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ON NON-COMMUTATIVE QUADRATIC EXTENSIONS OF A COMMUTATIVE RING

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Let A be a ring with the identity element and B an extension ring of A with the common identity element. B is called a quadratic extension of A, if the residue module B/A is an invertible A-A-bimodule, i.e. $B/A \otimes_A \mathrm{Hom}_A(B/A, A) \approx \mathrm{Hom}_A(B/A, A) \otimes_A B/A \approx A$. In [4], [5] and [9], one has studied about commutative quadratic extensions. We like to extend these results to non-commutative quadratic extensions. But, in general, it is difficult. In this note, we shall study non-commutative quadratic extensions of a commutative ring. Let A be a commutative ring with the identity element. Let D be an A-algebra with the identity element such that D is a quadratic extension of a commutative subring B and B is a separable quadratic extension of A. In the section 1, we shall show that if A has no idempotents other than 0 and 1, then such an A-algebra D is either a commutative ring or a central A-algebra having B as a maximal commutative subring. We shall say that A-algebra D is a *quaternion A-algebra* with a maximal commutative and separable subalgebra B, if D is an A-algebra mensioned above and is a central separable A-algebra. In the section 2, we shall show that a quaternion A-algebra with a maximal commutative and separable subalgebra B is characterized by the separable quadratic extension B of A and a non-degenerate hermitian B-module (V, Φ) of rank one. Let (V, q) be a non-degenerate quadratic A-module such that V is a finitely generated projective A-module with a constant rank two. Then the Clifford algebra $C(V, q) = C_0(V, q) \oplus C_1(V, q)$ is a quaternion A-algebra with a maximal commutative and separable subalgebra $C_0(V, q)$. And, the quadratic A-module (V, q) is hyperbolic if and only if $[C_0(V, q)] = 1$ in $Q_S(A)$, where $Q_S(A)$ is the group of separable quadratic extensions of A (cf. [4], [5] and [9]).

1. Let A be a commutative ring with the identity element, and B a commutative and separable quadratic extension of A. Then B is characterized by an invertible A-module U, an A-linear map $f\colon U \to A$ and a quadratic form $q\colon U \to A$, as $B = A \oplus U$ and $x^2 = f(x)x + q(x)$ for $x \in U$ (cf. [4]). Let τ be an A-algebra automorphism of B defined by $\tau(a+x) = a + f(x) - x$ for $a \in A$, $x \in U$. Then we have $B^\tau = A$. Because, if x is in U and $\tau(x) = x$, then $f(x) = 0$, and $2x = 0$. From the fact that a bilinear form $D_{f,q}\colon U \times U \to A$; $(x, y) \rightsquigarrow f(x)f(y) + 2B_q(x, y)$

is non-degenerate (Theorem 1 in [4]), x is 0, consequently we have $B^{\tau}=A$. Therefore, B is a Galois extension of A with the Galois group $G(B/A)=\{I, \tau\}$. If A has no idempotents other than 0 and 1, then $G(B/A)=\{I, \tau\}$ is the group of all A-algebra automorphisms of B.

Let D be an A-algebra which is a quadratic extension of B. Then we have

(1.1) **Theorem.** *Let D and B be as above. If A has no idempotents other than* 0 *and* 1, *then D is either a commutative ring or a central A-algebra having the subalgebra B as a maximal commutative subring.*

