On non-Archimedean curves omitting few components and their arithmetic analogues

Julie Tzu-Yueh Wang

Academia Sinica, Taiwan

Distribution of Rational and Holomorphic Curves

in Algebraic Varieties

BIRS, Banff

A motivating example.

Let **k** an algebraically closed field complete with respect to a non-Archimedean absolute value $|\cdot|$ of arbitrary characteristic.

$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbf{k}[[z]]$ is analytic on $z \in \mathbf{k}$ if $|a_n z^n| \to 0$ as $n \to \infty$.

$f(z)$ is called a **k** entire function if f is analytic on **k**.

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k hyperbolic

A variety X is said to be **k** *hyperbolic* if any analytic map from **k** to X is constant.

Some examples of **k** hyperbolic varieties.

- $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{0, \infty\}$.
- curvesof genus ≥ 1. (Berkovich 1990)
- curve omitting two points
- semi-Abelian variety (Cherry 1994)
- $\mathbb{P}^n$ omitting $n+1$ hypersurfaces in g. p. (Ru 2001)
- projective variety omitting $n+1$ hypersurface divisors in g. p. (An 2007)
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Further results.

Theorem (Lin-W.2010, An-Levin-W. 2011)

Let X be a nonsingular projective variety over $\mathbf{k}$. Let $D_1, \ldots, D_m$ be effective divisors on X with empty intersection. Let $D = \sum_{i=1}^m D_i$.

1. *If $\kappa(D_i) > 0$ for all i, then the image of an analytic map $f : \mathbf{k} \to X \setminus D$ is contained in a proper subvariety of X.*
2. *If D_i is big for all i, then there exists a proper Zariski-closed subset $Z \subset X$ such that the image of any non-constant analytic map $f : \mathbf{k} \to X \setminus D$ is contained in Z.*
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Question:

What if the intersection of the divisors D_i's is not empty?

A simple observation.

Let P_1, P_2 be non-constant homogeneous polynomials in $n+1$ variables with degree d_1 and d_2 respectively. Assume that the divisors $D_1 = \{P_1 = 0\}$ and $D_2 = \{P_2 = 0\}$ in $\mathbb{P}^n$ are distinct.

Let $f = (f_0, \cdots, f_n) : \mathbf{k} \to \mathbb{P}^n \setminus D_1 \cup D_2$ be an analytic map.

Then $P_1^{d_2}(f)/P_2^{d_1}(f)$ is entire without zero and hence is constant.

Consequently, the image of f is contained in a subvariety of codimension one.

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Theorem (An-W.-Wong 2008)

Let X be a nonsingular projective subvariety of $\mathbb{P}^N$ of dimension n. Let $P_1, \ldots, P_q$ be non-constant homogeneous polynomials in $N+1$ variables. Let $D_i = X \cap \{P_i = 0\}$, $1 \le i \le q$, be divisors of X in general position. Let f be an analytic map from $\mathbf{k}$ to $X \setminus \cup_{i=1}^q D_i$. Then the image of f is contained in a subvariety of X of codimension $\min\{n+1, q\} - 1$ in X. In particular, f is algebraically degenerate if $q \ge 2$, and $X \setminus \cup_{i=1}^q D_i$ is $\mathbf{k}$-hyperbolic if $q \ge n+1$.

Theorem (An-Cherry-W. 2008 & 2015)

Let Y be a closed positive dimensional subvariety of a non-singular projective variety X. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible, effective, ample divisors in general position on X. Let r be the rank of the subgroup of $\mathrm{NS}(X)$ generated by $\{c_1(D_i)\}_{i=1}^{\ell}$. If there exists an algebraically non-degenerate analytic map from $\mathbf{k}$ to Y omitting each of the D_i that does not contain all of Y, then

$$\ell \leq \min\{r + \mathrm{codim}\ Y, \dim X\}.$$

n-component

Theorem (An-W.-Wong 2008)

Let $D_1, \ldots, D_n$ be nonsingular hypersurfaces in $\mathbb{P}^n$ intersecting transversally. Then $\mathbb{P}^n \setminus \cup_{i=1}^n D_i$ is $\mathbf{k}$ hyperbolic if $\deg D_i \geq 2$ for each $1 \leq i \leq n$.

