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On noetherian subrings of an affine domain

Dedicated to Professor Yoshikazu Nakai on his sixtieth birthday

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Introduction

Let A be an affine domain over a field k and let R be a subring of A containing k. It is well-known that R is again an affine domain over k when $\dim R=1$. But, when $\dim R \geqq 2$, R is not necessarily noetherian, and not necessarily an affine domain over k even when R is noetherian. The purpose of the present paper is to find certain conditions for R to be an affine domain over k.

In the first section we define an ideal $\mathscr{A}(R)$ of R and by making use of this ideal we prove that R is an affine domain over k if and only if $R_{\mathfrak{m}}$ is a locality over k for any maximal ideal $\mathfrak{m}$ of R, where a locality over k is a local ring which is a localization of an affine domain over k (cf. Theorem 1.6).

It is known that R is an affine domain over k if R is a neotherian normal subring of dimension 2 and $\mathrm{tr.deg}_k R/\mathfrak{p}=1$ for any prime ideal $\mathfrak{p}$ of height 1 (cf. [2], [5]). If R satisfies these conditions it is seen that R is equidimensional, that is, we have $\dim R_{\mathfrak{m}}=2$ for any maximal ideal $\mathfrak{m}$ of R. In the third section we generalize this result as follows: if R is a noetherian subring of an affine domain over a field k such that the integral closure R' of R in its quotient field is equidimensional then R is an affine domain over k (cf. Theorem 3.2).

As a corollary of this theorem we prove that if R is a universally catenary and equidimensional subring then R is an affine domain over k. For the proof of this corollary we need the following theorem: the finiteness of the integral closure R' of R in its quotient field is a local property, that is, R' is a finite R-module if and only if $R'_{\mathfrak{m}}$ is a finite $R_{\mathfrak{m}}$-module for any maximal ideal $\mathfrak{m}$ of R. We prove this theorem in the second section (cf. Theorem 2.5).

Throughout this paper we fix a field k. All rings under consideration are commutative k-algebras and all affine domains are assumed to be defined over k.

1. The ideal $\mathscr{A}(R)$

Let A be an affine domain over a field k, that is, an integral domain which is finitely generated over k. We are mainly interested in subrings R of A, and we shall study when R is again an affine domain over k. For this purpose we define an ideal $\mathscr{A}(R)$ of R as follows:

PROPOSITION 1.1. *Let R be a subring of an affine domain A over k. Define $\mathscr{A}(R)$ by*

$$\mathscr{A}(R):=\{a;\, a\in R \text{ and } R[1/a] \text{ is an affine domain over } k\}\cup\{0\}.$$

Then $\mathscr{A}(R)$ is a non-zero radical ideal of R.

PROOF. By virtue of Propositon (2.1) in [3], we see that $\mathscr{A}(R)\neq 0$. We shall prove that $\mathscr{A}(R)$ is an ideal of R. Since $R[1/ax]=R[1/a][1/x]$, we have $ax\in\mathscr{A}(R)$ for any $a\in\mathscr{A}(R)$ and $x\in R$. We shall show $a+b\in\mathscr{A}(R)$ for any non-zero elelments a and b of $\mathscr{A}(R)$. Since $R[1/a]$ and $R[1/b]$ are affine domains over k, there exist elements $a_1,\dots, a_s$ and $b_1,\dots, b_t$ of R such that $R[1/a]=k[1/a, a_1,\dots, a_s]$ and $R[1/b]=k[1/b, b_1,\dots, b_t]$. Let $C=k[a, b, a_1,\dots, a_s, b_1,\dots, b_t]$. Then C is an affine domain over k and we have $C\subseteq R$. Let x be an arbitrary element of R. Since $R[1/a]\subseteq C[1/a]$ and $R[1/b]\subseteq C[1/b]$, we have $a^n x\in C$ and $b^n x\in C$ for a sufficiently large positive integer n. Then, as is easily seen, we have $(a+b)^{2n}x\in C$, whence we have $x\in C[1/(a+b)]$. Since x is an arbitrary element of R, we have $C[1/(a+b)]\supseteq R[1/(a+b)]$ and hence $C[1/(a+b)]=R[1/(a+b)]$. Therefore $R[1/(a+b)]$ is an affine domain over k and we have $a+b\in\mathscr{A}(R)$. Thus $\mathscr{A}(R)$ is an ideal of R. Finally we prove that $\mathscr{A}(R)$ is a radical ideal. Let x be an element of R with $x^n\in\mathscr{A}(R)$ for some positive integer n. Since $R[1/x]=R[1/x^n]$, we have $x\in\mathscr{A}(R)$. Therefore $\mathscr{A}(R)$ is a radical ideal. Q.E.D.