Proof. Since the residue B-B-bimodule D/B is invertible, there exists an A-algebra automorphism σ of B such that $xb\equiv\sigma(b)x$ (mod B) for all $x\in D$ and $b\in B$. Then σ is either I or τ. If $\sigma=I$, then for each x in D, $d_x(b)=xb-bx$ is in B for all $b\in B$. The map d_x: $B\rightarrow B$ becomes a derivation of B over A. B is seprable over over A, hence every derivation of B over A is 0, and so $d_x=0$. Therefore, D is a B-algebra. Since D is a quadratic extension of B, D is a commutative ring. If $\sigma=\tau$, then for each $x\in D$, $d_x(b)=xb-\tau(b)x$ is in B for all $b\in B$, and the map $d_x:B\rightarrow B$ is a (τ, I)-derivation of B over A, i.e. $d_x(b_1 b_2)=d_x(b_1)b_2+\tau(b_1)d_x(b_2)$ for b_1, b_2 in B, (cf. p. 170 in [6]). Since D/B is a projective left B-module, the exact sequence $0\rightarrow B\rightarrow D\rightarrow D/B\rightarrow 0$ is split, i.e. there exists an invertible left B-submodule V of D such that $D=B\oplus V$. We consider the commutator ring $V_D(B)=\{x\in D;\ xb=bx \text{ for all } b\in B\}$, then $V_D(B)\supset B$. Now, we shall show $V_D(B)\cap V=0$. If x is in $V_D(B)\cap V$, we have $d_x(b)=xb-\tau(b)x=bx-\tau(b)x\in B\cap V=0$, and so $\tau(b)x=bx$ for all $b\in B$. Since $B\supset A$ is a Galois extension with the Galois group $G(B/A)=\{I, \tau\}$, there exist $b_1, b_2,\cdots b_r$ and $c_1, c_2,\cdots c_r$ in B such that $\sum_i c_i b_i=1$ and $\sum_i c_i\tau(b_i)=0$. Then $x=\sum c_i b_i x=\sum_i c_i\tau(b_i)x=0$. Consequently, we get $V_D(B)=B$, i.e. B is a maximal commutative subring of D. Finally, we shall show that the center of D is A. Let c be an element of the center. c is contained in $B=V_D(B)$. For any $x\in V$, $cx=xc=d_x(c)+\tau(c)x$ in $B\oplus V=D$. Therefore, we have $cx=\tau(c)x$. Since V is faithful over B, $c=\tau(c)$, and c is contained in $B^{G(B/A)}=A$. Therefore, A is the center of D.

2. Let B be a commutative and separable quadratic extension of A, and D an A-algebra such that D is a quadratic extension of B. If D is central separable over A, then B is a maximal commutative subring of D. Because, when we regard D as $D\otimes_A B$-left module by $d\otimes b\cdot x=dxb$ for $d\otimes b\in D\otimes_A B$ and $x,\in D$, D is a finitely generated projective $D\otimes_A B$-module and $\mathrm{Hom}_{D\otimes B}(D, D)\approx V_D(B)\supset B$. For every maximal ideal $\mathfrak{m}$ of A, $\mathrm{Hom}_{D\otimes B}(D, D)\otimes_A A_{\mathfrak{m}}\approx\mathrm{Hom}_{D_{\mathfrak{m}}\otimes B_{\mathfrak{m}}}(D_{\mathfrak{m}}, D_{\mathfrak{m}})\approx V_{D_{\mathfrak{m}}}(B_{\mathfrak{m}})\supset B_{\mathfrak{m}}$. But, $A_{\mathfrak{m}}$ has no idempotents wihout 0 and 1, $V_{D_{\mathfrak{m}}}(B_{\mathfrak{m}})=B_{\mathfrak{m}}$. Therefore, $V_D(B)=B$. B is a maximal commutative subring of D. We shall say that D is a *quaternion A-algebra* with a maximal commutative and separable subalgebra B, if D is an A-algebra defined above and is central separable over A. If A has no idempotents other than 0 and 1, and if D is non-commutative

and separable over A, then by (1.1), D is a quaternion A-algebra with a maximal commutative and separable subalgebra B.

(2.1) **Proposition.** *Let D be a quaternion A-algebra with a maximal commutative and separable subalgebra B. Then D is a generalized crossed product of B and $G(B/A)$ (defined in* [3]*). Therefore, there exists an invertible B-B-submodule of D such that $D=B\oplus V$ and $B=V\cdot V\approx V\otimes_B V$.*

Proof. D is a central separable A-algebra and contains a maximal commutative subalgebra B which is a Galois extension of A with the Galois group $\mathrm{G}(B/A)=\{\mathrm{I}, \tau\}$. By Proposition 3 in [3], D is a generalized crossed product of B and $\mathrm{G}(B/A)$, and so D is written as $D=J_I\oplus J_\tau$, where $J_I=B$ and $J_\tau=\{x\in D;$ $\tau(b)x=xb$ for all $b\in B\}$ are invertible B-B-bimodules. Furthermore, the map $\mathrm{f}_{\tau,\tau}$: $\mathrm{J}_\tau\otimes_B \mathrm{J}_\tau\to \mathrm{J}_I$; $x\otimes y \rightsquigarrow xy$ is a B-B-isomorphism. Put $V=\mathrm{J}_\tau$, V is the required B-B-bimodlue.

