Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

WeA02.4

On non-asymptotic observation of nonlinear systems

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Abstract— In this contribution, we use the so-called algebraic derivative method for a non-asymptotic state observation of nonlinear SISO systems. We derive a general formula of a time-varying filter that allows a non-model-based, quasi-instantaneous estimation of time derivatives of arbitrary analog time signals. The estimates are based on integrals of measured signals alone. For preserving accuracy, the estimation process has to be re-initialized after some period of time. Besides resetting the estimation time-interval equidistantly, estimated absolute error bounds and integral error bounds are also used for improving the efficiency of the estimation process. As an example, we use Chen's chaotic oscillator to illustrate the velocity and the robustness of our observation method with respect to uncertainty of initial values and uniformly distributed measurement noise.

I. INTRODUCTION

Nonlinear observer design is considered a difficult task. From a theoretical prospect, difficulties originate from singularities within in the observation mapping (locality of the results), the input dependence of the observability property, and from the necessity of extended order observers in cases where the observability mapping fails the rank criterion; to mention but a few. From a practical point of view, reasons are, for instance, the presence of disturbances and measurement noise, insufficient model accuracy, and unknown initial conditions, what all turns out crucially challenging when striving for robustness in design. To tackle these problems, Gauthier et al. [7], Esfandiari and Khalil [9], among others, proposed high gain observers that ensure an asymptotic state estimation under attenuation of disturbances while keeping low the influences of uncertainties within the mathematical model and initial values. In principle, all these observers share Luenberger's idea of adjusting the observer model by weighing the observer output deviations from the plant's output. Thus, these methods naturally rely on an adequate modelling of the plant and yield, subject to certain requirements, asymptotical convergence of the state estimation error.

In this paper, we take an algebraic perspective. This way, a (local) observable system can be regarded as a system whose state variables can be expressed in terms of input and output variables and up to finite order of its time derivatives. Consequently, if the input variables cannot interfere with the

observability property of the system (uniform observability), an estimation of the input and output derivatives via input and output measurements is sufficient for constructing a local observer. To this end, we determine local, non-model-based, close approximations to time derivatives of arbitrary analytical time signals (e. g. inputs or outputs) which we use for a fast (i. e. non-asymptotic) state observation of, for simplicity, SISO differentially flat systems. The proposed estimation scheme relies on a truncated Taylor series of an analytical time signal and its representation as a chain of integrators. Using operational calculus, in particular the algebraic derivative, we eliminate the influence of initial values and obtain a triangular system of equations which allows to solve for the respective time signal derivatives up to a desired order. The main result is a general representation of a linear, time-varying filter together with an associated singular time-varying output equation that allow an online-estimation of the desired time derivatives. As this approximation is valid only on a certain time interval, a resetting is proposed for sustaining accuracy. For this purpose, we reset the calculations either after some "small" time interval of fixed length or we choose the reset time in terms of specific error bounds, i. e. the estimated absolute error and estimated integral error, respectively. For earlier developments of the algebraic derivative method refer to the work of Fliess and Sira-Ramírez [3], [4], [5]; an application to automatic control is explored in Sira-Ramírez and Fliess [10].

Our paper is organized as follows: Section II presents the general derivation of the filter formula for estimating the time derivatives of a measured, arbitrary, analog time signal. The closing part of this section is devoted to the exposition of different strategies for resetting the calculation intervals. In Section III we briefly recall the notion of local observability and its implication on differentially flat SISO systems. Section IV presents an application of the estimation scheme on the state observation of Chen's chaotic oscillator. The conclusion points out possible extensions of our setting.

II. AN ESTIMATE FOR THE DERIVATIVES OF AN ANALOG SIGNAL

A. *General derivation of the filter coefficients*

An analytic time signal $y(t)$ can be approximated after some time instant t_i by its truncated Taylor series expansion,

$$\tilde{y}(t) = \sum_{j=1}^{k} \frac{1}{(j-1)!} \, y^{(j-1)}(t_i)(t-t_i)^{j-1} \, 1(t-t_i) \,. \quad (1)$$

of sufficient order of approximation k. This power series can be interpreted as a k-fold integrating system with zero

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input and the corresponding initial conditions $y^{(j-1)}(t_i)$, $j = 1, 2, \ldots, k$. In the operational calculus frame this reads

$$\left(s^k \tilde{y}(s) - \sum_{j=1}^{k} s^{k-j} y^{(j-1)}(t_i) \right) e^{-st_i} = 0. \quad (2)$$

The k-fold derivation, of the expression within parentheses, with respect to the operator s yields the expression

$$\frac{d^k}{ds^k} \left(s^k \tilde{y}(s) \right) = 0 \quad (3)$$

which is independent of any initial values of the signal.