COROLLARY 1.2. *Let R be a subring of an affine domain over k. Then we have* $\dim R=\mathrm{tr.deg}_k R$.

PROOF. Let $n=\mathrm{tr.deg}_k R$ and let a be a non-zero element of $\mathscr{A}(R)$. Then $R[1/a]$ is an affine domain over k and hence we have $\dim R[1/a]=n$ (cf. [4, (14.G)]). Therefore we have $\dim R\geqq\dim R[1/a]=n$. Since $\dim R\leqq n$ in general, we have $\dim R=n$. Q.E.D.

We call a local ring S a locality over k if S is a localization of an affine domain over k with respect to a prime ideal (cf. [6, Ch. VI]).

LEMMA 1.3. *Let R be a subring of an affine domain A over k and let $\mathfrak{p}$ be a prime ideal of R. Then $R_\mathfrak{p}$ is a locality over k if and only if $\mathscr{A}(R)\not\subseteq\mathfrak{p}$.*

PROOF. Let x be a non-zero element of $\mathscr{A}(R)$. Replacing A by $R[1/x]$, we may assume that R and A are birational. Assume that $\mathscr{A}(R)\not\subseteq\mathfrak{p}$ and take an element a of $\mathscr{A}(R)\backslash\mathfrak{p}$. Then $R[1/a]$ is an affine domain over k and $\mathfrak{p}[1/a]$ is a prime ideal of $R[1/a]$. Hence $R_\mathfrak{p}=R[1/a]_{\mathfrak{p}[1/a]}$ is a locality over k. Conversely, assume that $R_\mathfrak{p}$ is a locality over k. Then there exist an affine domain B over k and a prime ideal P of B such that $R_\mathfrak{p}=B_P$. We may assume that $B\subseteq R$. Let K

be the quotient field of A and let B' and R' be the integral closures of B and R in K, respectively. Let $\mathfrak{p}'$ be a prime ideal of R' lying over $\mathfrak{p}$ and let $P'=\mathfrak{p}'\cap B'$. Since the integral closure of $R_{\mathfrak{p}}=B_P$ in K coincides with $R'_{\mathfrak{p}}=B'_P$, we have $R'_{\mathfrak{p}'}=B'_{P'}$. Let $F=\{Q;\ Q\in \operatorname{Spec} B',\ \operatorname{ht}Q=1 \text{ and } B'_Q\not\supseteq R'\}$. We claim that F is a finite set. In fact, since A is an affine domain, the integral closure A' of A in K is also an affine domain. Therefore A' is finitely generated over B', whence there exists an element b of B' such that $B'[1/b]\supseteq A'\supseteq R'$. Then it is easy to see that F is a subset of the set of the minimal prime divisors of b. Thus F is a finite set. Let $F=\{P'_1,\dots,P'_n\}$ and let $P_i=P'_i\cap B$ for $1\leqq i\leqq n$. Suppose that $P_1\cap\cdots\cap P_n\subseteq P$. Then we have $P_i\subseteq P$ for some i, and hence $B'_{P_i}\supseteq B'_P=R'_{\mathfrak{p}}\supseteq R'$, which is a contradiction. Thus we have $P_1\cap\cdots\cap P_n\not\subseteq P$. Take an element a of $P_1\cap\cdots\cap P_n\backslash P$ and let $\Lambda=\{Q;\ Q\in\operatorname{Spec}B',\ \operatorname{ht}Q=1 \text{ and } a\notin Q\}$. Then, as is easily seen, we have $B'_Q\supseteq R'$ for any element Q of Λ, whence we have $B'[1/a]=\bigcap_{Q\in\Lambda}B'_Q\supseteq R'$. Therefore we have $B'[1/a]=R'[1/a]$, and $R'[1/a]$ is an affine domain over k. Hence $R[1/a]$ is also an affine domain over k by the following well-known lemma. Since $a\in B\backslash P\subseteq R\backslash\mathfrak{p}$, we have $\mathscr{A}(R)\not\subseteq\mathfrak{p}$. Q.E.D.

LEMMA 1.4. *Let a ring R' be an integral extension of a ring R. If R' is an affine domain over k, then R is also an affine domain over k.*

PROOF. See [1, Ch. V, § 1.9, Lemma 5].

