_k)}(x)$$

so that each sum over m_i is differentiated p_i times with respect to α. The p_i can therefore be restricted by

$$1 \leq p_i \leq m_i - 1 \; ; \qquad \sum_{i=1}^{k} p_i = p \; .$$

We obtain

$$s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial \beta^{n-1}}$$

$$\left\{\beta^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{\substack{p_i \geq 1 \\ \Sigma p_i = p}} \prod_{i=1}^{k} \sum_{m_i \geq p_i + 1}^{\infty} (-1)^{m_i} \frac{\zeta(m_i)}{m_i} \binom{m_i}{p_i} \beta^{m_i - p_i}\right\}_{\beta=0} .$$

Applying the Leibniz formula again for differentiation with respect to β and setting $\beta = 0$ gives the condition

$$\sum_{i=1}^{k} m_i = n + p$$

so that

$$s_{n,p} = \sum_{k} \frac{(-1)^{k+1}}{k!} \sum_{\substack{p_i \geq 1 \\ \Sigma p_i = p}} \sum_{\substack{m_i \geq p_i + 1 \\ \Sigma m_i = n+p}} \prod_{i=1}^{k} \frac{\zeta(m_i)}{m_i} \binom{m_i}{p_i}$$

from which Eq. (3.21) follows.

As an example we give the calculation of $s_{4,4}$.

$$n = 4,\ p = 4; \qquad \Sigma\, p_i = 4; \qquad \Sigma\, m_i = 8; \qquad \zeta(q) = s_q$$

$$s_{4,4} = H_4(8)\,\frac{s_8}{8} - \frac{1}{2}\left\{[H_4(2,6) + H_4(6,2)]\,\frac{s_2 s_6}{2\times 6} + [H_4(3,5) + H_4(5,3)]\,\frac{s_3 s_5}{3\times 5} + \right.$$

$$\left. + H_4(4,4)\,\frac{s_4^2}{4^2}\right\} + \frac{1}{6}\left\{[H_4(2,2,4) + H_4(2,4,2) + H_4(4,2,2)]\,\frac{s_2^2 s_4}{2^2\times 4} + \right.$$

$$\left. + [H_4(2,3,3) + H_4(3,2,3) + H_4(3,3,2)]\,\frac{s_2 s_3^2}{2\times 3^2}\right\} - \frac{1}{24}\,H_4(2,2,2,2)\,\frac{s_2^4}{2^4}\ .$$

We obtain

$$H_4(8) = \binom{8}{4} = 70$$

$$H_4(2,6) = H_4(6,2) = \binom{2}{1}\binom{6}{3} = 40$$

$$H_4(3,5) = H_4(5,3) = \binom{3}{2}\binom{5}{2} + \binom{3}{1}\binom{5}{3} = 60$$

$$H_4(4,4) = \binom{4}{2}\binom{4}{2} + 2\binom{4}{1}\binom{4}{3} = 68$$

$$H_4(2,2,4) = H_4(2,4,2) = H_4(4,2,2) = \binom{2}{1}\binom{2}{1}\binom{4}{2} = 24$$

$$H_4(2,3,3) = H_4(3,2,3) = H_4(3,3,2) = 2\binom{2}{1}\binom{3}{1}\binom{3}{2} = 36$$

$$H_4(2,2,2,2) = \binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1} = 16$$

so that

$$s_{4,4} = \frac{35}{4}\,s_8 - \frac{10}{3}\,s_2 s_6 - 4\,s_3 s_5 - \frac{17}{8}\,s_4^2 + \frac{3}{4}\,s_2^2 s_4 + s_2 s_3^2 - \frac{1}{24}\,s_2^4\ .$$

APPENDIX 2

We give here the explicit forms of the reflection formula (3.5) and the inversion formula (3.11) for $1 \leq n \leq 4$, $1 \leq p \leq 4$, $n + p \leq 5$.

The reflection formulae (3.5) for $x \leq 1$

$$S_{1,1}(x) = s_2 - S_{1,1}(1-x) - \log x \log(1-x)$$

$$S_{1,2}(x) = s_3 - S_{2,1}(1-x) + \log(1-x)\, S_{1,1}(1-x) + \frac{1}{2} \log x \log^2(1-x)$$

$$S_{2,1}(x) = s_3 - S_{1,2}(1-x) + \log x\, [s_2 - S_{1,1}(1-x)] - \frac{1}{2} \log^2 x \log(1-x)$$

$$S_{1,3}(x) = s_4 - S_{3,1}(1-x) + \log(1-x)\, S_{2,1}(1-x) - \frac{1}{2} \log^2(1-x)\, S_{1,1}(1-x) - \frac{1}{6} \log x \log^3(1-x)$$

$$S_{2,2}(x) = \frac{1}{4} s_4 - S_{2,2}(1-x) + \log(1-x)\, S_{1,2}(1-x) + \log x\, [s_3 - S_{2,1}(1-x) + \log(1-x)\, S_{1,1}(1-x)] + \frac{1}{4} \log^2 x \log^2(1-x)$$

$$S_{3,1}(x) = s_4 - S_{1,3}(1-x) + \log x\, [s_3 - S_{1,2}(1-x)] + \frac{1}{2} \log^2 x\, [s_2 - S_{1,1}(1-x)] - \frac{1}{6} \log^3 x \log(1-x)$$

$$S_{1,4}(x) = s_5 - S_{4,1}(1-x) + \log(1-x)\, S_{3,1}(1-x) - \frac{1}{2} \log^2(1-x)\, S_{2,1}(1-x) + \frac{1}{6} \log^3(1-x)\, S_{1,1}(1-x) + \frac{1}{24} \log x \log^4(1-x)$$