Left multiplication by s^{-j}, $j = k-1, k-2, \ldots, k-d$ yields d equations

$$0 = s^{-j} \frac{d^k}{ds^k} \left(s^k \tilde{y}(s) \right), \quad j = k-1, k-2, \ldots, k-d \quad (4)$$

constituting a triangular linear system of equations for computing an estimate of the time derivatives of $y(t)$ up to order $d \le k-1$. To see this, recall Leibniz' formula

$$\frac{d^k}{ds^k} \left(x(s) y(s) \right) = \sum_{i=0}^{k} \binom{k}{i} \frac{d^{k-i}}{ds^{k-i}} x(s) \frac{d^i}{ds^i} y(s) \quad (5)$$

which with

$$\frac{d^{k-i}}{ds^{k-i}} x(s) = \frac{d^{k-i}}{ds^{k-i}} s^k = \frac{k!}{i!} s^i \quad (6)$$

applied in equation (4) results in

$$s^{-j} \frac{d^k}{ds^k} \left(s^k \tilde{y}(s) \right) = \sum_{i=0}^{k} \frac{(k!)^2}{(k-i)!(i!)^2} s^{i-j} \frac{d^i}{ds^i} \tilde{y}(s)$$

$$= \sum_{i=0}^{k} \binom{k}{i}^2 (k-i)! \, s^{i-j} \frac{d^i}{ds^i} \tilde{y}(s) \quad (7)$$

for $j = k-1, k-2, \ldots, k-d$. In view of the backward transformation into the time domain we expand (7) as per

$$0 = \sum_{i=j}^{k} \binom{k}{i}^2 (k-i)! \, s^{i-j} \frac{d^i}{ds^i} \tilde{y}(s)$$

$$+ \sum_{i=0}^{j-1} \binom{k}{i}^2 (k-i)! \, s^{i-j} \frac{d^i}{ds^i} \tilde{y}(s). \quad (8)$$

Hence, by the usual backward transformation formulae we obtain d equations in the time domain, i. e.

$$0 = \sum_{i=j}^{k} \binom{k}{i}^2 (k-i)! \, \frac{d^{i-j}}{dt^{i-j}} \left((-t)^i \tilde{y}(t) \right)$$

$$+ \sum_{i=0}^{j-1} \binom{k}{i}^2 (k-i)! \, \overset{(j-i)}{\int} (-t)^i \tilde{y}(t), \quad (9)$$

for $j = k-1, k-2, \ldots, k-d$. Here we have used, for simplicity, the abbreviation:

$$\overset{(m)}{\int} t^i \tilde{y}(t) := \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_{m-1}} (\sigma_m)^i \tilde{y}(\sigma_m) \, d\sigma_m d\sigma_{m-1} \cdots d\sigma_1. \quad (10)$$

Applying Leibniz' formula on (9), $\tilde{y}(t)$ and its time derivatives $\tilde{y}^{(1)}(t), \ldots, \tilde{y}^{(k-d)}(t)$ become isolated. Hence

$$0 = \sum_{i=j}^{k} \binom{k}{i}^2 (k-i)! \, (-1)^i \sum_{l=0}^{i-j} \binom{i-j}{l} \frac{i!}{(j+l)!} \, t^{j+l} \tilde{y}^{(l)}(t)$$

$$+ \sum_{i=0}^{j-1} \binom{k}{i}^2 (k-i)! \, (-1)^i \overset{(j-i)}{\int} t^i \tilde{y}(t)$$

$$= \sum_{i=j}^{k} \sum_{l=0}^{i-j} \binom{k}{i} \binom{i-j}{l} \frac{k! \, (-1)^i}{(j+l)!} \, t^{j+l} \tilde{y}^{(l)}(t)$$

$$+ \sum_{i=0}^{j-1} \binom{k}{i}^2 (k-i)! \, (-1)^i \overset{(j-i)}{\int} t^i \tilde{y}(t). \quad (11)$$

for $j = k-1, k-2, \ldots, k-d$. With the maximum derivative index $\delta := k-j$, setting $n := i + \delta - k$ and introducing the integration counter index $m := k - \delta - i$ in the first and second sum, respectively, the above equations read

$$0 = \underbrace{\sum_{n=0}^{\delta} \sum_{l=0}^{n} \binom{k}{\delta-n} \binom{n}{l} \frac{k! \, (-1)^{n+k-\delta}}{(k+l-\delta)!} \, t^{k+l-\delta} \tilde{y}^{(l)}(t)}_{=: \, \alpha(k, \delta, t)}$$