Theorem (An-W.-Wong 2008)

Let D_1 and D_2 be nonsingular projective curves in $\mathbb{P}^2$. Assume that D_1 and D_2 intersect transversally and $\deg D_1 \leq \deg D_2$. Then $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is $\mathbf{k}$-hyperbolic if and only if either $\deg D_1, \deg D_2 \geq 2$ or $\deg D_1 = 1$, $\deg D_2 \geq 3$ and D_1 does not intersect D_2 at any maximal inflexion point.

Corollary

If D_1 and D_2 are two generic curves in $\mathbb{P}^2(\mathbf{k})$ with $\deg D_1 + \deg D_2 \geq 4$, then $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is $\mathbf{k}$ hyperbolic.

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Main Theorem 1.

Theorem (Levin-W.)

Let $D_1, \ldots, D_n$ be effective nef divisors intersecting transversally in an n-dimensional nonsingular projective variety X over $\mathbf{k}$. Let K_X denote the canonical divisor on X.

1. *Assume that either $D_i^n > 1$ or that $D_i^n = 1$ and $K_X.D_i^{n-1} < 1 - n$ for each $1 \leq i \leq n$. Then the image of an analytic map $f : \mathbf{k} \to X \setminus \cup_{i=1}^n D_i$ is contained in a proper subvariety of X.*
2. *If $D_i^n > 1$ for all i, then there exists a proper Zariski-closed subset $Z \subset X$ such that the image of any non-constant analytic map $f : \mathbf{k} \to X \setminus \cup_{i=1}^n D_i$ is contained in Z.*

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Corollary

Let X be an n-dimensional nonsingular projective variety over $\mathbf{k}$. Suppose that $-K_X$ is nef and

$$(-K_X)^n > (n-1)^n.$$

Let $D_1, \ldots, D_n$ be effective nef and big divisors on X intersecting transversally. Then the image of an analytic map $f : \mathbf{k} \to X \setminus \cup_{i=1}^n D_i$ is contained in a proper subvariety of X.

Remark

Let X be a smooth projective surface over an algebraically closed field. Then $-K_X$ is nef and $(-K_X)^2 > 1$ if and only if X is one of the following:

1. $X \cong \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$;
2. $X \cong \mathbb{P}^2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$;
3. X is obtained from $\mathbb{P}^2$ by successively blowing up at most 7 points.

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Question.

Is it possible to impose conditions on the self-intersection numbers of the divisors or the degrees of the divisors (in some fixed projective embedding) to ensure that $X \setminus (D_1 \cup \cdots \cup D_n)$ is **k** hyperbolic?

Ans. Unlikely!

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Example.

Let m and n be positive integers. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let D_1 and D_2 be effective divisors of type $(1, m)$ and $(1, n)$, respectively. Let $D = D_1 + D_2$. Let P be a point in the intersection of the supports of D_1 and D_2 and let L be the line on X of type $(0, 1)$ through P. Then since $L \setminus D = L \setminus \{P\} \cong \mathbb{A}^1$, there exists a non-constant analytic map $f : \mathbf{k} \to X \setminus D$. Note that D_1 and D_2 are very ample and $D_1^2 = 2m$, $D_2^2 = 2n$.

Main Theorem 2.

Theorem (Levin-W.)

Let X be a rational ruled surface over $\mathbf{k}$. Let D_1 and D_2 be effective divisors intersecting transversally in X.

1. *If D_1 and D_2 are big, then the image of an analytic map $f : \mathbf{k} \to X \setminus D_1 \cup D_2$ is contained in a proper subvariety of X.*
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Nevanlinna theory *v.s.* Diophatine approximation

Let X be a variety over a number field K.

existence of a nonconstant analytic curve in X

$$\updownarrow$$

existence of an infinite set of integral points on X

(over a finite extension of K)

Definition.