$$S_{2,3}(x) = 2\, s_5 - s_2 s_3 - S_{3,2}(1-x) + \log(1-x)\, S_{2,2}(1-x) - \frac{1}{2} \log^2(1-x)\, S_{1,2}(1-x) + \log x\, [s_4 - S_{3,1}(1-x) + \log(1-x)\, S_{2,1}(1-x) - \frac{1}{2} \log^2(1-x)\, S_{1,1}(1-x)] - \frac{1}{12} \log^2 x \log^3(1-x)$$

$$S_{3,2}(x) = 2\, s_5 - s_2 s_3 - S_{2,3}(1-x) + \log(1-x)\, S_{1,3}(1-x) +$$
$$+ \log x \left[\frac{1}{4} s_4 - S_{2,2}(1-x) + \log(1-x)\, S_{1,2}(1-x)\right] +$$
$$+ \frac{1}{2}\log^2 x\, [s_3 - S_{2,1}(1-x) + \log(1-x)\, S_{1,1}(1-x)] +$$
$$+ \frac{1}{12}\log^3 x \log^2(1-x)$$

$$S_{4,1}(x) = s_5 - S_{1,4}(1-x) + \log x\, [s_4 - S_{1,3}(1-x)] +$$
$$+ \frac{1}{2}\log^2 x\, [s_3 - S_{1,2}(1-x)] + \frac{1}{6}\log^3 x\, [s_2 - S_{1,1}(1-x)] -$$
$$- \frac{1}{24}\log^4 x \log(1-x)$$

The reflection formulae (3.5) for $x \geq 1$

The formulae for the real part Re $S_{n,p}(x)$ can easily be found by replacing in the above expressions:

$$\log\ (1-x) \text{ by } \log\ (x-1)$$
$$\log^2\ (1-x) \text{ by } \log^2\ (x-1) - \pi^2$$
$$\log^3\ (1-x) \text{ by } \log^3\ (x-1) - 3\pi^2 \log\ (x-1)$$
$$\log^4\ (1-x) \text{ by } \log^4\ (x-1) - 6\pi^2 \log^2\ (x-1) + \pi^4 .$$

For the imaginary part Im $S_{n,p}(x)$, the following expressions hold:

$$-\frac{1}{\pi}\,\mathrm{Im}\, S_{1,1}(x) = -\log x$$

$$-\frac{1}{\pi}\,\mathrm{Im}\, S_{1,2}(x) = S_{1,1}(1-x) + \log x \log(x-1)$$

$$-\frac{1}{\pi}\,\mathrm{Im}\, S_{2,1}(x) = -\frac{1}{2}\log^2 x$$

$$-\frac{1}{\pi}\,\mathrm{Im}\, S_{1,3}(x) = S_{2,1}(1-x) - \log(x-1)\, S_{1,1}(1-x) -$$
$$- \frac{1}{2}\log x\, [\log^2(x-1) - 2\, s_2]$$

$$-\frac{1}{\pi}\,\mathrm{Im}\, S_{2,2}(x) = S_{1,2}(1-x) + \log x\, S_{1,1}(1-x) + \frac{1}{2}\log^2 x \log(x-1)$$

$$-\frac{1}{\pi}\,\mathrm{Im}\, S_{3,1}(x) = -\frac{1}{6}\log^3 x$$

$$-\frac{1}{\pi}\,\mathrm{Im}\,S_{1,4}(x) = S_{3,1}(1-x) - \log(x-1)\,S_{2,1}(1-x) + $$
$$+\frac{1}{2}\left[\log^2(x-1) - 2\,s_2\right] S_{1,1}(1-x) +$$
$$+\frac{1}{6}\log x\left[\log^3(x-1) - 6\,s_2\log(x-1)\right]$$

$$-\frac{1}{\pi}\,\mathrm{Im}\,S_{2,3}(x) = S_{2,2}(1-x) - \log(x-1)\,S_{1,2}(1-x) +$$
$$+\log x\left[S_{2,1}(1-x) - \log(x-1)\,S_{1,1}(1-x)\right] -$$
$$-\frac{1}{4}\log^2 x\left[\log^2(x-1) - 2\,s_2\right]$$

$$-\frac{1}{\pi}\,\mathrm{Im}\,S_{3,2}(x) = S_{1,3}(1-x) + \log x\,S_{1,2}(1-x) + \frac{1}{2}\log^2 x\;S_{1,1}(1-x) +$$
$$+\frac{1}{6}\log^3 x\log(x-1)$$

$$-\frac{1}{\pi}\,\mathrm{Im}\,S_{4,1}(x) = -\frac{1}{24}\log^4 x$$

<u>The inversion formulae (3.11) for $x > 1$</u>

$$\mathrm{Re}\,S_{1,1}(x) = 2\,s_2 - S_{1,1}\left(\frac{1}{x}\right) - \frac{1}{2}\log^2 x$$

$$\mathrm{Re}\,S_{1,2}(x) = s_3 - S_{1,2}\left(\frac{1}{x}\right) + S_{2,1}\left(\frac{1}{x}\right) + \log x\,S_{1,1}\left(\frac{1}{x}\right) + \frac{1}{6}\log^3 x - 3\,s_2\log x$$

$$\mathrm{Re}\,S_{2,1}(x) = S_{2,1}\left(\frac{1}{x}\right) - \frac{1}{6}\log^3 x + 2\,s_2\log x$$