$$+ \underbrace{\sum_{m=1}^{k-\delta} \binom{k}{m+\delta}^2 (m+\delta)! \, (-1)^{k-\delta-m} \overset{(m)}{\int} t^{k-m-\delta} \tilde{y}(t)}_{=: \, \beta(k, \delta, t)} \quad (12)$$

for $\delta = 1, \ldots, d$. Note that the triangular summation structure in $\alpha(k, \delta, t)$ implies that

$$\sum_{n=0}^{\delta} \sum_{l=0}^{n} \cdots = \sum_{l=0}^{\delta} \sum_{n=l}^{\delta} \cdots \quad (13)$$

which allows to evaluate

$$\alpha(k, \delta, t) =$$

$$= \sum_{l=0}^{\delta} \frac{k! \, (-1)^{k-\delta}}{(k+l-\delta)!} \left(\sum_{n=l}^{\delta} \binom{k}{\delta-n} \binom{n}{l} (-1)^n \right) t^{k+l-\delta} \tilde{y}^{(l)}(t)$$

$$= \sum_{l=0}^{\delta} \frac{k! \, (-1)^{k-\delta}}{(k+l-\delta)!} \left(\frac{(k-l-1)! \, (-1)^l}{(k-\delta-1)!(\delta-l)!} \right) t^{k+l-\delta} \tilde{y}^{(l)}(t)$$

$$= \sum_{l=0}^{\delta-1} \binom{k}{\delta-l} \frac{(k-l-1)!}{(k-\delta-1)!} (-1)^{k+l-\delta} \, t^{k+l-\delta} \tilde{y}^{(l)}(t)$$

$$+ (-1)^k t^k \tilde{y}^{(\delta)}(t). \quad (14)$$

Thus, with (12) we obtain the recursive formula

$$\tilde{y}^{(\delta)}(t) = \sum_{l=0}^{\delta-1} \underbrace{\binom{k}{\delta-l} \frac{(k-l-1)!}{(k-\delta-1)!} (-1)^{l-\delta+1} \, \frac{1}{t^{\delta-l}} \tilde{y}^{(l)}}_{=: \, a(k, \delta, l)}$$

$$+ \sum_{m=1}^{k-\delta} \underbrace{\binom{k}{m+\delta}^2 (-1)^{1-m-\delta} (m+\delta)! \, \frac{1}{t^k} \overset{(m)}{\int} t^{k-m-\delta} \tilde{y}(t)}_{=: \, b(k, \delta, m)}.$$

$$(15)$$

for determining the $\delta = 1, \ldots, d \leq k - 1$ derivatives of the signal $\tilde{y}(t)$. As the derivatives $\tilde{y}^{(\delta)}(t)$ depend on all lower order derivatives $\tilde{y}^{(\delta-1)}(t), \tilde{y}^{(\delta-2)}(t), \ldots, \dot{\tilde{y}}(t)$ this formula amounts to a triangular linear system of equations. As a consequence, the estimates $\tilde{y}^{(\delta-1)}(t), \tilde{y}^{(\delta-2)}(t), \ldots, \dot{\tilde{y}}(t)$ for the real derivatives $y^{(\delta-1)}(t), y^{(\delta-2)}(t), \ldots, \dot{y}(t)$ follow just from integrals of $\tilde{y}(t)$ or, since this signal may be measured, by replacing $\tilde{y}(t)$ with the measured value of $y(t)$.

The linear system of equations (15) can be interpreted as a time-varying linear filter with time varying inputs. In order to illustrate this perspective, we consider (15) rewriting the equivalent time-varying triangular linear system of equations

$$A(k,t)\,\bar{y}^{(d)}(t) = x(k,t) \tag{16}$$

with the help of the notation:

$$\bar{y}^{(d)}(t) = \left(\tilde{y}^{(1)}(t), \tilde{y}^{(2)}(t), \ldots, \tilde{y}^{(d)}(t)\right)^{\mathrm{T}},$$

$$A(k,t) = \begin{pmatrix} t^k & & & \\ -a(k,2,1)\,t^{k-1} & t^k & & \\ -a(k,3,1)\,t^{k-2} & -a(k,3,2)\,t^{k-1} & & \\ \vdots & \vdots & \ddots & \\ -a(k,d,1)\,t^{k-d+1} & -a(k,d,2)\,t^{k-d+2} & \cdots & t^k \end{pmatrix},$$

$$x(k,t) = \begin{pmatrix} a(k,1,0)\,t^{k-1} \\ a(k,2,0)\,t^{k-2} \\ a(k,3,0)\,t^{k-3} \\ \vdots \\ a(k,\delta,0)\,t^{k-d} \end{pmatrix} y(t) + \begin{pmatrix} \sum\limits_{m=1}^{k-1} b(k,1,m) \int\limits^{(m)} t^{k-m-1} y(t) \\ \sum\limits_{m=1}^{k-2} b(k,2,m) \int\limits^{(m)} t^{k-m-2} y(t) \\ \sum\limits_{m=1}^{k-3} b(k,3,m) \int\limits^{(m)} t^{k-m-3} y(t) \\ \vdots \\ \sum\limits_{m=1}^{k-d} b(k,d,m) \int\limits^{(m)} t^{k-m-d} y(t) \end{pmatrix}$$