Let X be an affine variety over a number field k. Let S be a finite set of places of k containing the archimedean places and let $\mathcal{O}_{k,S}$ denote the ring of S-integers of k. We define a set $R \subset X(k)$ to be a set of $\mathcal{O}_{k,S}$-integral points on X if there exists an affine embedding $\phi : X \hookrightarrow \mathbb{A}^n$ such that $\phi(R) \subset X \cap \mathbb{A}^n(\mathcal{O}_{k,S})$.

non-Archimedean Nevanlinna theory $\leftrightarrow$?

Note that **k** entire functions without zeros are constant.

It suggests that we should look at ring of integers of a number field k with finite unit groups.

Then $k = \mathbb{Q}$ or imaginary quadratic field.

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An Example

$C : x^2 - 2y^2 = 1$. Over $\overline{\mathbb{Q}}$, we have $C \cong \mathbb{G}_m$. C does admit infinitely many $\mathbb{Z}$-integral points, while C is **k** hyperbolic.

Let $\tilde{C}$ be the projective closure of C defined in the projective plane by $x^2 - 2y^2 = z^2$. Then C has two points at infinity $P_\pm = (\pm\sqrt{2}, 1, 0)$. Every regular function on C over $\mathbb{Q}$ has a pole at both P_+ and P_- on $\tilde{C}$. Over $\mathbb{Q}(\sqrt{2})$, however, there are regular functions on C with a pole only at, say, P_+ on $\tilde{C}$. One might view the problem here as being that C doesn't have enough regular functions over $\mathbb{Q}$ to have a good notion of $\mathbb{Z}$-integral points.

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A necessary condition

Let X be an affine variety over k where $k = \mathbb{Q}$ or an imaginary quadratic field.

Condition $(*)$: There exists a projective closure $\tilde{X}$ of X nonsingular at every point in $\tilde{X} \setminus X$ and such that every (geometric) irreducible component of $\tilde{X} \setminus X$ is defined over k.

Theorem

Let $k = \mathbb{Q}$ or an imaginary quadratic field and suppose that $\mathrm{char}\, \mathbf{k} = 0$. If X is an affine curve over k satisfying $(*)$ then X contains an infinite set of $\mathcal{O}_k$-integral points if and only if there exists a non-constant analytic map $f : \mathbf{k} \to X$ if and only if X is rational with a single point at infinity.

In the 2008 paper of An-Levin-W. more parallel statements and results (qualitative and quantitative) are made in non-Archimedean Nevanlinna theory and Diophantine approximation.

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Let $k = \mathbb{Q}$ or an imaginary quadratic field. Let X be an n-dimensional nonsingular projective variety over k. Let $D_1, \ldots, D_n$ be effective nef divisors on X, all defined over k, intersecting transversally in X. Let K_X denote the canonical divisor on X. Suppose that every point in the intersection $\cap_{i=1}^n D_i$ is k-rational.

1. *Assume that either $D_i^n > 1$ or that $D_i^n = 1$ and $K_X . D_i^{n-1} < 1 - n$ for each $1 \le i \le n$. Then any set R of $\mathcal{O}_k$-integral points on $X \setminus \cup_{i=1}^n D_i$ is contained in a proper Zariski-closed subset of X.*
2. *If $D_i^n > 1$ for all i, then there exists a proper Zariski-closed subset $Z \subset X$ such that for any set R of $\mathcal{O}_k$-integral points on $X \setminus \cup_{i=1}^n D_i$, the set $R \setminus Z$ is finite.*

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Corollary

Let $k = \mathbb{Q}$ or an imaginary quadratic field. Let X be an n-dimensional nonsingular projective variety over k. Suppose that $-K_X$ is nef and

$$(-K_X)^n > (n-1)^n.$$

Let $D_1, \ldots, D_n$ be effective nef and big divisors on X, all defined over k, intersecting transversally in X. Suppose that every point in the intersection $\cap_{i=1}^n D_i$ is k-rational. Then any set R of $\mathcal{O}_k$-integral points on $X \setminus \cup_{i=1}^n D_i$ is contained in a proper Zariski-closed subset of X.

Theorem

Let $k = \mathbb{Q}$ or an imaginary quadratic field. Let $\pi : X \to \mathbb{P}^1$ be a rational ruled surface over k. Let D_1 and D_2 be effective divisors intersecting transversally in X. Suppose that all the irreducible components of D_1 and D_2 are defined over k and that all of the points in the intersection $D_1 \cap D_2$ are k-rational.

