ON NON-COMMUTATIVE REGULAR LOCAL RINGS

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Let R be a ring (with identity). We shall call R a *local ring* if R is a right noetherian ring such that the Jacobson radical M is a maximal ideal (and so is the only maximal ideal), $\bigcap_{n=1}^{\infty} M^n = 0$ and R/M is a simple artinian ring. A local ring R with maximal ideal M is called *regular* if there exists a chain

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_n = 0$$

of ideals M_i of R such that M_{i-1}/M_i is generated by a central regular element of R/M_i ($1 \leqq i \leqq n$). For such a ring R, Walker [**6**, Theorem 2.7] proved that R is prime and n is the right global dimension of R, the Krull dimension of R, the homological dimension of the R-module R/M and the supremum of the lengths of chains of prime ideals of R. Such regular local rings will be called *n-dimensional.* The aim of this note is to give examples of regular local rings. These arise as localizations of universal enveloping algebras of nilpotent Lie algebras over fields and localizations of group algebras of certain finitely generated finite-by-nilpotent groups.

An ideal I of a ring R is called a *central* ideal if I can be generated by central elements of R. More generally an ideal I is *polycentral* if there exists a chain

$$0 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_m = I$$

of ideals I_j of R such that I_j/I_{j-1} is a central ideal of R/I_{j-1} ($1 \leqq j \leqq m$). The ring R is *polycentral* if every ideal of R is polycentral. If P is a prime ideal of a right noetherian polycentral ring R, and

$$\mathscr{C}(P) = \{c \in R : c+P \text{ is a regular element of } R/P\},$$

then R satisfies the right Ore condition with respect to $\mathscr{C}(P)$ [**5**, Theorem 2.2]. It follows that if

$$K = \{r \in R : rc = 0 \text{ for some } c \in \mathscr{C}(P)\}$$

then K is an ideal of R and $c+K$ is a regular element of R/K for all c in $\mathscr{C}(P)$. Moreover the partial right quotient ring of R/K with respect to $\{c+K : c \in \mathscr{C}(P)\}$ is a local ring and will be denoted by R_P. A right noetherian polycentral ring R will be called *super-regular* (after [**2**, p. 120]) if R_P is a regular local ring for all prime ideals P of R. We shall prove the following result.

THEOREM A. *Let U be the universal enveloping algebra of a finite dimensional nilpotent Lie algebra g over a field k. Then U is a right and left noetherian super-regular polycentral integral domain. Moreover for any prime ideal P of U the dimension of U_P is at most* $\dim_k g$.

The result we shall prove for group rings is similar. If we make the convention that a simple artinian ring is a 0-dimensional regular local ring then it is well known that any semisimple artinian ring is a super-regular polycentral ring. In particular if k is a field and G is a finite group whose order is a unit in k then the group algebra kG is super-regular. Our second theorem is the following one.

THEOREM B. *Let R be a commutative noetherian super-regular ring and G be a finitely generated group with a normal subgroup H such that G/H is torsion-free nilpotent and the order of H is a unit in R. Then the group ring $S = RG$ is a right and left noetherian super-regular polycentral ring.*

1. Proof of Theorem A. Let U be the universal enveloping algebra of a finite dimensional nilpotent Lie algebra g over a field k. Then U is isomorphic to the ring $k[X_1, X_2, \ldots, X_n]$ of polynomials in non-commuting indeterminates $X_1, X_2, \ldots, X_n$ where for all $1 \leqq i < j \leqq n$,

$$[X_i, X_j] = X_iX_j - X_jX_i \in k[X_1, X_2, \ldots, X_{i-1}]. \tag{1}$$

We are concerned with the following subrings of $k[X_1, X_2, \ldots, X_n]$:

$$S_0 = k,\ S_i = k[X_1, X_2, \ldots, X_i] \quad (1 \leqq i \leqq n).$$

That is, for each $1 \leqq i \leqq n$, S_i is an Ore extension of S_{i-1} by X_i and hence every element of S can be written uniquely as a polynomial in X_i over S_{i-1}.