$$\mathrm{Re}\,S_{1,3}(x) = -\frac{19}{4}\,s_4 - S_{1,3}\left(\frac{1}{x}\right) + S_{2,2}\left(\frac{1}{x}\right) - S_{3,1}\left(\frac{1}{x}\right) +$$
$$+\log x\left[S_{1,2}\left(\frac{1}{x}\right) - S_{2,1}\left(\frac{1}{x}\right)\right] + \frac{1}{2}\left(6\,s_2 - \log^2 x\right) S_{1,1}\left(\frac{1}{x}\right) -$$
$$-\frac{1}{24}\log^4 x + \frac{3}{2}\,s_2\log^2 x$$

$$\mathrm{Re}\,S_{2,2}(x) = 2\,s_4 + S_{2,2}\left(\frac{1}{x}\right) - 2\,S_{3,1}\left(\frac{1}{x}\right) - \log x\,S_{2,1}\left(\frac{1}{x}\right) + \frac{1}{24}\log^4 x -$$
$$-\frac{3}{2}\,s_2\log^2 x + s_3\log x$$

$$\mathrm{Re}\,S_{3,1}(x) = 2\,s_4 - S_{3,1}\left(\frac{1}{x}\right) - \frac{1}{24}\log^4 x + s_2\log^2 x$$

$$\begin{aligned}\text{Re } S_{1,4}(x) = {} & s_5 - S_{1,4}\left(\frac{1}{x}\right) + S_{2,3}\left(\frac{1}{x}\right) - S_{3,2}\left(\frac{1}{x}\right) + S_{4,1}\left(\frac{1}{x}\right) + \\ & + \log x \left[S_{1,3}\left(\frac{1}{x}\right) - S_{2,2}\left(\frac{1}{x}\right) + S_{3,1}\left(\frac{1}{x}\right)\right] + \\ & + \frac{1}{2}\,(\log^2 x - 6\, s_2)\left[S_{2,1}\left(\frac{1}{x}\right) - S_{1,2}\left(\frac{1}{x}\right)\right] + \\ & + \frac{1}{6}\,(\log^3 x - 18\, s_2 \log x)\; S_{1,1}\left(\frac{1}{x}\right) + \frac{1}{120}\log^5 x - \frac{1}{2}\, s_2 \log^3 x + \\ & + \frac{15}{4}\, s_4 \log x\end{aligned}$$

$$\begin{aligned}\text{Re } S_{2,3}(x) = {} & s_5 + s_2 s_3 + S_{2,3}\left(\frac{1}{x}\right) - 2\, S_{3,2}\left(\frac{1}{x}\right) + 3\, S_{4,1}\left(\frac{1}{x}\right) + \\ & + \log x \left[2\, S_{3,1}\left(\frac{1}{x}\right) - S_{2,2}\left(\frac{1}{x}\right)\right] + \\ & + \frac{1}{2}\,(\log^2 x - 6\, s_2)\; S_{2,1}\left(\frac{1}{x}\right) - \frac{1}{120}\log^5 x + \frac{1}{2}\, s_2 \log^3 x - \\ & - \frac{19}{4}\, s_4 \log x\end{aligned}$$

$$\begin{aligned}\text{Re } S_{3,2}(x) = {} & s_5 - 2\, s_2 s_3 - S_{3,2}\left(\frac{1}{x}\right) + 3\, S_{4,1}\left(\frac{1}{x}\right) + \log x\; S_{3,1}\left(\frac{1}{x}\right) + \\ & + \frac{1}{120}\log^5 x - \frac{1}{2}\, s_2 \log^3 x + \frac{1}{2}\, s_3 \log^2 x + 2\, s_4 \log x\end{aligned}$$

$$\text{Re } S_{4,1}(x) = S_{4,1}\left(\frac{1}{x}\right) - \frac{1}{120}\log^5 x + \frac{1}{3}\, s_2 \log^3 x + 2\, s_4 \log x$$

$$-\frac{1}{\pi}\,\text{Im } S_{1,1}(x) = -\log x$$

$$-\frac{1}{\pi}\,\text{Im } S_{1,2}(x) = -s_2 + S_{1,1}\left(\frac{1}{x}\right) + \frac{1}{2}\log^2 x$$

$$-\frac{1}{\pi}\,\text{Im } S_{2,1}(x) = -\frac{1}{2}\log^2 x$$

$$-\frac{1}{\pi}\,\text{Im } S_{1,3}(x) = S_{1,2}\left(\frac{1}{x}\right) - S_{2,1}\left(\frac{1}{x}\right) - \log x\; S_{1,1}\left(\frac{1}{x}\right) - \frac{1}{6}\log^3 x + s_2 \log x$$

$$-\frac{1}{\pi}\,\text{Im } S_{2,2}(x) = s_3 - S_{2,1}\left(\frac{1}{x}\right) + \frac{1}{6}\log^3 x - s_2 \log x$$

$$-\frac{1}{\pi}\,\text{Im } S_{3,1}(x) = -\frac{1}{6}\log^3 x$$

$$-\frac{1}{\pi}\,\mathrm{Im}\,S_{1,4}(x) = \frac{3}{4}s_4 + S_{1,3}\left(\frac{1}{x}\right) - S_{2,2}\left(\frac{1}{x}\right) + S_{3,1}\left(\frac{1}{x}\right) + $$
$$+ \log x\left[S_{2,1}\left(\frac{1}{x}\right) - S_{1,2}\left(\frac{1}{x}\right)\right] + \left(\frac{1}{2}\log^2 x - s_2\right) S_{1,1}\left(\frac{1}{x}\right) +$$
$$+ \frac{1}{24}\log^4 x - \frac{1}{2} s_2 \log^2 x$$