DEFINITION. Let $B\supset A$ be a commutative and separable quadratic extension which is a Galois extension with Galois group $\mathrm{G}(B/A)=\{\mathrm{I}, \tau\}$. For a left B-module M with an A-bilinear form Φ: $M\times M\to B$, we shall call (M, Φ) a *hermitian B-module* if it satisfies

1) $\Phi(bx, y)=b\Phi(x, y)$,
2) $\Phi(x, y)=\tau(\Phi(y, x))$ for every $b\in B$ and $x, y\in M$.

We shall say that a hermitian B-module (M, Φ) is non-degenerate, if the A-linear map $M\to \mathrm{Hom}_B(M, B)$; $x \rightsquigarrow \Phi(-, x)$ is an isomphism. Let (M_1, Φ_1) and (M_2, Φ_2) be hiermitian B-modules. The product $(M_1, \Phi_1)\otimes(M_2, \Phi_2)$ is defined by $(M_1\otimes_B M_2, \Phi_1\otimes\Phi_2)$ where $\Phi_1\otimes\Phi_2 : (M_1\otimes_B M_2)\times(M_1\otimes_B M_2)\to B$; $(x_1\otimes x_2, y_1\otimes y_2)\rightsquigarrow \Phi_1(x_1, y_1)\cdot\Phi_2(x_2, y_2)$. We denote by (B, I) a hermitian B-module defined by $I\,(b, b')=b\cdot\tau(b')$ for $b, b'\in B$.

If M_1 and M_2 are finitely generated projective B-modules, and if (M_1, Φ_1) and (M_2, Φ_2) are non-degenerate hermitian B-modules, then the product $(M_1, \Phi_1)\otimes(M_2, \Phi_2)$ is also non-degenerate.

(2.2) **Theorem.** *Let D be a quaternion A-algebra with a maximal commutative and separable subalgebra B. Then there exists a non-degenerate hermitian B-module (V, Φ) with an invertible B-bimodule V such that $D=B\oplus V$, $xb=\tau(b)x$ for $b\in B$, $x\in V$ and $xy=\Phi(x, y)$ for $x, y\in V$. Conversely, if (V, Φ) is any non-degenerate hermitian B-module with an invertible B-left module V, then an A-algebra $D=B\oplus V$ which is defined by $(b+x)\cdot(b'+x')=bb'+\Phi(x, x')+bx'+\tau(b')x$ for $b, b'\in B$ and $x, x'\in V$, is a quaternion A-algebra with a maximal commutative and separable subalgebra B.*

Proof. Let D be a quaternion A-algebra with a maximal commutative and separable subalgebra B. By (2.1), there exsists an invertible B-B-bimodule V

such that $D=B\oplus V$ and $V\cdot V=B$. We define an A-bilinear map $\Phi: V\times V\to B$ by $\Phi(x, y)=xy$ for $x, y\in V$. We shall show that (V, Φ) is a non-degenerate hermitian B-module. Put $\Psi(x, y)=\Phi(x, y)-\tau(\Phi(y, x))$ for x, y in V. For any maximal ideal $\mathfrak{m}$ of A, the localization $B_{\mathfrak{m}}$ is a semilocal ring, therefore $V_{\mathfrak{m}}$ is a free $B_{\mathfrak{m}}$-module of rank 1. Let $V_{\mathfrak{m}}=B_{\mathfrak{m}}v$, $\Psi_{\mathfrak{m}}=\Psi\otimes I_{\mathfrak{m}}$, $\Phi_{\mathfrak{m}}=\Phi\otimes I_{\mathfrak{m}}$ and $\tau_{\mathfrak{m}}=\tau\otimes I_{\mathfrak{m}}$ Then we have $\Psi_{\mathfrak{m}}(bv, b'v)v=\Phi_{\mathfrak{m}}(bv, b'v)v-\tau_{\mathfrak{m}}(\Phi_{\mathfrak{m}}(b'v, bv))v=(vb'v)v-v(b'vbv)=b\tau_{\mathfrak{m}}(b')v^3-\tau_{\mathfrak{m}}(b')bv^3=0$ in $D_{\mathfrak{m}}$. Therefore, we have $\Psi_{\mathfrak{m}}=0$ for any maximal ideal $\mathfrak{m}$ of A, and so $\Psi=0$, i.e. $\Phi(x, y)=\tau(\Phi(y, x))$ for every x, y in V. Since $V\otimes_B V\to B$; $x\otimes y \rightsquigarrow xy$ is B-B-isomorphism from (2.1), (V, Φ) is non-degenerate. Conversely, let (V, Φ) be any non-degenerate hermitian B-module with an invertible left B-module V. We can make a B-B-bimodule V by $xb=\tau(b)x$ for $b\in B$, $x\in V$. Since (V,Φ) is non-degenerate, the map $f_{\tau,\tau}: V\otimes_B V\to B$; $x\otimes y\rightsquigarrow\Phi(x, y)$ is a B-B-isomorphism as B-B-bimodules. By [3], we can construct a generalized crossed prduct $\Delta(f, B, \Psi, G)$ of B and $G=G(B/A)=\{I, \tau\}$ provided Ψ; $\Psi(I)=B$, $\Psi(\tau)=V$, and a factor set $f=\{I=f_{I,I}, f_{\tau,I}, f_{I,\tau}, f_{\tau,\tau}\}$, where $f_{I,\tau}: B\otimes_B V\to V$; $b\otimes x\rightsquigarrow bx$, $f_{\tau,I}: V\otimes_B B\to V$; $x\otimes b\rightsquigarrow xb$. To show the commutativity of the diagrams of the factor set, we need only to show the following commutative diagram:

$$\begin{array}{ccc} V\otimes_B V\otimes_B V & \xrightarrow{I\otimes f_{\tau,\tau}} & V\otimes_B B \\ \Big\downarrow{\scriptstyle f_{\tau,\tau}\otimes I} & & \Big\downarrow{\scriptstyle f_{\tau,I}} \\ B\otimes_B V & \xrightarrow{f_{I,\tau}} & V \end{array}$$

we shall show it by taking the localization with respect to a maximal ideal $\mathfrak{m}$ of A. Then we have $f_{\tau,I}\circ(I\otimes f_{\tau,\tau})(av\otimes bv\otimes cv)=f_{\tau,I}(av\otimes f_{\tau,\tau}(bv\otimes cv))=av\cdot\Phi(bv, cv)=a\tau(b)c\tau(\Phi(v, v))v=a\tau(b)c\Phi(v, v)v=\Phi(av, bv)cv=f_{\tau,\tau}(av\otimes bv)cv=f_{I,\tau}\circ(f_{\tau,\tau}\otimes I)(av\otimes bv\otimes cv)$ *for* all av, bv, cv in $V_{\mathfrak{m}}=A_{\mathfrak{m}}v$. Therefore, the diagram is commutative. Thus, $D=B\oplus V=\Delta(f, B, \Psi, G)$ is an A-algebra defined the multiplication by $(b+x)\cdot(b'+x')=bb'+\Phi(x, x')+bx'+\tau(b')x$ for $b+x$, $b'+x'$ in $B\oplus V=D$. By Proposition 3 in [3], D is an Azumaya A-algebra, accordingly $D=B\oplus V$ is a quaternion A-algebra with a maximal commutative and separable subalgebra B.

We shall call (V, Φ) a non-degenerate hermitian B-module of rank 1 if (V, Φ) is a non-degenerate hermitian B-module and V is an invertible left B-module. For a non-degenerate hermitian B-module of rank 1, we denote by $D(B, V, \Phi)$ the quaternion A-algebra D with a maximal commutative and separable sabalgebra B defined by (V, Φ) in (2.2)

(2.3) **Corollary.** *Let (V, Φ) and (V', Φ') be non-degenerate hermitian B-modules of rank* 1. *Then (V, Φ) and (V', Φ') are isometric if and only if there exists an A-algebra isomorphism of $D(B, V, \Phi)$ to $D(B, V'\ \Phi')$ which is idetity map on B.*

Let (P, q) be a quadratic A-module with a quadratic form $q: P \to A$. We shall call that (P, q) is a non-degenrate quadratic A-module of rank n, if P is a finitely generated projective A-module with constant rank n, i.e. $[P_{\mathfrak{m}}: A_{\mathfrak{m}}]=n$ for every maximal ideal $\mathfrak{m}$ of A, and $q: P \to A$ is non-degenetate.