COROLLARY 1.5. *Let R be a subring of an affine domain over k. Then we have* $\mathrm{V}(\mathscr{A}(R))=\{\mathfrak{p};\ \mathfrak{p}\in\operatorname{Spec}R$ *and $R_{\mathfrak{p}}$ is not a locality over k*$\}$.

The following theorem is an immediate cosequence of Corollary 1.5 which asserts that affineness is a local property for subrings of an affine domain.

THEOREM 1.6. *Let R be a subring of an affine domain over k. Then the following three conditions are equivalent to each other.*

(1) *R is an affine domain over k.*

(2) *$R_{\mathfrak{p}}$ is a locality over k for any prime ideal $\mathfrak{p}$ of R.*

(3) *$R_{\mathfrak{m}}$ is a locality over k for any maximal ideal $\mathfrak{m}$ of R.*

2. Open properties of a ring

Let $\boldsymbol{P}$ be a property for domains. For the sake of brevity, we use the symbol $[\boldsymbol{P}]$ to denote the class of domains which have the property $\boldsymbol{P}$. We say that a property $\boldsymbol{P}$ is stable under localization if a domain R belongs to $[\boldsymbol{P}]$ then $R_{\mathfrak{p}}$ belongs to $[\boldsymbol{P}]$ for any prime ideal $\mathfrak{p}$ of R. For a domain R, we define

$$\boldsymbol{P}(R)=\{a;\ a\in R \text{ and } R[1/a]\in[\boldsymbol{P}]\}\cup\{0\}$$

and

$$\Delta_{\boldsymbol{P}}(R) = \{\mathfrak{p}\,;\ \mathfrak{p} \in \operatorname{Spec} R \text{ and } R_{\mathfrak{p}} \notin [\boldsymbol{P}]\}.$$

We say that $\boldsymbol{P}$ is an open property for R if $\Delta_{\boldsymbol{P}}(R)$ is a closed set of Spec R.

LEMMA 2.1. *Let* $\boldsymbol{P}$ *be a property stable under localization and let* R *be a domain. Then*

(1) *If* $\boldsymbol{P}(R)$ *is an ideal of* R *then* $\boldsymbol{P}(R)$ *is a radical ideal of* R.

(2) $\Delta_{\boldsymbol{P}}(R) \subseteq \mathrm{V}(\boldsymbol{P}(R))$.

(3) *If* $\mathfrak{p} \in \Delta_{\boldsymbol{P}}(R)$ *then* $\mathfrak{q} \in \Delta_{\boldsymbol{P}}(R)$ *for any specialization* $\mathfrak{q}$ *of* $\mathfrak{p}$.

PROOF. (1): Since $R[1/a]=R[1/a^n]$, we have $a \in \boldsymbol{P}(R)$ if and only if $a^n \in \boldsymbol{P}(R)$. Hence the assertion is obvious.

(2): Let $\mathfrak{p}$ be a prime ideal of R with $\mathfrak{p} \not\supseteq \boldsymbol{P}(R)$ and let a be an element of $\boldsymbol{P}(R)\backslash\mathfrak{p}$. Then $R[1/a] \in [\boldsymbol{P}]$ and $\mathfrak{p}[1/a] \in \operatorname{Spec} R[1/a]$, hence we have $R_{\mathfrak{p}} = R[1/a]_{\mathfrak{p}[1/a]} \in [\boldsymbol{P}]$ because $\boldsymbol{P}$ is stable under localization. Thus we have $\mathfrak{p} \notin \Delta_{\boldsymbol{P}}(R)$.

(3): Let $\mathfrak{p} \in \Delta_{\boldsymbol{P}}(R)$ and let $\mathfrak{q}$ be a specialization of $\mathfrak{p}$. Suppose that $\mathfrak{q} \notin \Delta_{\boldsymbol{P}}(R)$. Then we have $R_{\mathfrak{q}} \in [\boldsymbol{P}]$, and hence we have $R_{\mathfrak{p}} \in [\boldsymbol{P}]$ because $R_{\mathfrak{p}}$ is a localization of $R_{\mathfrak{q}}$ with respect to the prime ideal $\mathfrak{p}R_{\mathfrak{q}}$. Thus we have $\mathfrak{p} \notin \Delta_{\boldsymbol{P}}(R)$, which is a contradiction. Therefore we have $\mathfrak{q} \in \Delta_{\boldsymbol{P}}(R)$. Q.E.D.