Let $1 \leqq i \leqq n$. If $a \in S_i$ and $1 \leqq j \leqq n$ then it follows immediately from (1) that $[X_j, a] \in S_{i-1} \subseteq S_i$. An ideal I of S_i is called a *g-ideal* if $[X_j, a] \in I$ for all $a \in I$, $1 \leqq j \leqq n$. The ring S_i is called *g-hypercentral* if whenever $I_1 \supset I_2$ are g-ideals of S_i there exists an element $c \in I_1 \backslash I_2$ such that $[X_j, c] \in I_2$ $(1 \leqq j \leqq n)$.

LEMMA 1.1. *If $i \geqq 1$ and S_{i-1} is g-hypercentral then S_i is g-hypercentral.*

Proof. Let $I_1 \supset I_2$ be g-ideals of S_i. For convenience let T denote S_{i-1} and X denote X_i. Let m be the least positive integer such that

$$I_1 \cap (T + XT + X^2T + \ldots + X^mT) \supset I_2 \cap (T + XT + X^2T + \ldots + X^mT).$$

For $j = 1, 2$ let

$$K_j = \{t \in T \colon X^m t \in T + XT + \ldots + X^{m-1}T + I_j\}.$$

Then it can easily be checked that $K_1 \supset K_2$ are g-ideals of T. By hypothesis there exists c in $K_1 \backslash K_2$ such that $[X_j, c] \in K_2$ $(1 \leqq j \leqq n)$. There exist elements $t_0, t_1, \ldots, t_{m-1}$ of T such that if

$$d = t_0 + Xt_1 + X^2t_2 + \ldots + X^{m-1}t_{m-1} + X^m c$$

then $d \in I_1$. Clearly $d \notin I_2$. If $1 \leqq j \leqq n$ then

$$[X_j, d] \in T + XT + \ldots + X^{m-1}T + X^m[X_j, c]$$

and so $[X_j, d] \in I_2$. The result follows.

COROLLARY 1.2 (see [3, 2.6 and 2.7]). *U is a right and left noetherian polycentral domain.*

Proof. That U is a right and left noetherian domain is well known. By the lemma and induction on i, U is g-hypercentral. Clearly it follows that U is polycentral.

LEMMA 1.3. *Let P be a prime ideal of U, $P_1 = P \cap S_i$ and $P_2 = P \cap S_{i-1}$ for some $1 \leqq i \leqq n$. Then P_1 and S_iP_2 are g-ideals of S_i. Moreover either $P_1 = S_iP_2$ or there exists an element p in $P_1 \backslash S_iP_2$ such that*

(i) $[X_j, p] \in S_iP_2$ $(1 \leqq j \leqq n)$,
(ii) *if $s \in S_i$ and $ps \in S_iP_2$ then $s \in S_iP_2$, and*
(iii) *for all $q \in P_1$ there exists $c \in \mathscr{C}(P)$ such that $qc \in S_iP_2 + S_ip$.*

Proof. That P_1 and S_iP_2 are g-ideals of S_i follows from (1) above. Clearly $P_1 \supseteq S_iP_2$. For convenience let T denote S_{i-1} and X denote X_i. If $P_1 \supset S_iP_2$ then let $m \geqq 1$ be the least positive integer such that

$$P_1 \cap (T + XT + \ldots + X^mT) \supset (S_iP_2) \cap (T + XT + \ldots + X^mT).$$

By the proof of Lemma 1.1 there exist elements $a_0, a_1, a_2, \ldots, a_m$ of T such that the element $p = \sum_{t=0}^{m} X^ta_t$ belongs to $P_1 \backslash S_iP_2$ and satisfies $[X_j, p] \in S_iP_2 = \sum_{t=0}^{\infty} X^tP_2 (1 \leqq j \leqq n)$.

By (1), $[X_j, a_m] \in P_2 \leqq P$ $(1 \leqq i \leqq n)$ and since $a_m \notin P$ it follows that $a_m \in \mathscr{C}(P)$. Suppose that (ii) is false and let $s = \sum_{t=0}^{u} X^ts_t \in S_i$, with $s_t \in T$ $(0 \leqq t \leqq u)$, be of least degree in X over T such that $ps \in S_iP_2$ and $s \notin S_iP_2$. Then $a_ms_u \in P_2$ by (1) and hence $s_u \in P_2$, contradicting the choice of s. Thus (ii) holds.