$$-\frac{1}{\pi}\,\mathrm{Im}\,S_{2,3}(x) = -\frac{7}{4}s_4 + 2\,S_{3,1}\left(\frac{1}{x}\right) - S_{2,2}\left(\frac{1}{x}\right) + \log x\, S_{2,1}\left(\frac{1}{x}\right) -$$
$$- \frac{1}{24}\log^4 x + \frac{1}{2} s_2 \log^2 x$$

$$-\frac{1}{\pi}\,\mathrm{Im}\,S_{3,2}(x) = -s_4 + S_{3,1}\left(\frac{1}{x}\right) + \frac{1}{24}\log^4 x - \frac{1}{2}s_2\log^2 x + s_3 \log x$$

$$-\frac{1}{\pi}\,\mathrm{Im}\,S_{4,1}(x) = -\frac{1}{24}\log^4 x$$

<u>The inversion formulae (3.11) for $x < -1$</u>

$$S_{1,1}(x) = -s_2 - S_{1,1}\left(\frac{1}{x}\right) - \frac{1}{2}\log^2|x|$$

$$S_{1,2}(x) = s_3 - S_{1,2}\left(\frac{1}{x}\right) + S_{2,1}\left(\frac{1}{x}\right) + \log|x|\; S_{1,1}\left(\frac{1}{x}\right) + \frac{1}{6}\log^3|x|$$

$$S_{2,1}(x) = S_{2,1}\left(\frac{1}{x}\right) - \frac{1}{6}\log^3|x| - s_2\log|x|$$

$$S_{1,3}(x) = -s_4 - S_{1,3}\left(\frac{1}{x}\right) + S_{2,2}\left(\frac{1}{x}\right) - S_{3,1}\left(\frac{1}{x}\right) +$$
$$+ \log|x|\left[S_{1,2}\left(\frac{1}{x}\right) - S_{2,1}\left(\frac{1}{x}\right)\right] - \frac{1}{2}\log^2|x|\; S_{1,1}\left(\frac{1}{x}\right) - \frac{1}{24}\log^4|x|$$

$$S_{2,2}(x) = -\frac{7}{4}s_4 + S_{2,2}\left(\frac{1}{x}\right) - 2\,S_{3,1}\left(\frac{1}{x}\right) - \log|x|\; S_{2,1}\left(\frac{1}{x}\right) +$$
$$+ \frac{1}{24}\log^4|x| + s_3\log|x|$$

$$S_{3,1}(x) = -\frac{7}{4}s_4 - S_{3,1}\left(\frac{1}{x}\right) - \frac{1}{24}\log^4|x| - \frac{1}{2}s_2\log^2|x|$$

$$S_{1,4}(x) = s_5 - S_{1,4}\left(\frac{1}{x}\right) + S_{2,3}\left(\frac{1}{x}\right) - S_{3,2}\left(\frac{1}{x}\right) + S_{4,1}\left(\frac{1}{x}\right) +$$
$$+ \log|x| \left[S_{1,3}\left(\frac{1}{x}\right) - S_{2,2}\left(\frac{1}{x}\right) + S_{3,1}\left(\frac{1}{x}\right)\right] +$$
$$+ \frac{1}{2}\log^2|x| \left[S_{2,1}\left(\frac{1}{x}\right) - S_{1,2}\left(\frac{1}{x}\right)\right] + \frac{1}{6}\log^3|x| \; S_{1,1}\left(\frac{1}{x}\right) +$$
$$+ \frac{1}{120}\log^5|x|$$

$$S_{2,3}(x) = s_5 + s_2 s_3 + S_{2,3}\left(\frac{1}{x}\right) - 2\, S_{3,2}\left(\frac{1}{x}\right) + 3\, S_{4,1}\left(\frac{1}{x}\right) +$$
$$+ \log|x| \left[2\, S_{3,1}\left(\frac{1}{x}\right) - S_{2,2}\left(\frac{1}{x}\right)\right] + \frac{1}{2}\log^2|x| \; S_{2,1}\left(\frac{1}{x}\right) -$$
$$- \frac{1}{120}\log^5|x| - s_4 \log|x|$$

$$S_{3,2}(x) = s_5 + s_2 s_3 - S_{3,2}\left(\frac{1}{x}\right) + 3\, S_{4,1}\left(\frac{1}{x}\right) + \log|x| \; S_{3,1}\left(\frac{1}{x}\right) + \frac{1}{120}\log^5|x| +$$
$$+ \frac{1}{2} s_3 \log^2|x| - \frac{7}{4} s_4 \log|x|$$

$$S_{4,1}(x) = S_{4,1}\left(\frac{1}{x}\right) - \frac{1}{120}\log^5|x| - \frac{1}{6} s_2 \log^3|x| - \frac{7}{4} s_4 \log|x|$$

