# On non-existence of static vacuum black holes with degenerate components of the event horizon


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**Abstract**

We present a simple proof of the non-existence of degenerate components of the event horizon in static, vacuum, regular, four-dimensional black hole spacetimes. We discuss the generalization to higher dimensions and the inclusion of a cosmological constant.


The classical proof of uniqueness of static vacuum black holes [3] assumes that all components of the event horizon are non-degenerate. The argument has been extended[1] to include degenerate components [5] by studying the orbit-space geometry near the event horizon, and applying an appropriate

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[1]A static configuration with all components degenerate is easy to exclude using Komar integrals and the positive energy theorem, see [5, Section 4] for precise statements. However, one also wants to exclude solutions with both degenerate and non-degenerate components.

version of the positive energy theorem [2]. The object of this note is to give an alternative proof of non-existence of degenerate components in four dimensional space-times. Our approach is inspired by the analysis of supersymmetric black holes in [14].

Consider, thus, a four-dimensional static vacuum black hole space-time $(\mathscr{M}, g)$. We shall assume that the regularity hypotheses needed in the *black hole topology theorem* hold, the reader is referred to points (1) and (2) of [7, Theorem 3] for details. It follows that each connected component of the horizon has spherical topology.

We assume, for contradiction, that $(\mathscr{M}, g)$ contains a *degenerate* component $\mathscr{N}$ of the event horizon. By [4] or [16] the static Killing vector field $X$ is tangent to the generators of $\mathscr{N}$. Following [11], we introduce Gaussian null coordinates near $\mathscr{N}$, in which the metric takes the form

$$g = r\varphi dv^2 + 2dvdr + 2rh_a dx^a dv + h_{ab}dx^a dx^b \ . \tag{1}$$

(These coordinates can be introduced for any Killing horizon, not necessarily static, in any number of dimensions). The horizon is given by the equation $\{r = 0\}$. The Killing vector $X$ equals $\partial_v$, with norm

$$g(X, X) = r\varphi \ ,$$

so that the surface gravity equals $\kappa = -\partial_r(r\varphi)$. The degeneracy condition is $\kappa = 0$, hence $\partial_r\varphi$ vanishes on $\mathscr{N}$. It follows that

$$\varphi = Ar$$

for some function $A = A(r, x^a)$.

To simplify the calculations it is convenient to consider the *near horizon geometry* $\mathring{g}$, defined as follows [14]. Let $\varepsilon > 0$ and consider the family of metrics $g^\varepsilon$ defined by replacing $r$ by $\varepsilon r$ and $v$ by $v/\varepsilon$ in (1):

$$g^\varepsilon = r^2 A^\varepsilon dv^2 + 2dvdr + 2rh^\varepsilon_a dx^a dv + h^\varepsilon_{ab}dx^a dx^b \ . \tag{2}$$

with

$$A^\varepsilon = A(\varepsilon r, x^a) \ , \quad \text{etc} \ .$$

Clearly the $g^\varepsilon$'s converge, as $\varepsilon$ tends to zero, to a metric $\mathring{g}$ of the form

$$\mathring{g} = r^2 \mathring{A} dv^2 + 2dvdr + 2r\mathring{h}_a dx^a dv + \mathring{h}_{ab}dx^a dx^b \ , \tag{3}$$

$$\partial_r\mathring{A} = \partial_r\mathring{h}_a = \partial_r\mathring{h}_{ab} = \partial_v\mathring{A} = \partial_v\mathring{h}_a = \partial_v\mathring{h}_{ab} = 0 \ . \tag{4}$$

We have $\mathring{h}_{ab} = h_{ab}|_{r=0}$, $\mathring{h}_a = h_a|_{r=0}$, $\mathring{A} = A|_{r=0}$, so that $\mathring{g}$ encodes information about the values of $h_{ab}$, $h_a$ and $A$ at $\mathscr{N}$.

The vacuum Einstein equations imply (see [11, eq. (2.9)] in dimension four and [12, eq. (5.9)] in higher dimensions)

$$\mathring{R}_{ab} = \frac{1}{2}\mathring{h}_a\mathring{h}_b - \mathring{D}_{(a}\mathring{h}_{b)} \ , \tag{5}$$

where $\mathring{R}_{ab}$ is the Ricci tensor of $\mathring{h}_{ab}$, and $\mathring{D}$ is the covariant derivative thereof. They also determine $\mathring{A}$ uniquely in terms of $\mathring{h}$ and $\mathring{h}_{ab}$ (this equations follows again e.g. from [11, eq. (2.9)] in dimension four, and can be checked by a calculation in all higher dimensions):

$$\mathring{A} = \frac{1}{2}\mathring{h}^{ab}\left(\mathring{h}_a\mathring{h}_b - \mathring{D}_a\mathring{h}_b\right) . \tag{6}$$

We should note that, so far, our analysis applies to any degenerate Killing horizon (not necessarily static) in four or more dimensions. Equation (5) also arises in the study of vacuum degenerate isolated horizons [1, 10, 12, 13].