LEMMA 2.2. *Let* $\boldsymbol{P}$ *be a property stable under localization and let* R *be a domain. Then the following two conditions are equivalent to each other.*

(1) $\Delta_{\boldsymbol{P}}(R) = \mathrm{V}(\boldsymbol{P}(R))$.

(2) *If* $\mathfrak{p}$ *is a prime ideal of* R *with* $R_{\mathfrak{p}} \in [\boldsymbol{P}]$, *then* $\boldsymbol{P}(R) \not\subseteq \mathfrak{p}$.

PROOF. (1)⇒(2): Let $\mathfrak{p}$ be a prime ideal of R with $R_{\mathfrak{p}} \in [\boldsymbol{P}]$. Then we have $\mathfrak{p} \notin \Delta_{\boldsymbol{P}}(R)$ by the definition. Since $\Delta_{\boldsymbol{P}}(R) = \mathrm{V}(\boldsymbol{P}(R))$, we have $\boldsymbol{P}(R) \not\subseteq \mathfrak{p}$.

(2) ⇒(1): If the condition (2) holds, we have $R_{\mathfrak{p}} \notin [\boldsymbol{P}]$, i.e., $\mathfrak{p} \in \Delta_{\boldsymbol{P}}(R)$ for any $\mathfrak{p} \in \mathrm{V}(\boldsymbol{P}(R))$. Thus we have $\mathrm{V}(\boldsymbol{P}(R)) \subseteq \Delta_{\boldsymbol{P}}(R)$. On the other hand, we have $\mathrm{V}(\boldsymbol{P}(R)) \supseteq \Delta_{\boldsymbol{P}}(R)$ by Lemma 2.1, whence $\Delta_{\boldsymbol{P}}(R) = \mathrm{V}(\boldsymbol{P}(R))$. Q.E.D.

COROLLARY 2.3. *Let* $\boldsymbol{P}$ *be a property stable under localization and let* R *be a domain. Assume that* R *satisfies the following three conditions.*

(1) $\boldsymbol{P}(R)$ *is an ideal of* R.

(2) $\Delta_{\boldsymbol{P}}(R) = \mathrm{V}(\boldsymbol{P}(R))$.

(3) $R_{\mathfrak{m}} \in [\boldsymbol{P}]$ *for any maximal ideal* $\mathfrak{m}$ *of* R.

Then R *has the property* $\boldsymbol{P}$.

PROOF. We have $\boldsymbol{P}(R) \not\subseteq \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R by Lemma 2.2. Since $\boldsymbol{P}(R)$ is an ideal of R, we have $\boldsymbol{P}(R) = R$, hence $R \in [\boldsymbol{P}]$. Q.E.D.

We shall denote by $\boldsymbol{I}$ and $\boldsymbol{F}$ the properties for domains defined by

$R \in [\boldsymbol{I}]$ implies that R is integrally closed.

$R \in [\boldsymbol{F}]$ implies that the integral closure R' of R in its quotient field is a finite R-module.

PROPOSITION 2.4. *Let R be a noetherian subring of an affine domain over k and let $\boldsymbol{P}$ be either $\boldsymbol{I}$ or $\boldsymbol{F}$.*

Then we have the following:

(1) *$\boldsymbol{P}(R)$ is a non-zero radical ideal of R.*

(2) $\Delta_{\boldsymbol{P}}(R) = \mathrm{V}(\boldsymbol{P}(R))$.

PROOF. (I) The case $\boldsymbol{P}=\boldsymbol{I}$.