We rewrite the $i = 1, \ldots, d$ components of $x(k,t)$ as

$$x_i(k,t) = a(k,i,0)\,t^{k-i}\,y(t) + z_i(k,t) \tag{17}$$

in which we define

$$z_i(k,t) := \sum_{m=1}^{k-i} b(k,i,m) \int\limits^{(m)} t^{k-m-i} y(t). \tag{18}$$

The variables $z_i(k,t)$ obey a differential relation since

$$\dot{z}_i(k,t) = \sum_{m=1}^{k-i} b(k,i,m) \int\limits^{(m-1)} t^{k-m-i} y(t)$$

$$= \sum_{\mu=0}^{k-i-1} b(k,i,\mu+1) \int\limits^{(\mu)} t^{k-\mu-i-1} y(t)$$

$$= b(k,i,1)\,t^{k-i-1}\,y(t) + \sum_{\mu=1}^{k-i-1} b(k,i+1,\mu) \int\limits^{(\mu)} t^{k-\mu-i-1} y(t).$$

For $i = 1, \ldots, d-1$ it results in

$$\dot{z}_i(k,t) = b(k,i,1)\,t^{k-i-1}\,y(t) + z_{i+1}(k,t). \tag{19}$$

This is a dynamic system with input $y(t)$. It can be extended uniformly so as to obtain a finite system order of $k-1$, thus

$$\dot{z}_i(k,t) = b(k,i,1)\,t^{k-i-1} y(t) + z_{i+1}(k,t), \ i = 1, \ldots, k-2$$

$$\dot{z}_{k-1}(k,t) = b(k,k-1,1)\,y(t). \tag{20}$$

The first $i = 1, \ldots, d$ states, $z_i(k,t)$, of this dynamic system are the inputs of (16), that is

$$x(k,t) = \begin{pmatrix} a(k,1,0)\,t^{k-1} \\ a(k,2,0)\,t^{k-2} \\ a(k,3,0)\,t^{k-3} \\ \vdots \\ a(k,d,0)\,t^{k-d} \end{pmatrix} y(t) + \begin{pmatrix} z_1(k,t) \\ z_2(k,t) \\ z_3(k,t) \\ \vdots \\ z_d(k,t) \end{pmatrix}. \tag{21}$$

Furthermore, we can derive an explicit formula for the time derivative estimates $\tilde{y}^{(d)}(t)$, $\tilde{y}^{(d-1)}(t)$, $\ldots, \dot{\tilde{y}}(t)$ employing equation (16). Since the elements of $A(k,t)$ are

$$[A(k,t)]_{ij} = \begin{cases} \binom{k}{i-j} \dfrac{(k-j-1)!}{(k-i-1)!}(-1)^{i-j} t^{k+j-i}, & i \geq j \\ 0 & , \ i < j \end{cases} \tag{22}$$

the elements of the corresponding inverse matrix A^{-1} can be shown to satisfy

$$[A^{-1}(k,t)]_{ij} = \begin{cases} \binom{k+i-j-1}{i-j} \dfrac{(k-j-1)!}{(k-i-1)!} t^{j-i-k}, & i \geq j \\ 0 & , \ i < j \end{cases} \tag{23}$$

Referring to this inverse matrix, we solve equation (16) for the $i = 1, \ldots, d$ derivative estimates and obtain

$$\tilde{y}^{(i)}(t) = \sum_{j=1}^{d} [A^{-1}(k,t)]_{ij}\, x_j(k,t) =$$

$$= \sum_{j=1}^{i} [A^{-1}(k,t)]_{ij}\, a(k,j,0)\, t^{k-j}\, y(t)$$

$$+ \sum_{j=1}^{i} \sum_{m=1}^{k-j} [A^{-1}(k,t)]_{ij}\, b(k,j,m) \int\limits^{(m)} t^{k-m-j} y(t). \tag{24}$$