(2.4) **Proposition.** *Let (V, q) be a non-degenerate A-module of rank 2. Then the Clifford algebra $C(V, q)=C_0(V, q) \oplus C_1(V, q)$ is a quaternion A-algebra with a maximal commutative and separable subalgebra $C_0(V, q)$, where $C_0(V, q)$ (resp. $C_1(V, q)$) is the subalgebra of $C(V, q)$ of homogeneous elements of degree* 0 (*resp. degree* 1.)

Proof. $C(V, q)$ is an Azumaya A-algrbra, and $C_0(V, q)$ is a commutative and separable quadratic extension of A (Lemma 6 in [7]). Therefore, $V \approx C_1(V, q)$ is a finitely generated projective $C_0(V, q)$-module. We shall show that $C_1(V, q)$ is an invertible $C_0(V, q)$-module. It suffices to show that for the case where A is a local ring. Assume that A is a local ring. Then, $V=Au \oplus Av$ is a free A-module of rank 2. Since (V, q) is non-degenerate, we may assume that $q(u)$ is invertible in A. Then we have $C_0(V, q)=A \oplus Auv$ and $C_1(V, q) \approx V= Au \oplus Av = C_1(V, q)\, u$. Since u is invertible in $C(V, q)$, $C_1(V, q)$ is a free $C_0(V, q)$ -module of rank 1.

(2.5) **Lemma.** *Let Λ be a Galois extension of a ring Γ with a Galois group G, and P a Λ-module. Then we have $Hom_\Gamma(P, \Gamma)=Tr \circ Hom_\Lambda(P, \Lambda)$, where $Tr(x)=\sum_{\sigma \in G} \sigma(x)$ for $x \in \Lambda$.*

Proof. Since $\Lambda \supset \Gamma$ is a Galois extension with a Galois group G, there exist $x_1, x_2, \cdots x_n$ and $y_1, y_2, \cdots y_n$ in Λ such that $\sum_i \sigma(x_i) y_i = \begin{cases} 1, & \sigma = I \\ 0, & \sigma \neq I \end{cases}$. Then, for f in $\mathrm{Hom}_\Gamma(P, \Gamma)$, $F(-)=\sum_i x_i f(y_i -)$ is contained in $\mathrm{Hom}_\Lambda(P, \Lambda)$, and $\mathrm{Tr} \circ F(z)=\sum_i \mathrm{Tr}(x_i f(y_i z))=f(\sum_i \mathrm{Tr}(x_i) y_i z)=f(z)$ for all $z \in P$. Therefore, f is in $\mathrm{Tr} \circ \mathrm{Hom}_\Lambda(P, \Lambda)$. The converse is clear.

(2.6) **Lemma.** *Let (P, Φ) be a non-degenerate hermitian B-dmodule. Then $(P, Tr \circ \Phi)$ is a non-degenerate bilinear A-module.*

Proof. $\mathrm{Tr} \circ \Phi: P \times P \to A$; $(x, y) \rightsquigarrow \mathrm{Tr}(\Phi(x, y))=\Phi(x, y)+\tau(\Phi(x, y))$ is an A-bilinear form. We show that $P \to \mathrm{Hom}_A(P, A)$; $x \rightsquigarrow \mathrm{Tr}(\Phi(-, x))$ is an A-isomorphism. If x is in P such that $\mathrm{Tr}(\Phi(-, x))=0$, $\Phi(P, x)$ is an ideal of B and $\mathrm{Tr}(\Phi(P, x))=0$. Let $b_1, b_2, \cdots b_n$ and $b_1', b_2', \cdots b_n'$ be elements in B such that $\sum_i b_i b_i'=1$ and $\sum_i \tau(b_i) b_i'=0$. Then, we have $b=\sum_i \mathrm{Tr}(b b_i) b_i'=0$ for every b in $\Phi(P, x)$, hence $\Phi(P, x)=0$. Therefore, $x=0$. From Lemma (2.5), $(P, \mathrm{Tr} \circ \Phi)$ is non-degenerate.