(1): Let R' be the integral closure of R in the quotient field K of R and let a be a non-zero element of $\mathscr{A}(R)$. Since $R[1/a]$ is an affine domain, the integral closure $R'[1/a]$ of $R[1/a]$ in K is a finite $R[1/a]$-module. Hence there exist elements $\alpha_1, \dots, \alpha_s$ of R' such that $R'[1/a] = R[1/a]\alpha_1 + \cdots + R[1/a]\alpha_s$. Take an element b of R such that $b\alpha_i \in R$ for $1 \leqq i \leqq s$. Then we have $R'[1/a] \subseteq R[1/a][1/b]$, whence we have $R'[1/ab] = R[1/ab]$. Thus $R[1/ab]$ is integrally closed, and hence we have $0 \neq ab \in \boldsymbol{I}(R)$. Therefore $\boldsymbol{I}(R) \neq 0$. Next we shall prove that $\boldsymbol{I}(R)$ is a radical ideal of R. It is easy to see that we have $ar \in \boldsymbol{I}(R)$ for any $a \in \boldsymbol{I}(R)$ and $r \in R$. Let a and b be two non-zero elements of $\boldsymbol{I}(R)$ and let $\mathfrak{p}$ be an element of $\mathrm{D}(a+b)$, where $\mathrm{D}(a+b) = \{\mathfrak{p}\,;\ \mathfrak{p} \in \operatorname{Spec} R$ and $a+b \notin \mathfrak{p}\}$. Then we have either $a \notin \mathfrak{p}$ or $b \notin \mathfrak{p}$, and we may assume that $a \notin \mathfrak{p}$. Then $R[1/a]$ is integrally closed and $\mathfrak{p}[1/a]$ is a prime ideal of $R[1/a]$, hence $R_{\mathfrak{p}} = R[1/a]_{\mathfrak{p}[1/a]}$ is integrally closed. Since $R[1/(a+b)] = \bigcap_{\mathfrak{p} \in \mathrm{D}(a+b)} R_{\mathfrak{p}}$, we see that $R[1/(a+b)]$ is integrally closed, i.e., $a+b \in \boldsymbol{I}(R)$. Therefore $\boldsymbol{I}(R)$ is an ideal of R, and hence $\boldsymbol{I}(R)$ is a radical ideal of R by Lemma 2.1.

(2): Let $\mathfrak{p}$ be a minimal element of $\Delta_{\boldsymbol{I}}(R)$. We claim that $\operatorname{depth} R_{\mathfrak{p}} = 1$. In fact, suppose that $\operatorname{depth} R_{\mathfrak{p}} > 1$ and let $\Lambda = \{\mathfrak{q}\,;\ \mathfrak{q} \in \operatorname{Spec} R,\ \operatorname{depth} R_{\mathfrak{q}} = 1$ and $\mathfrak{q} \subseteq \mathfrak{p}\}$. Then, for any element $\mathfrak{q}$ of Λ, we have $\mathfrak{q} \notin \Delta_{\boldsymbol{I}}(R)$, i.e., $R_{\mathfrak{q}}$ is integrally closed because $\mathfrak{p}$ is a minimal element of $\Delta_{\boldsymbol{I}}(R)$. Since $R_{\mathfrak{p}} = \bigcap_{\mathfrak{q} \in \Lambda} R_{\mathfrak{q}}$, we see that $R_{\mathfrak{p}}$ is integrally closed. Hence we have $\mathfrak{p} \notin \Delta_{\boldsymbol{I}}(R)$, which is a contradiction. By Lemma 2.1, we have $\Delta_{\boldsymbol{I}}(R) \subseteq \mathrm{V}(\boldsymbol{I}(R))$, hence we have $\mathfrak{p} \supseteq \boldsymbol{I}(R)$. Let a be a non-zero element of $\boldsymbol{I}(R)$. Since $\operatorname{depth} R_{\mathfrak{p}} = 1$ and $a \in \mathfrak{p}$, $\mathfrak{p}$ is a prime divisor of aR. Therefore we see that the number of the minimal elements of $\Delta_{\boldsymbol{I}}(R)$ is finite because R is noetherian. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be all the minimal elements of $\Delta_{\boldsymbol{I}}(R)$ and let $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$. Then we have $\Delta_{\boldsymbol{I}}(R) = \mathrm{V}(\mathfrak{a})$ by Lemma 2.1. We shall show that $\mathfrak{a} = \boldsymbol{I}(R)$. Since $\mathrm{V}(\mathfrak{a}) = \Delta_{\boldsymbol{I}}(R) \subseteq \mathrm{V}(\boldsymbol{I}(R))$ and $\mathfrak{a}$ is a radical ideal, we have $\boldsymbol{I}(R) \subseteq \mathfrak{a}$. Conversely, let x be an element of $\mathfrak{a}$ and let $\mathfrak{p}$ be an element of $\mathrm{D}(x)$, where $\mathrm{D}(x) = \{\mathfrak{p}\,;\ \mathfrak{p} \in \operatorname{Spec} R$ and $x \notin \mathfrak{p}\}$. Then we have $\mathfrak{p} \not\supseteq \mathfrak{a}$ and hence $\mathfrak{p} \notin \mathrm{V}(\mathfrak{a}) = \Delta_{\boldsymbol{I}}(R)$, i.e., $R_{\mathfrak{p}}$ is integrally closed. Since $R[1/x] = \bigcap_{\mathfrak{p} \in \mathrm{D}(x)} R_{\mathfrak{p}}$, we see that $R[1/x]$ is integrally closed, whence we have $x \in \boldsymbol{I}(R)$. Thus we have $\mathfrak{a} \subseteq \boldsymbol{I}(R)$, and hence $\mathfrak{a} = \boldsymbol{I}(R)$.