1. *If D_1 and D_2 are big, then any set R of $\mathcal{O}_k$-integral points on $X \setminus D_1 \cup D_2$ is contained in a proper Zariski-closed subset of X.*
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Basic ingredient for the non-Archimedean case

Theorem (Lin-W.2010, An-Levin-W.2011)

Let X be a nonsingular projective variety over $\mathbf{k}$. Let $D_1, \ldots, D_m$ be effective divisors on X with empty intersection. Let $D = \sum_{i=1}^{m} D_i$.

1. *If $\kappa(D_i) > 0$ for all i, then the image of an analytic map $f : \mathbf{k} \to X \setminus D$ is contained in a proper subvariety of X.*
2. *If D_i is big for all i, then there exists a proper Zariski-closed subset $Z \subset X$ such that the image of any non-constant analytic map $f : \mathbf{k} \to X \setminus D$ is contained in Z.*
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Ideas of the proof of Main Theorem 1.

Since $D_1, \ldots, D_n$ intersect transversally, their intersection contains only points. Let $\cap_{i=1}^n D_i = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$.

Let $\pi : \tilde{X} \to X$ be the blow-up along $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$, successively, and let $E_j := \pi^{-1}(\mathfrak{p}_j)$ be the exceptional divisor for $1 \le j \le m$.

For $1 \le i \le n$ and $1 \le j \le m$, we let

$G_{ij} := \pi^*(D_i) - E_j = \tilde{D}_i + E_1 + \cdots + E_{j-1} + E_{j+1} + \cdots + E_m$, where $\tilde{D}_i$ is the strict transform of D_i under π.

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Theorem

Let X be a nonsingular projective variety of dimension n over a field k and let D be a nef divisor on X. Let K_X denote the canonical divisor on X. Let $\pi : \tilde{X} \to X$ be the blow-up along m distinct points $P_1, \ldots, P_m$ of X, successively, and let $E_i := \pi^{-1}(P_i)$ be the exceptional divisor for $1 \leq i \leq m$.

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Sketch of Proof

The case of blowing-up one point: $\pi : \tilde{X} \to X$. Let $i : E \to \tilde{X}$ be the inclusion map. We have an exact sequence

$$0 \to H^0(\tilde{X}, \mathcal{O}(m\pi^* D - (j+1)E)) \to H^0(\tilde{X}, \mathcal{O}(m\pi^* D - jE)) \to H^0(E, i^*\mathcal{O}(m\pi^* D - jE)).$$

A standard computation shows
$h^0(m\pi^* D) - h^0(m\pi^* D - mE) \leq \frac{m^n}{n!} + \frac{1}{2}\frac{m^{n-1}}{(n-2)!} + O(m^{n-2})$.

Apply the following asymptotic Riemann-Roch formula.

Theorem (Matsusaka)

Let X be a nonsingular projective variety of dimension n and let D be a nef and big divisor on X. Then

$$h^0(mD) = \frac{D^n}{n!}m^n - \frac{K_X.D^{n-1}}{2(n-1)!}m^{n-1} + O(m^{n-2}).$$

Then

$$h^0(m\pi^*D - mE) \geq \frac{D^n - 1}{n!}m^n - \frac{K_X.D^{n-1} + (n-1)}{2(n-1)!}m^{n-1} + O(m^{n-2}).$$

Basics of rational ruled surfaces

Proposition

A rational ruled surface X (over $\bar{k}$) is isomorphic to $X_e := \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ over $\mathbb{P}^1$ for some nonnegative integer e. Let F denote a fiber on X and let C_0 denote a section of X such that $\mathcal{O}(C_0) \cong \mathcal{O}_{X_e}(1)$. Then

(a) $\mathrm{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ *generated by* $C_0 \subset X$ *and* F *with* $C_0^2 = -e$, $F^2 = 0$, *and* $C_0 \cdot F = 1$.

(b) *Let* K_X *be the canonical divisor on* X. *Then* $K_X \sim -2C_0 - (2+e)F$. *In particular,* $K_X^2 = 8$.