(2.7) **Theorem.** *Let $D=D(B, V, \Phi)$ be quaternion A-algebra with a maximal* *commutactive and separable subalgebra B. Then there exists an involution* $\sigma: D \rightarrow D$ *which is defined by* $\sigma(b+v)=\tau(b)-v$ *for* $b \in B$, $v \in V$. *We put* $N(x)=x \cdot \sigma(x)$ *and* $T(x)=x+\sigma(x)$ *for x in D. Then* $N: D \rightarrow A$ *is a qduaratic form,* (D, N) *is a non-degenerate quadratic A-module of rank 4, and* $D=B \perp V$. $T: D \rightarrow A$ *is an A-linear map and* $B_N(x, y)=T(x \cdot \sigma(y))$ *for* $x, y \in D$.

Proof. Let $x=b+v$ and $x'=b'+v'$ be elements in $D=B \oplus V$. Then we have $\sigma(xx')=\sigma(bb'+\Phi(v, v')+bv'+\tau(b')v)=\tau(bb'+\Phi(v, v'))-(bv'+\tau(b')v)=\tau(b) \cdot \tau(b')+\Phi(v, v')-v'\tau(b)-\tau(b')v=\sigma(x') \cdot \sigma(x)$, and $\sigma^2(x)=x$. Therefore, σ is an involution. Furthermore, $N(b+v)=b\sigma(b)-\Phi(v, v)$ and $\mathrm{T}(b+v)=b+\tau(b)$ are contained in $B^G=A$, hence $N: D \rightarrow A$ is a quadratic form, and the bilinear form is $B_N(x, x')=N(x+x')-N(x)-N(x')=x\sigma(x')+x'\sigma(x)=\mathrm{T}(x\sigma(x))$ for $x, x' \in D$. Therefore, we have $D=B \perp V$. To prove that (D, N) is non-degenerate, it suffices to show that $(B, N|B)$ and $(V, N|V)$ are non-degenerate. From Lemma (2.6), $\mathrm{Tr} \circ \Phi$ and $\mathrm{Tr} \circ I$ are non-degenerate, and $B_N(b, b')=\mathrm{T}(b\tau(b'))=\mathrm{Tr}(b\tau(b'))=\mathrm{Tr} \circ I$ (b, b') for $b, b' \in B$ and $\mathrm{B}_N(v, v')=\mathrm{T}(v(-v'))=\mathrm{T}(-\Phi(v, v'))=-\mathrm{Tr} \circ \Phi(v, v')$ for $v, v' \in V$, hence $(B, N|B)$ and $(V, N|V)$ are non-degenerate.

In Theorem (2.7), we put $Q=-N|V$. Then (V, Q) is a non-degenerate quadratic A-module of rank 2.

(2.8) **Theorem.** *Let* $D(B, V, \Phi)$ *be a quaternion A-algebra with a maximal commutative and separable subalgebra B, and* $N: D \rightarrow A$ *and* $Q=-N|V$ *as defined before. Then* $D(B, V, \Phi)$ *is a isomorphic to the Clifford algebra* $C(V, Q)$ *of the quadratic module* (V, Q) *as A-algebras.*

Proof. Since $Q(x)$ is equal to $x^2=N(x)$ in $D(B, V, \Phi)$ for every $x \in V$, the inclusion map $V \rightarrow \mathrm{D}(B, V, \Phi)=B \oplus V$ can be extended to an A-algebra homomorphism $\rho: C(V, Q) \rightarrow D(B, V, \Phi)$. From the fact that $C(V, Q)$ and $\mathrm{D}(B, V, \Phi)$ are Azumaya algebras over A and are generated by V, we obtain that ρ is an A-isomorphism.

(2.9) **Lemma.** *Let V be any invertible B-module. Then for any f in* Hom_B (V, B) *and x, y in V, we have* $f(x)y=f(y)x$.

Proof. Put $\Psi(x, y)=f(y)x-f(x)y$ for every $x, y \in V$, then $\Psi: V \times V \rightarrow V$ is a B-bilinear form. By taking the loclization of V with respect to a maximal ideal $\mathfrak{m}$ of A, we get easily $\Psi_{\mathfrak{m}}=0$. Therefore, $\Psi=0$.