(II) The case $\boldsymbol{P}=\boldsymbol{F}$.

(1): Since $\boldsymbol{I}(R)\subseteq\boldsymbol{F}(R)$ and $\boldsymbol{I}(R)\neq 0$, we have $\boldsymbol{F}(R)\neq 0$. We shall prove that $\boldsymbol{F}(R)$ is a radical ideal of R. It is easy to see that we have $ar\in\boldsymbol{F}(R)$ for any $a\in\boldsymbol{F}(R)$ and $r\in R$. Let a and b be two non-zero elements of $\boldsymbol{F}(R)$. Then there exist elements $\alpha_1,\dots,\alpha_s$ and $\beta_1,\dots,\beta_t$ of R' such that $R'[1/a]=R[1/a]\alpha_1+\cdots+R[1/a]\alpha_s$ and $R'[1/b]=R[1/b]\beta_1+\cdots+R[1/b]\beta_t$. Let $B=R[\alpha_1,\dots,\alpha_s,\beta_1,\dots,\beta_t]$. Then B is a finite R-module, and we have $R'[1/a]\subseteq B[1/a]$ and $R'[1/b]\subseteq B[1/b]$. Hence, for any element x of R', there exists a positive integer n such that $a^n x\in B$ and $b^n x\in B$. Then we have $(a+b)^{2n}x\in B$, whence $x\in B[1/(a+b)]$. Therefore we have $R'[1/(a+b)]=B[1/(a+b)]$, and hence $R'[1/(a+b)]$ is a finite $R[1/(a+b)]$-module, i.e., $a+b\in\boldsymbol{F}(R)$. Thus $\boldsymbol{F}(R)$ is an ideal of R, and $\boldsymbol{F}(R)$ is a radical ideal by Lemma 2.1.

(2): By Lemma 2.1, it suffices to show that $\mathrm{V}(\boldsymbol{F}(R))\subseteq\Delta_{\boldsymbol{F}}(R)$. Let $\mathfrak{p}$ be an element of $\mathrm{V}(\boldsymbol{F}(R))$ and suppose that $\mathfrak{p}\notin\Delta_{\boldsymbol{F}}(R)$. Then we have $R_{\mathfrak{p}}\in[\boldsymbol{F}]$, hence the integral closure $R'_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$ in its quotient field is a finite $R_{\mathfrak{p}}$-module. Thus there exist elements $\alpha_1,\dots,\alpha_r$ of R' such that $R'_{\mathfrak{p}}=R_{\mathfrak{p}}\alpha_1+\cdots+R_{\mathfrak{p}}\alpha_r$. Let $C=R[\alpha_1,\dots,\alpha_r]$. Then C is a finite R-module and we have $R'_{\mathfrak{p}}=C_{\mathfrak{p}}$. Let $P_1,\dots,P_n\in\operatorname{Spec}C$ be all the minimal elements of $\Delta_{\boldsymbol{I}}(C)$, and let $\mathfrak{p}_i=P_i\cap R$ for $1\leqq i\leqq n$. We shall show that $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n\not\subseteq\mathfrak{p}$. In fact, if $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n\subseteq\mathfrak{p}$ then we have $\mathfrak{p}_i\subseteq\mathfrak{p}$ for some i. Since $P_i\cap R=\mathfrak{p}_i\subseteq\mathfrak{p}$ and $C_{\mathfrak{p}}=R'_{\mathfrak{p}}$ is integrally closed, we see that $C_{\mathfrak{p}_i}$ is also integrally closed. Hence we have $P_i\notin\Delta_{\boldsymbol{I}}(C)$, which is a contradiction. Let x be an element of $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n\backslash\mathfrak{p}$, and let P be a prime ideal of C with $x\notin P$. Then we have $P_1\cap\cdots\cap P_n\not\subseteq P$, whence we have $P\notin\Delta_{\boldsymbol{I}}(C)$ because $P_1,\dots,P_n$ are all the minimal elements of $\Delta_{\boldsymbol{I}}(C)$. Thus C_P is integrally closed, and hence we see that $C[1/x]$ is also integrally closed. Therefore the integral closure of $R[1/x]$ in its quotient field coincides with $C[1/x]$. Since $C[1/x]$ is a finite $R[1/x]$-module, we have $x\in\boldsymbol{F}(R)$, whence $\boldsymbol{F}(R)\not\subseteq\mathfrak{p}$. Thus we have $\mathfrak{p}\notin\mathrm{V}(\boldsymbol{F}(R))$, which is a contradiction. Hence we have $\mathfrak{p}\in\Delta_{\boldsymbol{F}}(R)$, and $\mathrm{V}(\boldsymbol{F}(R))\subseteq\Delta_{\boldsymbol{F}}(R)$. Q.E.D.