The first summation leads to a closed expression. Concerning the second part, the number of integrations can be made explicit. Accordingly, we have

$$\tilde{y}^{(i)}(t) =$$

$$= \frac{1}{t^i} y(t) \frac{(k-1)!}{(k-i-1)!} \sum_{j=1}^{i} \binom{k}{j} \binom{k+i-j-1}{i-j}(-1)^{1-j}$$

$$+ \sum_{m=1}^{k-i} \sum_{j=1}^{i} [A^{-1}]_{ij}\, b(k,j,m) \int\limits^{(m)} t^{k-m-j} y(t)$$

$$+ \sum_{m=k-i+1}^{k-1} \sum_{j=1}^{k-m} [A^{-1}]_{ij}\, b(k,j,m) \int\limits^{(m)} t^{k-m-j} y(t) \tag{25}$$

which, after using the notation:

$$c(k,i,j,m) :=$$

$$= \binom{k+i-j-1}{i-j}\binom{k}{m+j}^2 \frac{(k-j-1)!}{(k-i-1)!}(-1)^{1-m-j}(m+j)! \tag{26}$$

results in an explicit expression for the estimates of the $i = 1, \ldots, d$ derivatives

$$\tilde{y}^{(i)}(t) = \frac{(k+i-1)!}{i!(k-i-1)!} \frac{1}{t^i} y(t)$$

$$+ \sum_{m=1}^{k-i} \sum_{j=1}^{i} c(k,i,j,m) \frac{1}{t^{k+i-j}} \overset{(m)}{\int} t^{k-m-j} y(t)$$

$$+ \sum_{m=k-i+1}^{k-1} \sum_{j=1}^{k-m} c(k,i,j,m) \frac{1}{t^{k+i-j}} \overset{(m)}{\int} t^{k-m-j} y(t). \quad (27)$$

With the latter derivation, we are in the position of stating our main result.

B. Main result

By introducing the filter states $z_i(k,t)$ from equation (20) we may rephrase the derivative estimator equation (27) as

$$\tilde{y}^{(i)}(t) = \frac{(k+i-1)!}{i!(k-i-1)!} \frac{1}{t^i} y(t)$$

$$+ \sum_{j=1}^{i} \binom{k+i-j-1}{i-j} \frac{(k-j-1)!}{(k-i-1)!} \frac{1}{t^{k+i-j}} z_j(k,t) \quad (28)$$

for $i = 1, \ldots, d$.

Remark: For resolving the sums over products of binomial coefficients in the afore-presented steps of derivation we have employed Zeilberger's algorithm [11] which is realized within the computer algebra package Maple®. In any case, the algorithm was capable of returning factorial expressions without involving hypergeometric series.

Example: We determine estimates of the first three derivatives of an analog time signal $y(t)$ for some sufficiently small time $t > 0$. To this end, we choose an approximation of order $k = 7$ in the Taylor-expansion. With equation (20), we are given the corresponding filter equation

$$\begin{aligned}
\dot{z}_1(t) &= -882\, t^5\, y(t) + z_2(t) \\
\dot{z}_2(t) &= 7350\, t^4\, y(t) + z_3(t) \\
\dot{z}_3(t) &= -29400\, t^3\, y(t) + z_4(t) \\
\dot{z}_4(t) &= 52920\, t^2\, y(t) + z_5(t) \\
\dot{z}_5(t) &= -35280\, t\, y(t) + z_6(t) \\
\dot{z}_6(t) &= 5040\, y(t).
\end{aligned} \quad (29)$$

and the desired derivative estimates follow from the respective filter output equation (28), hence

$$\begin{aligned}
\dot{\tilde{y}}(t) &= \tfrac{42}{t} y(t) + \tfrac{1}{t^7} z_1(t) \\
\ddot{\tilde{y}}(t) &= \tfrac{840}{t^2} y(t) + \tfrac{35}{t^8} z_1(t) + \tfrac{1}{t^7} z_2(t) \\
\dddot{\tilde{y}}(t) &= \tfrac{10080}{t^3} y(t) + \tfrac{560}{t^9} z_1(t) + \tfrac{28}{t^8} z_2(t) + \tfrac{1}{t^7} z_3(t).
\end{aligned} \quad (30)$$

Note that, at time $t = 0$, the above formulae yield an *indetermination*. In fact, due to the finite precision of the numerical processors, the computation will not be appropriately defined over a small interval of time of the form: $[0, \varepsilon)$. Thus, the formulae for $\dot{\tilde{y}}(t)$, $\ddot{\tilde{y}}(t)$, and $\dddot{\tilde{y}}(t)$ are valid for $t \geq \varepsilon$. During the interval of time $[0, \varepsilon)$, we may replace their values by arbitrary constant values or by appropriate polynomial spline approximations (see the discussion at the end of this section).

It is also clear that for any $t \geq \varepsilon > 0$ the expressions found yield suitable approximations for the first three time derivatives of $y(t)$ during an open time interval of the form $[\varepsilon, t)$. We now examine the issue of how and when to update, or re-initialize, the computations.