In the static case of interest here, note that staticity of $g$ implies staticity of $\mathring{g}$. If we let

$$X^\flat := \mathring{g}_{\mu\nu}X^\nu dx^\mu = \mathring{g}_{\mu v}dx^\mu = r^2\mathring{A}dv + dr + r\mathring{h}_a dx^a \ ,$$

then the staticity condition $X^\flat \wedge dX^\flat = 0$ leads to

$$\begin{aligned} 0 &= (r^2\mathring{A}dv + dr + r\mathring{h}_a dx^a) \wedge (d(r^2\mathring{A}dv) + dr \wedge (\mathring{h}_a dx^a) + rd(\mathring{h}_a dx^a)) \\ &= rdr \wedge d(\mathring{h}_a dx^a) + O(r^2) \ , \end{aligned}$$

implying $d(\mathring{h}_a dx^a) = 0$. We now return to four dimensions. Simple connectedness of $S^2$ guarantees the existence of a function $\lambda$ such that

$$\mathring{h}_a dx^a = d\lambda \ . \tag{7}$$

Equation (5) can thus be rewritten as

$$\mathring{R}_{ab} = \frac{1}{2}\mathring{D}_a\lambda\mathring{D}_b\lambda - \mathring{D}_a\mathring{D}_b\lambda \ . \tag{8}$$

Set $\psi = e^{-\lambda/2}$, then

$$\psi\mathring{R}_{ab} = 2\mathring{D}_a\mathring{D}_b\psi \ , \tag{9}$$

and taking a trace gives

$$2\Delta_{\mathring{h}}\psi = \mathring{R}\psi \ . \tag{10}$$

In dimension two we have $\mathring{R}_{ab} = \mathring{R}\mathring{h}_{ab}/2$; inserting this into (9), applying $\mathring{D}^a$ to the resulting equation, and commuting derivatives one obtains[2]

$$\mathring{D}_b(\mathring{R}\psi^3) = 0 \ . \tag{11}$$

It follows that $\mathring{R}\psi^3$ has constant sign or vanishes, so that $\mathring{R}$ has constant sign or vanishes since $\psi$ is strictly positive by definition. On a compact manifold this is compatible with (10) only if $\mathring{R}\psi = 0$ and $\psi$ is a constant, thus $\lambda$ is constant and (8) shows that $\mathring{h}$ is flat. This gives a contradiction, as there are no flat metrics on $S^2$ by the Gauss-Bonnet theorem. Hence, no degenerate components are possible, as claimed.

Recall that the Curzon-Scott-Szekeres [15] black holes provide examples of vacuum static black holes with flat degenerate horizons. However, those space-times are nakedly singular, so that the topology theorem does not apply. We also note that while the near horizon geometry of those black holes is compatible with our result, this fact can *not* be deduced from our analysis, as the horizon there does not have compact cross-sections.

Some comments about higher dimensions are in order. First, the proof in [5] generalises immediately to any space-time dimension greater than or equal to four. On the other hand, the hypothesis of space-time dimension four was essential in several steps of the current argument so in higher dimensions we must proceed differently. Without the approach in [5], there is no reason to expect the horizon to be simply connected, so that a globally defined potential $\lambda$ might fail to exist. Assuming, first, that $\lambda$ in (7) exists, one finds again (8). Taking a divergence of (9) and using the contracted Bianchi identity $\mathring{D}_a\mathring{R}^a{}_b = \mathring{D}_b\mathring{R}/2$ one obtains

$$\mathring{D}_a(|\mathring{D}\psi|^2 + \frac{1}{2}\mathring{R}\psi^2) = 0 \ . \tag{12}$$

Hence there exists a constant $C$ such that

$$\mathring{R} = \psi^{-2}(C - 2|\mathring{D}\psi|^2) \ .$$

Inserting this into (10) one is led to

$$\Delta_{\mathring{h}}\psi^2 = C \ , \tag{13}$$

which is possible on a compact manifold if and only if $C$ vanishes, $\psi$ is constant, and then $\mathring{h}_{ab}$ is Ricci flat by (9).

[2]This is a special case of a result for stationary degenerate horizons obtained in [10].

In general, let $\{\mathscr{O}_i\}_{i\in I}$ be an open cover by simply connected sets, with associated potentials $\lambda_i$ and $\psi_i$. Equation (12) shows that there exist constants $C_i$ such that on each $\mathscr{O}_i$ we can write

$$\mathring{R} = C_i\psi_i^{-2} - 2|\mathring{D}\ln(\psi_i)|^2 = C_i\psi_i^{-2} - \frac{1}{2}\mathring{h}^{ab}\mathring{h}_a\mathring{h}_b \; . \tag{14}$$

It follows that on each intersection $\mathscr{O}_i \cap \mathscr{O}_j$ we have

$$C_i\psi_i^{-2} = C_j\psi_j^{-2} \; , \tag{15}$$

which implies that all the $C_i$'s have the same sign.