By virtue of Proposition 2.4 and Corollary 2.3, we have the following:

THEOREM 2.5. *Let R be a noetherian subring of an affine domain over k and let R' be the integral closure of R in its quotient field. Then R' is a finite R-module if and only if $R'_{\mathfrak{m}}$ is a finite $R_{\mathfrak{m}}$-module for any maximal ideal $\mathfrak{m}$ of R.*

3. Affineness of noetherian subrings of an affine domain

In this section we shall prove that a noetherian subring R of an affine domain over k will be an affine domain over k provided the integral closure R' of R in its quotient field is equidimensional. For the proof we need the following:

THEOREM 3.1. *Let R be a d-dimensional subring of an affine domain over k and let R' be the integral closure of R in its quotient field K. Let $\mathfrak{M}$ be a maximal ideal of R' with* $\mathrm{ht}\,\mathfrak{M}=d$. *If R is noetherian then $R'_{\mathfrak{M}}$ is a locality over k.*

PROOF. Let $\mathfrak{m}=\mathfrak{M}\cap R$. Since R is noetherian, $\mathfrak{m}$ is finitely generated, say $\mathfrak{m}=(x_1,\dots, x_t)R$. Let B be an affine domain over k contained in R such that R and B are birational and $x_1,\dots, x_t\in B$. Let $M=\mathfrak{m}\cap B$. Then we have $x_1,\dots, x_t\in M$ and hence $MR=\mathfrak{m}$. Since $\mathrm{ht}\,\mathfrak{M}=d$ and $\mathrm{tr.deg}_k R'/\mathfrak{M}\leqq \mathrm{tr.deg}_k R'-\mathrm{ht}\,\mathfrak{M}$, we have $\mathrm{tr.deg}_k B/M\leqq \mathrm{tr.deg}_k R'/\mathfrak{M}=0$. Thus B/M is algebraic over k, hence B/M is a field and M is a maximal ideal of B. Let B' be the integral closure of B in K and let $\bar{R}=R[B']$. Since B is an affine domain over k, B' is a finite B-module. Whence $\bar{R}$ is a finite R-module, especially $\bar{R}$ is noetherian. Let $\mathfrak{N}=\mathfrak{M}\cap\bar{R}$ and let $M'=\mathfrak{N}\cap B'$. Since R' is integral over $\bar{R}$, we have $\mathrm{ht}\,\mathfrak{N}\geqq \mathrm{ht}\,\mathfrak{M}=d$, hence we have $\mathrm{ht}\,\mathfrak{N}=d$. On the other hand, B' is integral over B and M' lies over the maximal ideal M of B. Hence M' is a maximal ideal of B', and we have $\mathrm{ht}\,M'=d$ because B' is an affine domain over k. Thus we have $\dim B'_{M'}=\dim\bar{R}_{\mathfrak{N}}$. Notice that $M'\bar{R}_{\mathfrak{N}}\supseteq M\bar{R}_{\mathfrak{N}}=\mathfrak{m}\bar{R}_{\mathfrak{N}}$ and $\mathfrak{m}\bar{R}_{\mathfrak{N}}$ is a $\mathfrak{N}\bar{R}_{\mathfrak{N}}$-primary ideal. Therefore there exists a positive integer r such that $\mathfrak{N}^r\bar{R}_{\mathfrak{N}}\subseteq M'\bar{R}_{\mathfrak{N}}$. Let $k'=B'/M'$ and let $L=\bar{R}/\mathfrak{N}$. Then we have $\mathrm{length}_{k'}\bar{R}_{\mathfrak{N}}/M'\bar{R}_{\mathfrak{N}}\leqq(\mathrm{length}_{k'}L)(\mathrm{length}_{\bar{R}}\bar{R}/\mathfrak{N}^r)$. Since $\mathrm{ht}\,\mathfrak{N}=d$, we have $\mathrm{tr.deg}_k\bar{R}/\mathfrak{N}=0$, and hence $L=\bar{R}/\mathfrak{N}$ is a subfield of a certain affine domain over k (cf. [7, Theorem 2]). Thus L is a finite algebraic extension field of k, whence $\mathrm{length}_k L$ is finite, a fortiori, $\mathrm{length}_{k'}L$ is finite. On the other hand, since $\bar{R}$ is noetherian, we have $\mathrm{length}_{\bar{R}}\bar{R}/\mathfrak{N}^r$ is finite. Thus we have $\mathrm{length}_{k'}\bar{R}_{\mathfrak{N}}/M'\bar{R}_{\mathfrak{N}}$ is finite. Moreover, since B' is a normal affine domain, $B'_{M'}$ is analytically normal by Theorem (37.5) in [6], and obviously $\bar{R}_{\mathfrak{N}}$ and $B'_{M'}$ are birational. Hence we have $B'_{M'}=\bar{R}_{\mathfrak{N}}$ by Theorem (37.4) in [6]. Thus $\bar{R}_{\mathfrak{N}}$ is integrally closed, whence we have $\bar{R}_{\mathfrak{N}}=R'_{\mathfrak{N}}$ because $R'_{\mathfrak{N}}$ is integral and birational over $\bar{R}_{\mathfrak{N}}$. Therefore $R'_{\mathfrak{N}}$ is a local ring, and hence we have $R'_{\mathfrak{N}}=R'_{\mathfrak{M}}$. Whence we have $R'_{\mathfrak{M}}=B'_{M'}$ and $R'_{\mathfrak{M}}$ is a locality over k. Q.E.D.