APPENDIX 3

$$
(s_{n,p}) = \begin{pmatrix}
1.64493\ 40668\ 48226 & 1.20205\ 69031\ 59594 & 1.08232\ 32337\ 11138 & 1.03692\ 77551\ 43370 \\
1.20205\ 69031\ 59594 & 2.70580\ 80842\ 77845\ (-1) & 9.65511\ 59989\ 44373\ (-2) & 4.05368\ 97271\ 51974\ (-2) \\
1.08232\ 32337\ 11138 & 9.65511\ 59989\ 44373\ (-2) & 1.74898\ 53169\ 01140\ (-2) & 4.12316\ 51524\ 32536\ (-3) \\
1.03692\ 77551\ 43370 & 4.05368\ 97271\ 51974\ (-2) & 4.12316\ 51524\ 32536\ (-3) & 6.02891\ 53283\ 31914\ (-4)
\end{pmatrix}
$$

$$
(\sigma_{n,p}) = \begin{pmatrix}
8.22467\ 03342\ 41132\ (-1) & 1.50257\ 11289\ 49493\ (-1) & 2.37523\ 66322\ 61849\ (-2) & 3.13500\ 96016\ 80862\ (-3) \\
9.01542\ 67736\ 96957\ (-1) & 8.77856\ 71568\ 65530\ (-2) & 9.60156\ 84431\ 29833\ (-3) & * \\
9.47032\ 82949\ 72459\ (-1) & 4.89363\ 97049\ 96906\ (-2) & * & * \\
9.72119\ 77044\ 69093\ (-1) & * & * & *
\end{pmatrix}
$$

$$
(a_{n,p}) = \begin{pmatrix}
5.82240\ 52646\ 50125\ (-1) & 9.47530\ 04230\ 12771\ (-2) & 1.41342\ 37214\ 99001\ (-2) & 1.80165\ 37870\ 38018\ (-3) \\
5.37213\ 19360\ 80402\ (-1) & 4.07582\ 39159\ 30925\ (-2) & 3.87606\ 73146\ 65264\ (-3) & * \\
5.17479\ 06167\ 38994\ (-1) & 1.85307\ 86065\ 46661\ (-2) & * & * \\
5.08400\ 57924\ 22687\ (-1) & * & * & *
\end{pmatrix}
$$

REFERENCES

1) N. Nielsen, Der Eulersche Dilogarithmus und seine Verallgemeinerungen, Nova Acta, Abh.d.Kaiserl.Leop.-Carol. Deutschen Akademie d. Naturforscher 90 (1909) 123-211.

2) J.A. Mignaco and E. Remiddi, Fourth order vacuum polarization contribution to the 6th order electron magnetic moment, Nuovo Cim. 60A (1969) 519-528.

3) R. Karplus and N.M. Kroll, Fourth-order corrections in quantum electrodynamics and the magnetic moment of the electron, Phys.Rev. 77 (1950) 536-549.

4) G. Källén and A. Sabry, Fourth order vacuum polarization, K.Danske Vidensk. Selsk.Mat.-Fys.Medd. 29 (1955) 1-20.

5) A. Petermann, Fourth order magnetic moment of the electron, Helv.Phys.Acta. 30 (1957) 407-408.

6) L. Lewin, Dilogarithms and associated functions, MacDonald London (1958).

7) A. Fletcher, J.C.P. Miller, L. Rosenhead and L.J. Comrie, An index of mathematical tables, Vol. 1, Blackwell Scientific Publications, Oxford (1962).

8) M. Abramowitz and I.A. Stegun (Edit.), Handbook of mathematical functions, Chapts. 24 and 27, NBS Appl.Math.Ser. 55, U.S. Government Printing Office, Washington, D.C. (1965).

9) R. Karplus and M. Neuman, The scattering of light by light, Phys.Rev. 83 (1951) 776-784.

10) M.S.P. Eastham, On polylogarithms, Proc. Glasgow Math.Assoc. 6 (1964) 169-171.

11) W. Maier and H. Zahn, Über die Funktionalgleichung 2. Stufe des Trilogarithmus, Wiss.Z. Friedrich Schiller Univ. Jena 11 (1962) 109-110.

12) W. Magnus, F. Oberhettinger and R.P. Soni, Formulas and theorems for the special functions of mathematical physics, 3rd Edn., Springer-Verlag, Berlin-Heidelberg-New York (1966).

13) A. Erdélyi (Ed.), Higher transcendental functions, Vol. I, McGraw-Hill, New York-Toronto-London (1953).

14) K. Mitchell, Tables of the function $\int_0^z (-\log|1-y|/y)\, dy$, with an account of some properties of this and related functions, Phil.Mag. 40 (1949) 351-368.

15) A. Jonquière, Über eine Klasse von Transzendenten, welche durch mehrmalige Integration rationaler Funktionen entstehen, Öfversigter of the Stockholm Academy (1888) 522-531.

16) A. Jonquière, Über einige Transzendenten, welche bei der wiederholten Integration rationaler Funktionen auftreten, Bihang of the Stockholm Academy 15 (1889) 1-50.

17) W. Rühl, private communication.

18) G. Wechsung, Über Kummers Funktionalgleichung für den Pentalogarithmus, Jber.Deutsch.Math. Verein 68 (1966) 140-143.

19) K.S. Kölbig, Algorithm 327, Dilogarithm, Comm. ACM 11 (1968) 270.

20) T. Håvie, CHECOF - double precision calculation of the coefficients in a Chebyshev expansion, CERN 6600 Series Program Library C320 (1968), unpublished.

21) K.S. Kölbig, DGAUSS - double precision Gaussian integration, CERN 6600 Series Program Library D109 (1968), unpublished.

22) C.W. Clenshaw, G.F. Miller and M. Woodger, Algorithms for special functions I, Handbook Series Special Functions, Numer.Math. 4 (1963) 403-419.

23) Mathematisch Centrum Amsterdam, Polylogarithms, Report R24 (1954).

24) R.E. Shafer, Tables of polylogarithms etc., M.S. Lawrence Radiation Lab., University of California.

25) I.M. Ryshik and I.S. Gradstein, Tafeln-Tables, VEB Deutscher Verlag d. Wissensch. Berlin 1957.