C. Reset Intervals

The validity of the formulae for the estimates of $y^{(i)}$ in the open time interval $[\varepsilon, t)$ becomes questionable as t grows, due to the approximate nature of the adopted truncated Taylor series expansion. The calculations need to be reset at some finite time t_r. It is not difficult to see that for any resetting time, t_r, we also have the following approximation formulae valid. For the output equation we have

$$\tilde{y}^{(i)}(t) = \frac{(k+i-1)!}{i!(k-i-1)!} \frac{1}{(t-t_r)^i} y(t)$$

$$+ \sum_{j=1}^{i} \binom{k+i-j-1}{i-j} \frac{(k-j-1)!}{(k-i-1)!} \frac{1}{(t-t_r)^{k+i-j}} z_j(k,t) \quad (31)$$

for $i = 1, \ldots, d$, and the associated filter equation becomes

$$\dot{z}_i(k,t) = b(k,i,1)\,(t-t_r)^{k-i-1} y(t) + z_{i+1}(k,t) \quad (32)$$

for $i = 1, \ldots, k-2$, and

$$\dot{z}_{k-1}(k,t) = b(k,k-1,1)\, y(t), \quad (33)$$

respectively.

1) Equidistant Intervals: A simple way of maintaining the estimation accuracy is to choose intervals of time whose length h may be arbitrarily fixed to be "small" at the outset. Thus we assume validity of the formulae in each interval of the form $[t_r + \varepsilon, t_r + h]$. Clearly, $h \gg \varepsilon$. Naturally, this entitles choosing a small value for h. The determination of h may require some additional off-line trial and error runs.

This straight-forward procedure is, obviously, highly dependent on the encoding system and requires judgment, rather than an objective criterion evaluation. Therefore, we propose objective criteria for determining a reasonable time instant for resetting the derivative calculations, when the actual values of such derivatives are not known beforehand.

2) Estimated Absolute Error Bound: At any time t, we may easily generate an estimate $\hat{y}(t)$ of the actual signal $y(t)$ on the basis of the computed time derivatives so far. Any deviation of this estimated value from the actual (measured) value of the original signal $y(t)$ is due to the fact that the computed values of the time derivatives are drifting from the actual ones. Hence, we propose to operate a calculation resetting each time when an absolute error estimate surpasses a small constant threshold value $\delta > 0$, i. e. when

$$e(t) = |y(t) - \hat{y}(t)| \geq \delta \quad (34)$$

with $\hat{y}(t)$ being a generated estimate of $y(t)$ itself, which is computed by means of known data as per

$$\hat{y}(t) = y(t_r) + \dot{\tilde{y}}(t_r)(t-t_r) + 1/2 \ddot{\tilde{y}}(t_r)(t-t_r)^2. \quad (35)$$

Here $\dot{\tilde{y}}(t_r)$ and $\ddot{\tilde{y}}(t_r)$ denote the computed first and second estimated time derivatives of $y(t)$ at time t_r. Consequently,

strong deviations of the actual signal $y(t)$ from a polynomial evolution will lead to more resettings. However, in cases where more than two time derivatives of a signal are to be computed, one may propose a more general error criterion by involving higher order time derivative estimates.

3) Estimated Integral Error Bound: If the accumulative deviations are to be stressed, the reset time may be automatically determined by resorting to an integral error criterion. For this purpose, we may employ the integrated error $\int e(t)\,dt$, the integrated absolute error $\int |e(t)|\,dt$ or an integrated squared error $\int (e(t))^2\,dt$. Along the ideas from before, we may refer to the generated estimate $\hat{y}(t)$ of the signal $y(t)$ from (35) and use it for calculating the integral over the error estimate $e(t)$, defined in (34).

4) Estimation Shortly After Resetting: As already mentioned, the formulae for the derivative estimates are valid after a small ε-time, i. e. we consider the estimation valid if $t \geq t_r + \varepsilon$ and if the above-stated error bounds are not surpassed. In the interval $[t_r, t_r + \varepsilon)$, we may adopt as temporary values for the time derivative estimates $\tilde{y}^{(i)}(t)$ either constant values of the form $\tilde{y}^{(i)}(t_r^-)$, i. e. the last computed values of the time derivatives at $t = t_r$, or, alternatively, we may use polynomial extrapolations on the basis of the last values of the previously computed time derivatives at time t_r. With respect to the example from above, we may opt, for instance, for a square extrapolation for the first time derivative $\dot{\tilde{y}}(t)$, a linear extrapolation for $\ddot{\tilde{y}}(t)$, and a constant extrapolation for $\dddot{\tilde{y}}(t)$. Hence, for all t in the time interval $[t_r, t_r + \varepsilon)$ we have