To establish (iii) let $q = \sum_{t=0}^{v} X^tq_t \in P_1$. If $q \notin S_iP_2$ then $v \geqq m$. It is clear that

$$qa_m - X^{v-m}q_vp \in \sum_{t=0}^{v-1} X^tT.$$

By induction on v there exists $f \in S_i$ such that

$$qa_m^{v-m+1} - fp \in \sum_{t=0}^{m-1} X^tT.$$

By the choice of m, $qa_m^{v-m+1} \in S_iP_2 + S_ip$. Since $a_m \in \mathscr{C}(P)$, (iii) holds and the element p has the properties (i), (ii), (iii).

A prime ideal P of a ring R is *regularly localizable* if there is a chain

$$P = P_0 \supseteq P_1 \supseteq \ldots \supseteq P_m = 0 \tag{2}$$

of ideals of R such that if $0 \leqq i \leqq m-1$ and P_i/P_{i+1} is not generated by a central regular element of R/P_{i+1} then for all $p \in P_i$ there exists $c \in \mathscr{C}(P)$ such that $pc \in P_{i+1}$. The proof of [5, Theorem 2.2] can easily be adapted to show that if P is a regularly localizable prime ideal of a right noetherian ring R then R has the right Ore condition with respect to $\mathscr{C}(P)$.

LEMMA 1.4. *If a prime ideal P of a prime right noetherian ring R is regularly localizable then R_P is a regular local ring.*

Proof. With the above notation the chain (2) gives rise to the chain

$$PR_P = P_0R_P \supseteq P_1R_P \supseteq \ldots \supseteq P_mR_P = 0$$

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of ideals of R_P such that for all $0 \leqq i \leqq m-1$, $P_i R_P = P_{i+1} R_P$ or $P_i R_P / P_{i+1} R_P$ is generated by a central regular element of R_P. It follows that R_P is a regular local ring.

Proof of Theorem A. Let P be a prime ideal of U and consider the chain

$$P = P_0 \supseteq P_1 \supseteq \ldots \supseteq P_n = 0$$

of ideals of U where $P_i = U(P \cap S_{n-i})$. Suppose $0 \leqq i \leqq n-1$ and $P_i \supset P_{i+1}$. Then $P \cap S_{n-i} \supset S_{n-i}(P \cap S_{n-i-1})$. Since U is a free S_{n-i}-module we infer from Lemma 1.3 (i), (ii) that there exists $p \in (P \cap S_{n-i}) \backslash S_{n-i}(P \cap S_{n-i-1})$ such that $p + P_{i+1}$ is a central regular element of U/P_{i+1}. It follows by Lemma 1.3 (iii) that P is regularly localizable. By Lemma 1.4, U_P is a regular local ring and it is clear that the dimension of U_P is at most n.

In our consideration of the ring $k[X_1, X_2, \ldots, X_n]$ the fact that k is a field has not played a prominent role. Indeed the above methods give the following generalization of [**2**, Theorem 171].

THEOREM 1.5. *Let R be a right noetherian super-regular polycentral ring and S be the ring $R[X_1, X_2, \ldots, X_n]$ of polynomials in the non-commuting indeterminates $X_1, X_2, \ldots, X_n$ where $[X_i, X_j] \in R[X_1, X_2, \ldots, X_{i-1}]$ $(1 \leqq i < j \leqq n)$. Then S is a right noetherian super-regular polycentral ring.*

2. Proof of Theorem B. Let R be a commutative noetherian super-regular ring, G be a group and S be the group ring RG. If P is a prime ideal of S then $Q = P \cap R$ is a prime ideal of R. By passing to the group ring $R_Q G$ it is clear that we can suppose that R is a regular local ring with maximal ideal $Q = P \cap R$. It is also clear that, by passing to the group ring $(R/Q)G$, we can suppose in Theorem B that R is a field which we shall denote by k.