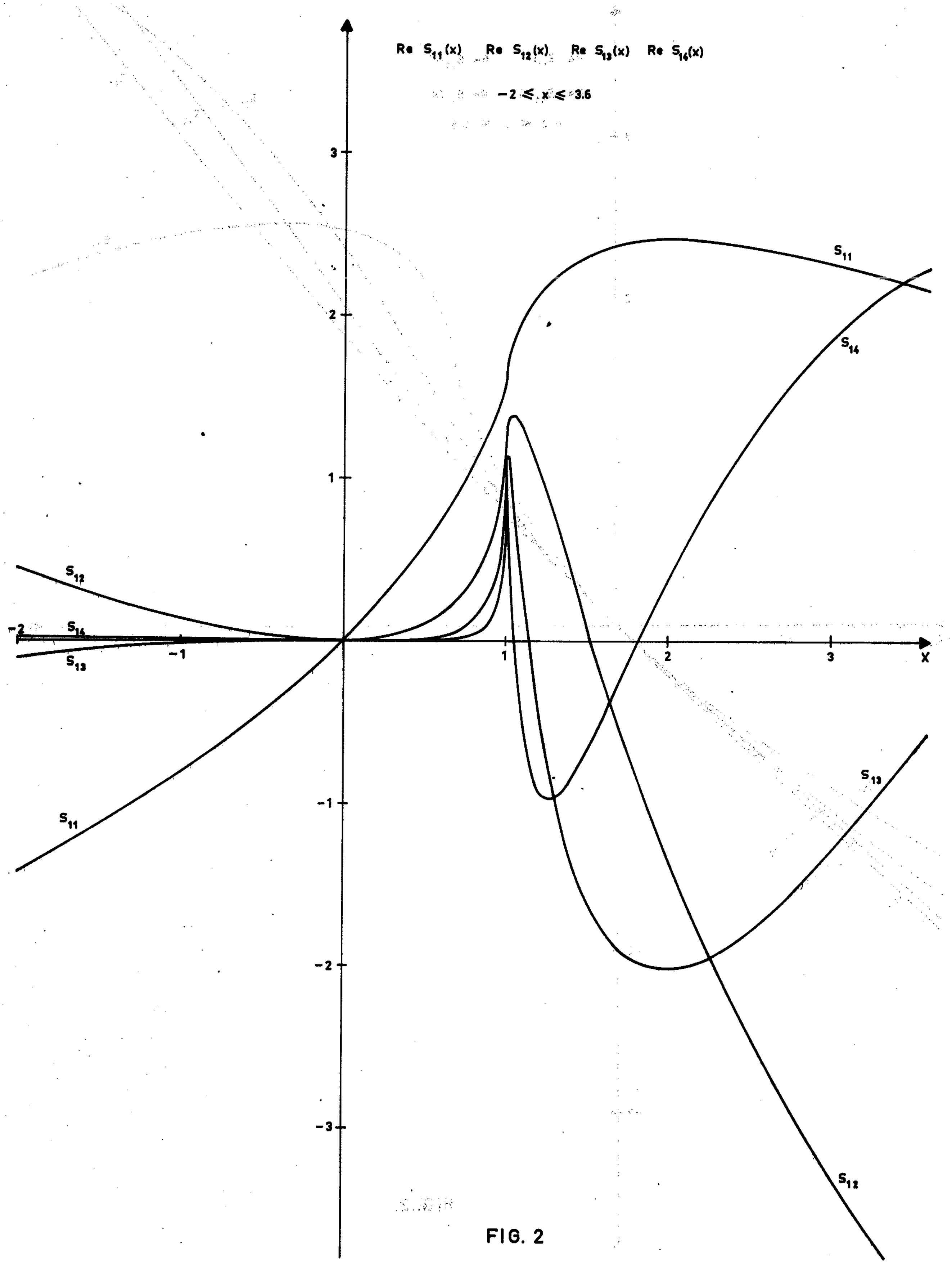

FIG. 2

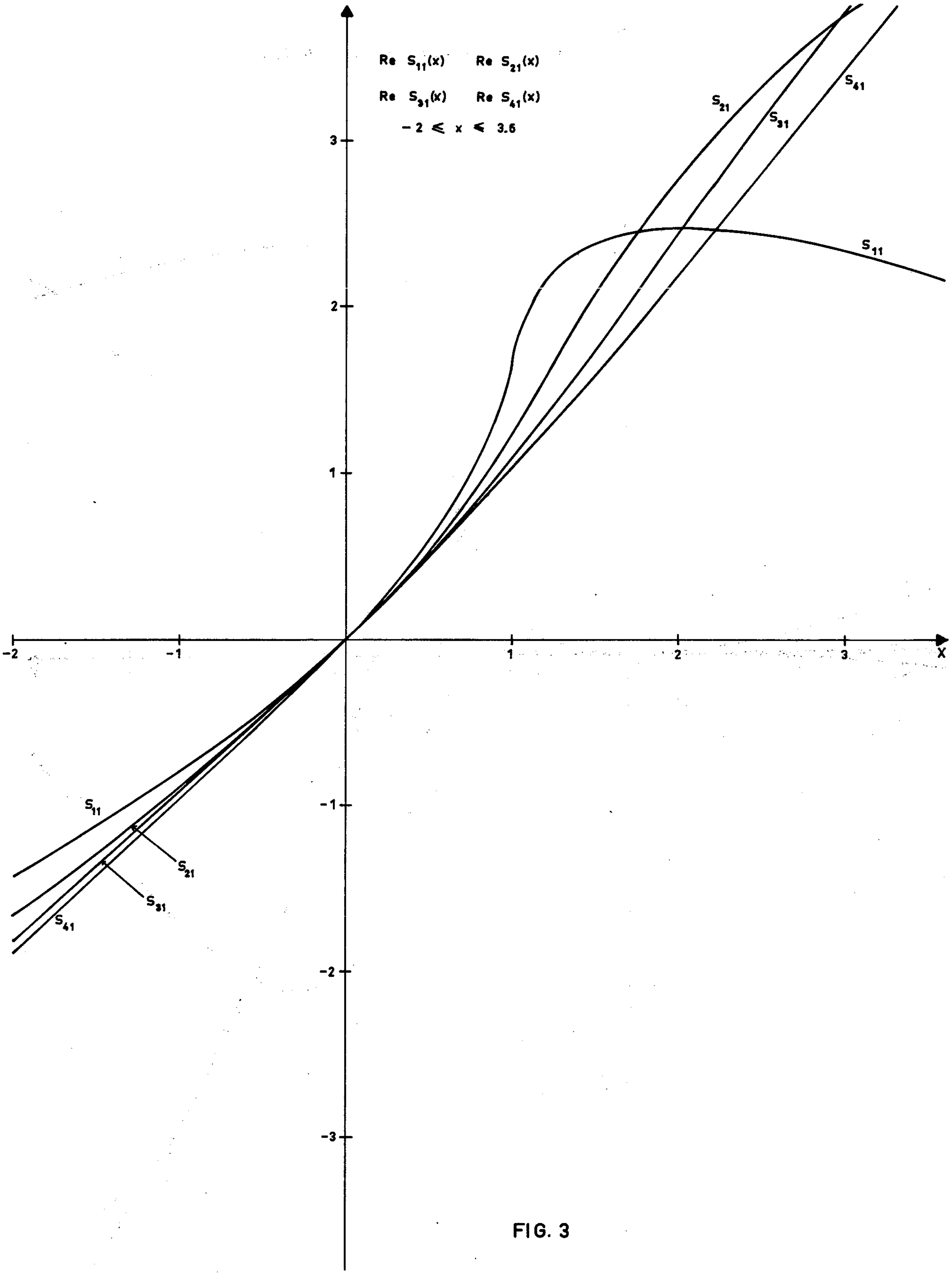
Re $S_{11}(x)$ Re $S_{21}(x)$
Re $S_{31}(x)$ Re $S_{41}(x)$
$-2 \leq x \leq 3.6$
3
2
1
-1
-2
-3
-2
-1
1
2
3
X
S_{21}
S_{31}
S_{41}