$$
\begin{aligned}
\dot{\tilde{y}}(t) &= \dot{\tilde{y}}(t_r^-) + \ddot{\tilde{y}}(t_r^-)(t - t_r) + 1/2\,\dddot{\tilde{y}}(t_r^-)(t - t_r)^2, \\
\ddot{\tilde{y}}(t) &= \ddot{\tilde{y}}(t_r^-) + \dddot{\tilde{y}}(t_r^-)(t - t_r), \\
\dddot{\tilde{y}}(t) &= \dddot{\tilde{y}}(t_r^-).
\end{aligned}
\tag{36}
$$

If a larger number of computed time derivatives is available at time t_r we may use extrapolations of higher order.

III. OBSERVABILITY OF AUTONOMOUS NONLINEAR SYSTEMS

The example system in the next section is autonomous, thus for brevity, we confine our examination on observability of autonomous nonlinear SISO systems characterized by

$$
\begin{aligned}
\dot{x} &= f(x), \\
y &= h(x).
\end{aligned}
\tag{37}
$$

Here $x \in \mathbb{R}^n$ is the system state, and $y \in \mathbb{R}$ is the output of the system. We assume that $g(\cdot)$ and $h(\cdot)$ are sufficiently smooth. The system is said to be *locally observable* from the output $y = h(x)$ if the map

$$
\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}
\tag{38}
$$

is locally full rank n (see Hermann and Krener [8]). In this map, the expression $L_f^k h(x)$ is the Lie-derivative, which is recursively defined by $\frac{\partial L_f^{k-1} h(x)}{\partial x} f(x)$ with $L_f^0 h(x) = h(x)$.

A well known result establishes that if the above map is locally full rank n, then the state vector, x, of the system can be locally expressed as a smooth *differential function* of $y(t)$ i. e. a smooth function of y and a finite number (in fact $n - 1$) of its time derivatives (see Diop and Fliess [2] and also Fliess [6]). We also address this type of function as a *differential parametrization* of the state x in terms of the observable output y. We have then that x can be uniquely expressed as

$$
x = \Phi(y, \dot{y}, \ddot{y}, \cdots, y^{(n-1)})
\tag{39}
$$

for some smooth function Φ.

Remark: The differential parametrization (39) can be identified with one of the main properties of a differentially flat SISO system of the form $\dot{x} = f(x, u)$, $y = h(x)$, in which y is the observable, flat output. Note that this parametrization does not involve the inputs, nor input time derivatives. That is, these systems are uniformly locally observable.

IV. EXAMPLE: CHEN'S CHAOTIC OSCILLATOR

Consider Chen's chaotic oscillator [1], given by

$$
\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= (c - a)x_1 + c\,x_2 - x_1 x_3 \\
\dot{x}_3 &= x_1 x_2 - b\,x_3
\end{aligned}
\tag{40}
$$

with output $y = x_1$ which we assume measurable; a, b, c are known parameters.

The state x is parametrized in terms of the output y as

$$
\begin{aligned}
x_1 &= y \\
x_2 &= \frac{1}{a}\dot{y} + y \\
x_3 &= -\frac{1}{y}\left(\frac{1}{a}\ddot{y} + \left(1 - \frac{c}{a}\right)\dot{y} + (a - 2c)y \right)
\end{aligned}
\tag{41}
$$

Thus, the system is observable from $y(t)$ except on the line $y = 0$.

We propose a derivative estimator that uses $y(t)$ for reconstructing $x_2(t)$ and $x_3(t)$. For generating the time derivatives of $y(t)$ we propose a Taylor-approximation of order $k = 7$. Hence, we may refer to the equations (29) and (30) from the example given in Section 2.

For maintaining accuracy we shift to a reset time t_r, if need be, and circumvent the singularity at $t = t_r$ by approximating $\tilde{y}^{(i)}(t)$ in each small interval $[t_r, t_r + \varepsilon)$ by taking the polynomial extrapolation from equation (36).

Invoking an absolute error bound for the estimated absolute error

$$
e(t) = |y(t) - \hat{y}(t)|
$$

in which

$$
\hat{y}(t) = y(t_r) + \dot{\tilde{y}}(t_r)(t - t_r) + \frac{1}{2}\ddot{\tilde{y}}(t_r)(t - t_r)^2 + \frac{1}{6}\dddot{\tilde{y}}(t_r)(t - t_r)^3
$$

we calculate a new reset time t_r each time when the error estimate $e(t)$ surpasses the bound δ.