(c) *Let* D *be a divisor on* X *equivalent to* $aC_0 + bF$ *in* $\mathrm{Pic} X$. *Then*

  (i) *If* D *is an irreducible curve* $\neq C_0, F$, *then* $a, b > 0$, $b \geq ae$, *and* $D^2 > 0$.

  (ii) D *is big if and only if* $a > 0$ *and* $b > 0$.

  (iii) D *is ample if and only if* $a > 0$ *and* $b > ae$.

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Let D_1 and D_2 be big effective divisors on X.

Suppose first that D_1 and D_2 have irreducible components E_1 and E_2, respectively, with $E_i \not\sim C_0, F$, for $i = 1, 2$.

Then by Proposition $E_i^2 \geq 1$ for $i = 1, 2$.

$(K_X.E_i)^2 \geq K_X^2 E_i^2 \geq K_X^2 = 8$ and $K_X.E_i \leq -3$.

Then by Theorem 1, the image of an analytic map $f : \mathbf{k} \to X \setminus (D_1 \cup D_2) \subset X \setminus (E_1 \cup E_2)$ is contained in a proper subvariety of X.

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Suppose now that, say, D_1 has every irreducible component linearly equivalent to either C_0 or F.

Since D_1 is big, by Proposition, D_1 must contain at least two irreducible components C and F' with $C \sim C_0$ and F' a fiber.

Since D_2 is linearly equivalent to a positive integral linear combination of C and F', there exists a non-constant function $\phi \in \mathbf{k}(X)^*$ with poles and zeros only in the support of D_1 and D_2.

Consider an analytic map $f : \mathbf{k} \to X \setminus (D_1 \cup D_2)$. Then $\phi \circ f : \mathbf{k} \to \mathbb{A}^1 \setminus \{0\}$ is analytic, and hence constant. It follows that the image of f is contained in a proper subvariety of X.

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Assume furthermore that D_1 and D_2 are ample.

By (a), the image of a non-constant analytic map $f : \mathbf{k} \to X \setminus (D_1 \cup D_2)$ is contained in a curve in X.

Let C be the Zariski closure of the image of f in X.

If $C \cap (D_1 \cup D_2)$ contains more than one point, then f must be constant.

On the other hand, $D_i \cap C \neq \emptyset$ for $i = 1, 2$, since D_1 and D_2 are ample.

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Let $\pi : \tilde{X} \to X$ be the blow-up of X at x with exceptional divisor E.

Let $\tilde{C}$ be the strict transform of C and $\tilde{D}_i$ the strict transform of D_i, $i = 1, 2$.

Then $f : \mathbf{k} \to X \setminus D_1 \cup D_2$ lifts to $\tilde{f} : \mathbf{k} \to \tilde{X} \setminus \tilde{D}_1 \cup \tilde{D}_2$ with $f = \pi \circ \tilde{f}$ and the image of $\tilde{f}$ is contained in $\tilde{C}$.

Denote by $m = m_x(C)$ the multiplicity of C at x.

If $(C.D_i)_x > m = m_x(C) \cdot m_x(D_i)$ for $i = 1, 2$, then each $\tilde{D}_i$ must intersect $\tilde{C}$ at some point on $\tilde{X}$ lying above x.

Since D_1 and D_2 intersect transversally, $\cap_{i=1}^2 \tilde{D}_i \cap E = \emptyset$. Thus, there must be at least two points on $\tilde{C}$ lying above x.

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Assume that C is not linearly equivalent to C_0 or F.

From Proposition, we have that $a, b - ae, c, d > 0$.

By taking F to be the fiber passing through x, we see that

$c = C.F \geq m$.

Then $m = C.D_1 = ad + c(b - ae) > c \geq m$, a contradiction.

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Then $(C.D_1)_x = C.D_1 = ad + c(b - ae)$.

Assume that C is not linearly equivalent to C_0 or F.

From Proposition, we have that $a, b - ae, c, d > 0$.

By taking F to be the fiber passing through x, we see that

$c = C.F \geq m$.

Then $m = C.D_1 = ad + c(b - ae) > c \geq m$, a contradiction.