S_{11}
S_{11}
S_{21}
S_{31}
S_{41}

FIG. 3

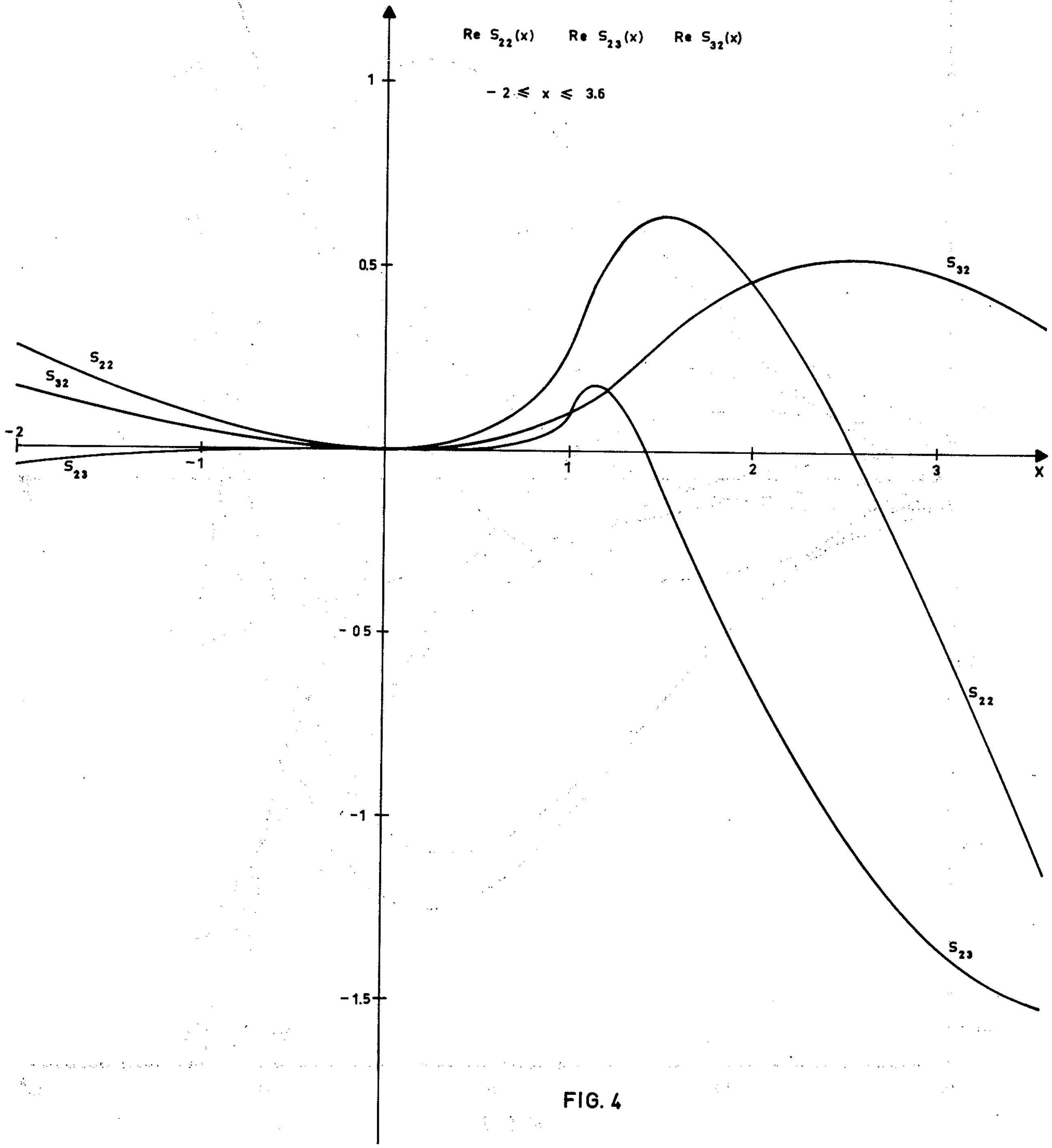
Re $S_{22}(x)$ Re $S_{23}(x)$ Re $S_{32}(x)$
$-2 \leq x \leq 3.6$
1
0.5
-0.5
-1
-1.5
-2
-1
1
2
3
X
S_{22}
S_{32}
S_{23}
S_{32}
S_{22}
S_{23}