For the simulations we used the following parameters

$$
a = 35, \; b = 3, \; c = 28
$$

and initial values

$$x_1(0) = -10, \ x_2(0) = 0, \ x_3(0) = 20\,.$$

Within the estimation, however, we consider $x_2(0)$ and $x_3(0)$ unknown and set $\dot{\hat{y}}(0) = \ddot{\hat{y}}(0) = 0$, for simplicity. Moreover, we fix the interval length after resetting to $\varepsilon = 0.0013$ sec. and specify the error bound to $\delta = 1$.

Fig. 1 shows a resetting at $t \approx 0.42$ sec., i. e. when $\delta = 1$ is surpassed. Note that the exact value of $y(t)$ and $x_3(t)$, resp., is taken quasi-instantaneously. The singularity at $t \approx 0.35$ sec. is due to the lack of observability at $y = 0$.
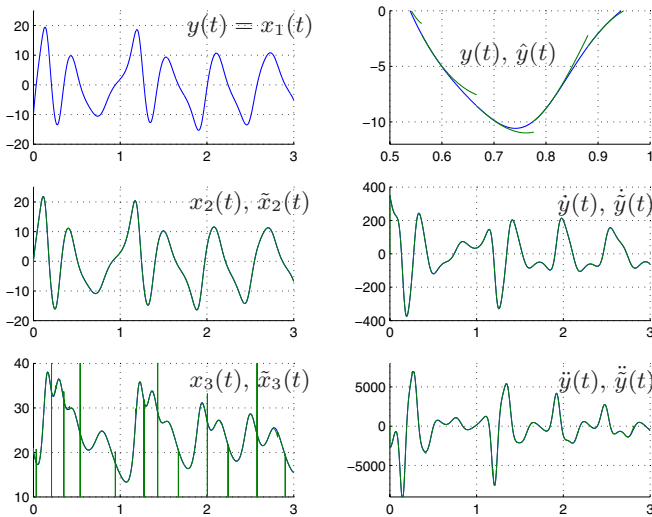
Fig. 1. Resetting, rapid estimation, and singularities
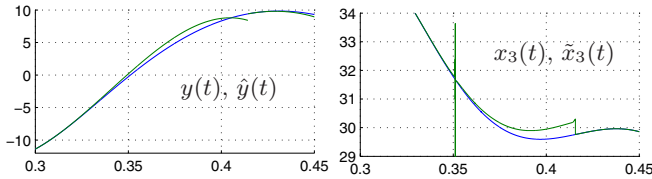
Fig. 2 illustrates the estimated time derivatives of $y(t)$, and the respective reconstructions of $x_2(t)$ and $x_3(t)$ by use of (41). Remarkably, even if a resetting takes place only at about every 0.1 seconds, original and estimated values can hardly be distinguished.

Fig. 2. Original values and its estimates

We also reset the estimation at equidistant time intervals $h = 0.01$ sec. Additionally, we perturbed the measured signal $y(t)$ with zero mean computer generated noise $\xi(t)$, uniformly distributed in the interval $[-0.0025, 0.0025]$, and chose a measurement sampling rate of 100Hz. The robustness of the estimation with respect to noise is depicted in Fig. 3.

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we have derived general formulae for the estimation of time derivatives of an arbitrary analytic time signal. These formulae, which are non-model-based, yield time-varying filter realizations with suitable output equations. The derivative estimate values were shown to be rather

Fig. 3. Robustness against uniformly distributed noise
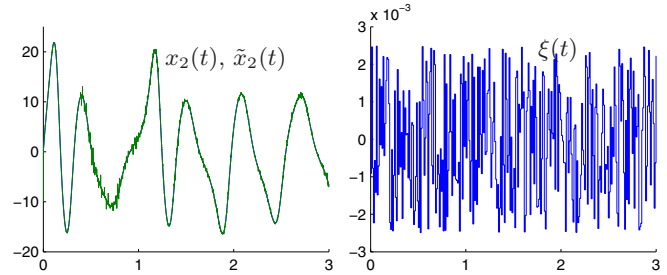
accurate after some short ε-time interval. Since the derivative estimation is based on truncated Taylor-series expansions of a given time signal, the estimation process needs resettings. These resettings are either periodical or dependent on error bounds. For the second option, one assesses the accuracy of the time derivatives estimates in terms of the reconstruction error of the signal itself. The application of the estimation scheme on the state reconstruction of Chen's chaotic oscillator, which is shown to be locally observable, yields very precise and fast results. In fact, the estimation is much faster than asymptotical convergence. Moreover, the estimation turns out to be quite robust with respect to measurement noise and unknown initial values. Further work will be conducted towards the state estimation of a broader class of nonlinear observable systems.

VI. ACKNOWLEDGMENTS

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