(2.10) **Proposition.** *Let* (V, Φ) *be a non-degenerate hermitian B-module of rank* 1. *Then, the quaternion A-algebra* $D(B, V \otimes_B V, \Phi \otimes \Phi)$ *which is determined by* $(V, \Phi) \otimes (V, \Phi)=(V \otimes_B V, \Phi \otimes \Phi)$, *is A-algebra isomorphic to* $Hom_A(V, V)$, *and this isomorphism preserves the structure of B-modules.*

Proof. We can define a map θ: $D(B, V\otimes_B V, \Phi\otimes\Phi)=B\oplus V\otimes_B V\to\mathrm{Hom}_A(V, V)$ as follows: $\theta(b)(x)=bx$ for $b\in B$, $x\in V$, and $\theta(u\otimes v)(x)=\Phi(u, x)v$ for $u\otimes v\in V\otimes_B V$, $x\in V$. Then θ is an A-algebra homomorphism. Because, for $b\in B$, $u\otimes v\in V\otimes_B V$ and $x\in V$, we have $\theta(bu\otimes v)(x)=\Phi(bu, x)v=b\Phi(u, x)v=\theta(b)\circ\theta(u\otimes v)(x)$ and $\theta(u\otimes v b)(x)=\Phi(u, x)vb=\tau(b)\Phi(u, x)v=\Phi(u, bx)v=\theta(u\otimes v)\circ\theta(b)(x)$. And, for $u\otimes v$, $u'\otimes v'\in V\otimes_B V$ and $x\in V$, $\theta(u\otimes v)\circ\theta(u'\otimes v')(x)=\theta(u\otimes v)(\Phi(u', x')v')=\Phi(u, \Phi(u', x)v')v=\Phi(u, v')\Phi(x, u'))v$. On the other hand, $\Phi(-, v')$ and $\Phi(-, u')$ are in $\mathrm{Hom}_B(V, B)$, by Lemma (2.9) we get $\Phi(x, u')\Phi(u, v')v=\Phi(x, u')\Phi(v, v')u=\Phi(u, u')\Phi(v, v')x=\theta(\Phi(u, u')\Phi(v, v'))(x)$ $=\theta((u\otimes v)(u'\otimes v'))(x)$. Thus, θ is an A-algebra homomorphism. Now we check that θ is an epimorphism. From Lemma (2.5), we have $\mathrm{Hom}_A(V, V)\approx\mathrm{Hom}_A(V, A)\otimes_A V\approx\mathrm{Tr}\circ\mathrm{Hom}_B(V, B)\otimes_A V\approx(\mathrm{Tr}\circ\Phi(-, V))\otimes_A V$. Therefore, any element f in $\mathrm{Hom}_A(V, V)$ is written as $f=\sum_i\mathrm{Tr}\circ\Phi(-, u_i)v_i=\sum_i(\Phi(-, u_i)v+\Phi(u_i, -)v_i)$ for some $u_i, v_i\in V$, and by Lemma (2.9), $f(x)=\sum_i\Phi(x, u_i)v_i+\sum_i\Phi(u_i, x)v_i=\sum_i\Phi(v_i, u_i)x+\theta(\sum_i u_i\otimes v_i)(x)$ for $x\in V$. Thus, we get $f=\theta(\sum_i\Phi(v_i, u_i)+\sum u_i\otimes v_i)$. Since $D(B, V\otimes_B V, \Phi\otimes\Phi)$ and $\mathrm{Hom}_A(V, V)$ are Azumaya A-algrebras, θ is an A-algebra isomorphism.

(2.11) **Corollary.** *$D(B, B, I)\approx Hom_A(B, B)$ as A-algebras.*

(2.12) **Corollary.** *For any non-degenerate hermitian B-modules of rank* 1 *(V, Φ) and (V, Φ'), $(V\otimes_B V, \Phi\otimes\Phi)$ and $(V\otimes_B V, \Phi'\otimes\Phi')$ are isometric.*

(2.13) **Theorem.** *Let $D(B, V, \Phi)$ be a quaternion A-algebra with a maximal commutative and separable subalgebra B, and (V, Q) a non-degenerate quadratic A-module of rank 2 defined by $D(B, V, \Phi)$ in (2.8). Then, (V, Q) is hyperbolic if and only if $[B]=1$ in $Q_S(A)$ (cf.* [4]).