THEOREM 3.2. *Let R be a d-dimensional subring of an affine domain over k and let R' be the integral closure of R in its quotient field. If R is noetherian and R' is equidimensional, that is,* $\dim R'_{\mathfrak{M}}=d$ *for any maximal ideal $\mathfrak{M}$ of R', then R is an affine domain over k.*

PROOF. Since R' is equidimensional, $R'_{\mathfrak{M}}$ is a locality over k for any maximal ideal $\mathfrak{M}$ of R' by Theorem 3.1. Thus R' is an affine domain over k by Theorem 1.6, and hence R is also an affine domain over k by Lemma 1.4. Q.E.D.

Recall that a ring R is called catenary if, for any pair of prime ideals $\mathfrak{p}$, $\mathfrak{q}$ with $\mathfrak{p}\supseteq\mathfrak{q}$, we have $\mathrm{ht}\,\mathfrak{p}=\mathrm{ht}\,\mathfrak{q}+\mathrm{ht}(\mathfrak{p}/\mathfrak{q})$. A ring R is called universally catenary if R is noetherian and if every R-algebra of finite type is catenary (cf. [4, (14.B)]).

COROLLARY 3.3. *Let R be a subring of an affine domain over k. If R is universally catenary and equidimensional then R is an affine domain over k.*

PROOF. Let $\mathfrak{m}=(x_1,\dots, x_t)R$ be an arbitrary maximal ideal of R and let B be an affine domain over k contained in R such that R and B are birational and $x_1,\dots, x_t \in B$. Let B' be the integral closure of B in its quotient field and let $\bar{R}=R[B']$. Then $\bar{R}$ is a finite R-module. Let $\mathfrak{M}_1,\dots, \mathfrak{M}_n$ be all the maximal ideals of $\bar{R}$ lying over $\mathfrak{m}$. Since R is universally catenary and $\bar{R}$ is a finite R-module, the dimension formula holds between R and $\bar{R}$, whence we have $\mathrm{ht}\,\mathfrak{M}_i=\mathrm{ht}\,\mathfrak{m}$ for each i (cf. [4, (14.C)]). Thus, as is shown in the proof of Theorem 3.1, we have $\bar{R}_{\mathfrak{M}_i}=B'_{M_i}$ for each i, where $M_i=\mathfrak{M}_i \cap B'$. Therefore $\bar{R}_{\mathfrak{M}_i}$ is integrally closed for each i, hence $\bar{R}_{\mathfrak{m}}$ is integrally closed. Thus the integral closure of $R_{\mathfrak{m}}$ in its quotient field is equal to $\bar{R}_{\mathfrak{m}}$ which is a finite $R_{\mathfrak{m}}$-module. Whence, by Theorem 2.5, the integral closure R' of R in its quotient field is a finite R-module, and hence the dimension formula holds between R and R'. Since R is equidimensional, we see that R' is also equidimensional. Thus the assertion follows from Theorem 3.2. Q. E. D.

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