Let k be a field and G be a finitely generated group which has a finite normal subgroup H such that G/H is a torsion-free nilpotent group and the order of H is a unit in k. The group ring $S = kG$ is right and left noetherian by methods of P. Hall [**1**, Theorem 1] and is polycentral by [**4**].

Since G/H is a finitely generated torsion-free nilpotent group, there exists a chain

$$H = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = G$$

of normal subgroups F_i of G, such that for each $1 \leqq i \leqq n$, $[F_i, G] \subseteq F_{i-1}$ and F_i/F_{i-1} is an infinite cyclic group. Let S_i denote the group ring kF_i $(0 \leqq i \leqq n)$. An ideal I of S_i is a *G-ideal* if $I^x = x^{-1}Ix = I$ for all x in G. The following analogue of Lemma 1.3 can be proved by adapting and combining the proofs of Lemma 1.3 and [**4**, Lemma 7].

LEMMA 2.1. *Let P be a prime ideal of S, $P_1 = P \cap S_i$ and $P_2 = P \cap S_{i-1}$ for some $1 \leqq i \leqq n$. Then P_1 and $S_i P_2$ are G-ideals of S_i. Moreover either $P_1 = S_i P_2$ or there exists an element p in $P_1 \backslash S_i P_2$ such that*

(i) *$p^y - p \in S_i P_2$ for all y in G,*
(ii) *if $s \in S_i$ and $ps \in S_i P_2$ then $s \in S_i P_2$, and*
(iii) *for all $q \in P_1$ there exists $c \in \mathscr{C}(P)$ such that $qc \in S_i P_2 + S_i p$.*

If A is an ideal of S then we denote by $\mathscr{C}(A)$ the set of elements s of S such that $s + A$ is a regular element of S/A. If P is a prime ideal of S and $A = \{s \in S : sc = 0$ for some c in $\mathscr{C}(P)\}$ then, as we remarked above, $\mathscr{C}(P) \subseteq \mathscr{C}(A)$. It follows that $A = \{s \in S : cs = 0$ for some c in $\mathscr{C}(P)\}$. Since kH is a semisimple artinian ring there exists a unique central idempotent element e of kH which generates $P \cap kH$. From the uniqueness of e it follows that $e^x = e$ for all x in G and hence e is a central element of S. Thus $B = S(P \cap kH)$ is an ideal of S and $B(1-e) = 0$. We have the following lemma.

LEMMA 2.2. *With the above notation* $B \subseteq A$ *and* $\mathscr{C}(B) \subseteq \mathscr{C}(A)$.

Proof. It is clear that $B \subseteq A$. If $c \in \mathscr{C}(B)$ and $x \in S$ satisfy $cx \in A$ then there exists $d \in \mathscr{C}(P)$ such that $cxd = 0$. Therefore $xd \in B \subseteq A$ and hence $x \in A$. The result follows.

Theorem B follows from Lemma 2.1 in roughly the same way as Theorem A followed from Lemma 1.3. Let P be a prime ideal of S. If P is generated by the element e introduced above then certainly P is regularly localizable. Otherwise, since in the above notation S is a free S_i-module $(0 \leqq i \leqq n)$ it follows by Lemmas 2.1 and 2.2 that there exist $i \geqq 1$ and $c \in P \cap S_i$ such that $c + A$ is a central regular element of S/A, where by A we mean the set defined above. It follows by Lemma 2.1 that P is regularly localizable. By Lemma 1.4 S_P is a regular local ring. Thus S is a right and left noetherian super-regular polycentral ring. This completes the proof of Theorem B.

Let R be a commutative noetherian super-regular ring, G be a finitely generated group which has a finite normal subgroup H such that the order of H is a unit in R and G/H is torsion-free nilpotent. If S is the group ring RG, let P be a prime ideal of S and $Q = P \cap R$. Then the proof of Theorem B makes it clear that the dimension of S_P does not exceed the sum of the dimension of R_Q and the Hirsch number of G.

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