Proof. In (2.8), we obtained $D(B, V, \Phi)=C(V, Q)=B\oplus V$, $C_0(V, Q)=B$ and $C_1(V, Q)=V$. We assume that (V, Q) is hyperbolic. Then we may assume that $V=P\oplus P^*$ for some invertible A-module P and $P^*=\mathrm{Hom}_A(P, A)$, and $Q(x+f)=f(x)$ for $x\in P, f\in P^*$. Since $P\cdot P=P^*\cdot P^*=0$ in $C(V, Q)$, we get $C_0(V, Q)=A\oplus P\cdot P^*\approx A\oplus P\otimes_A P^*$. For any $\sum_i x_i f_i$ in $P\cdot P^*$, we have $(\sum_i x_i f_i)^2=\sum_{i,j} x_i f_i x_j f_j=\sum_{i,j} x_i(f_i(x_j)-x_j f_i)f_j=\sum_{i,j} f_i(x_j)x_i f_j=\sum_{i,j} f_i(x_i)x_j f_j=(\sum_i f_i(x_i))(\sum_i x_i f_i)$ using Lemma (2.9). We condisder an A-isomorphism μ: $P\cdot P^*(\approx P\otimes_B P^*)\to A$ defined by $\mu(\sum_i x_i f_i)=\sum_i f_i(x_i)$ for $\sum_i x_i f_i$ in $P\cdot P^*$. Then we have $(\sum_i x_i f_i)^2=\mu(\sum_i x_i f_i)\sum_i x_i f_i$ for every $\sum_i x_i f_i$ in $P\cdot P^*$, hence $B=C_0(V, Q)\approx(P\otimes_A P^*, \mu, 0)\approx(A, 1, 0)$ as A-algebras (cf. [4]). Accordingly, $[B]=1$ in $Q_S(A)$. Conversely, we assume $[B]=1$ in $Q_S(A)$. Then the quadratic extension B of A has idempotents e_1 and e_2 such that $1=e_1+e_2$, $e_1e_2=0$ and $B=Ae_1\oplus Ae_2$. Furthermore, A-module V is written as a direct sum of A-submodules e_1V and e_2V. Since the Galois group $G=G(B/A)=\{I, \tau\}$ is permutations of $\{e_1, e_2\}$, we have $Q(e_1x)=\Phi(e_1x, e_1x)=e_1\tau(e_1)\Phi(x, x)=e_1e_2\Phi(x, x)=0$ for every

$x \in V$. Therefore, $e_1 V$ is totally isotropic. We have $(e_1 V)^{\perp} = e_1 V$, because, for $e_1 y + e_2 z \in (e_2 V)^{\perp}$, $0 = B_Q(e_1 y + e_2 z, e_1 x) = \Phi(e_1 y + e_2 z, e_1 x) + \Phi(e_1 x, e_1 y + e_2 z) = \Phi(e_2 z, e_1 x) + \Phi(e_1 x, e_2 z) = e_2 \Phi(z, x) + e_1 \Phi(x, z)$ in $e_2 A \oplus e_1 A = B$, hence $e_1 \Phi(x, z) = \Phi(x, e_2 z) = 0$ for all z in V. Therefore, we get $e_2 z = 0$. Accordingly, (V, Q) is hyperbolic (cf. [2]).

(2.14) **Corollary.** *Let (P, q) be any non-degenerate quadratic A-module of rank 2. Then (P, q) is hyperbolic if and only if $[C_0(P, q)] = 1$ in $Q_S(A)$.*

(2.15) **Corollary.** *If B is a quadratic extension of A such that $[B] = 1$ in $Q_S(A)$, then every quaternion A-algebra $D(B, V, \Phi)$ with a maximal commutative and separable subalgebra B is split, i.e. $[D(B, V, \Phi)] = 1$ in the Brauer group $B(A)$.*

(2.16) **Corollary.** *If A is commutative ring such that $Q_S(A) = 1$, then every non-degenerate quadratic A-module of rank 2 is hyperbolic.*

(2.17) EXAMPLE. *If A is the integers Z or the gaussian intgers $Z[i]$, then every non-degenerate quadratic A-module of rank 2 is hyperbolic (cf. [5], [7]).*

(2.18) REMARK. Let K be a field, and (V, q) and (V', q') non-degenerate quadratic K-mdoules of rank 2. Then, (V, q) and (V', q') are isometric if and only if $[C(V, q)] = [C(V', q')]$ in the Brauer group $B(K)$ and $[C_0(V, q)] = [C_0(V', q')]$ in $Q_S(A)$.

Proof. For a field of characteristic $\neq 2$, this is obtained from Theorem 58:4 in [8], and for a field of charcteristic 2, is obtained from Theorem 3 in [1].

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