FIG. 4

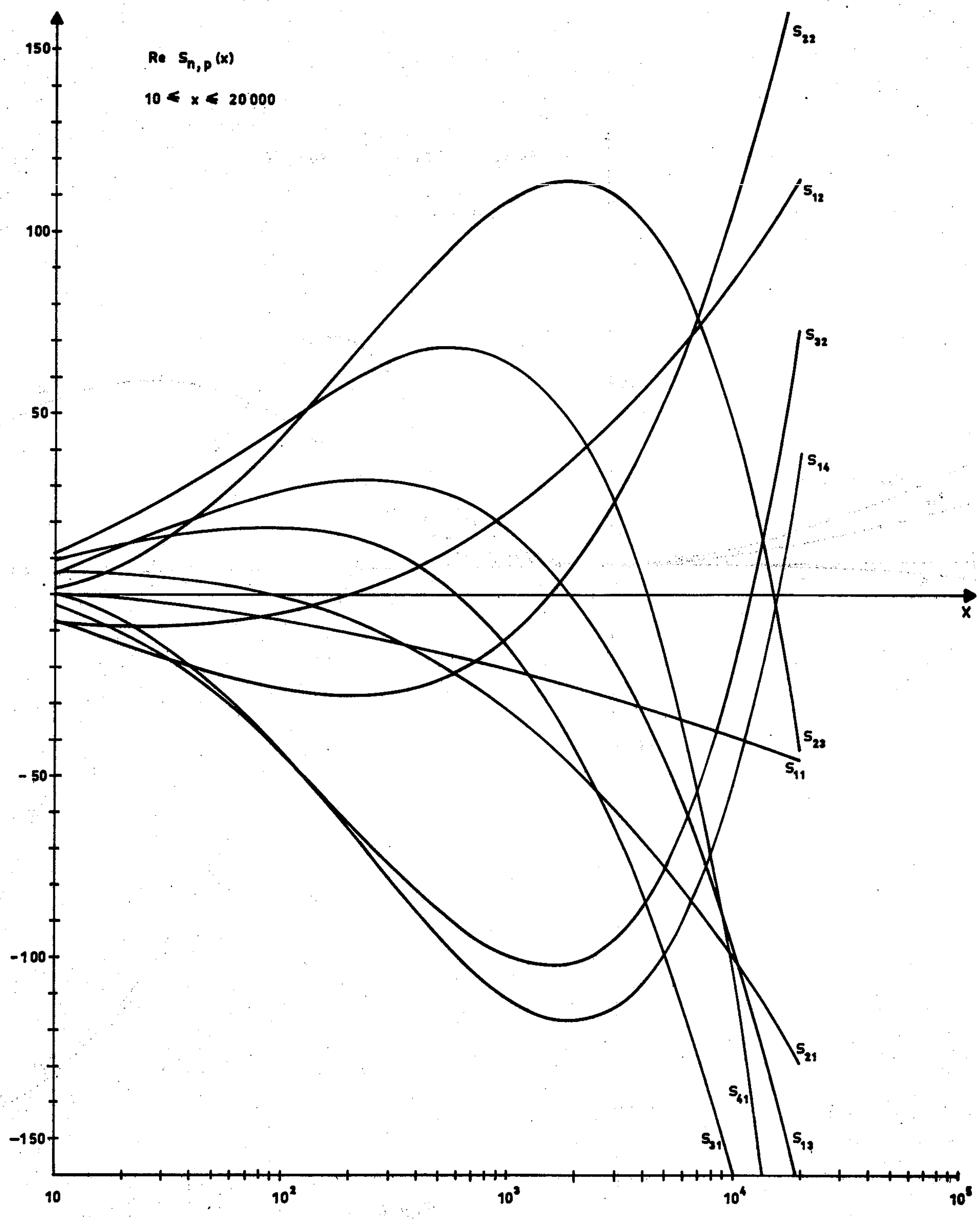
Re $S_{n,p}(x)$
$10 \leq x \leq 20000$
150
100
50
- 50
-100
-150
10
10^2
10^3
10^4
10^5
X
S_{22}
S_{12}
S_{32}
S_{14}
S_{23}
S_{11}
S_{21}
S_{41}
S_{31}
S_{13}

FIG. 5