Suppose that there exists $i_0$ such that $C_{i_0} \neq 0$, then $C_i \neq 0$ for all $i$ by (15). The $\lambda_i$'s are defined up to the addition of a constant, which implies that the $\psi_i$'s are defined up to a multiplicative constant, and by rescaling we can obtain either $C_i = 1$ for all $i$, or $C_i = -1$ for all $i$. It then follows from (15) that $\psi_i = \psi_j$ on each intersection, i.e., $\psi$ is globally defined after all. But then the previous argument shows $C = 0$, hence $C_i = 0$ for all $i$, a contradiction.

So in fact all the $C_i$'s vanish and, by (14),

$$\mathring{R} = -\frac{1}{2}\mathring{h}^{ab}\mathring{h}_a\mathring{h}_b \; . \tag{16}$$

But the trace of (5) gives $\mathring{R} = \mathring{h}^{ab}\mathring{h}_a\mathring{h}_b/2 + \mathring{D}^a\mathring{h}_a$, which upon integration on a cross-section of the horizon gives

$$\int \mathring{R} = \frac{1}{2}\int \mathring{h}^{ab}\mathring{h}_a\mathring{h}_b \geq 0 \; .$$

This is compatible with (16) if and only if $\mathring{h}_a = 0$.

Thus, we have shown, in all space-time dimensions, that *static, degenerate, vacuum Killing horizons with compact spacelike sections have vanishing scalar* $\mathring{A}$ *and rotation form* $\mathring{h}_a dx^a$ *and are spatially Ricci flat*[3]. The near-horizon geometry is the product of flat space with a compact Ricci flat space. This is not known to lead to a contradiction with asymptotic flatness except in space-time dimension four (compare [8]), unless one invokes the arguments in [5], which we wanted to sidestep to start with.

It turns out that one can derive the local form of the metric $\mathring{h}_{ab}$ in space-time dimension four, when compactness of the space-sections of the horizon

[3]Space-time dimension three is covered by applying the four dimensional result to $M \times S^1$ with the product metric.

is not assumed. In four dimensions, equations (11) and (12) establish that, regardless of compactness, there exist constants $\alpha$ and $\beta$ such that

$$|\mathring{D}\psi|^2 = \alpha + \frac{\beta}{\psi} \ , \qquad \Delta_{\mathring{h}}\psi = -\frac{\beta}{\psi^2} \ .$$

Assume $d\psi \not\equiv 0$, then on any open set on which $d\psi$ has no zeros the metric can be written in the form

$$\mathring{h}_{ab}dx^a dx^b = \frac{d\psi^2}{|\mathring{D}\psi|^2} + H(\varphi,\psi)d\varphi^2 \ .$$

Calculating $\Delta_{\mathring{h}}\psi$ one finds $H = \gamma(\varphi)|\mathring{D}\psi|^2$ for some function $\gamma$. Redefining $\varphi$ one obtains, locally

$$\mathring{h}_{ab}dx^a dx^b = \frac{d\psi^2}{\alpha + \frac{\beta}{\psi}} + (\alpha + \frac{\beta}{\psi})d\varphi^2 \ . \tag{17}$$

It is straightforward now to check that (9) holds for all $\alpha^2 + \beta^2 \neq 0$.

Now, the full near-horizon metric $\mathring{g}$ is closely related to a generalised Schwarzschild solution. To see this, note that we can use the freedom to rescale $\psi$ and $\phi$ to arrange $\alpha = k \in \{1, 0, -1\}$. We shall also introduce the suggestive notation $\beta = -2M$, and consider the following metric:

$$ds^2 = -U(R)dt^2 + U(R)^{-1}dR^2 + R^2 d\Sigma_k^2, \tag{18}$$

where $U(R) = k - 2M/R$ and $d\Sigma_k^2$ is a two-dimensional Riemannian metric with Ricci scalar $2k$. Setting $R = \psi$ and $t = i\varphi$, the first two terms in the metric reproduce the local solution for $\mathring{h}_{ab}$ obtained above. The full near-horizon geometry $\mathring{g}$ is obtained by a further analytic continuation in which $d\Sigma_k^2$ becomes a *Lorentzian* metric of Ricci scalar $2k$, i.e., two-dimensional de Sitter, Minkowski or anti-de Sitter space-time for $k = 1, 0, -1$ respectively. The degenerate horizon corresponds to a Killing horizon in this two-dimensional space-time. Note that $\mathring{h}_{ab}$ is singular except in the trivial (flat[4]) case $k = 1, M = 0$ and the case $k = 1, M > 0$ in which it describes the familiar "cigar" geometry of the "Euclideanized" Schwarzschild solution.

Our analysis can be extended to include a cosmological constant $\Lambda$. This produces additional terms $\Lambda\mathring{h}_{ab}$ and $\Lambda$ on the right-hand-sides of (5) and (6) respectively. Then

[4]The corresponding space-time metric (1) is also flat, with a singularity at $\psi = 0$ which can be gotten rid of by a coordinate transformation.

- For spatially compact horizons, in all space-time dimensions greater than or equal to four, for negative $\Lambda$ we again obtain the existence of a globally defined potential $\psi$. Applying a maximum principle to the $\Lambda$-analogue of (13) one finds that $\psi$ is constant and $\mathring{R}_{ab} = \Lambda \mathring{h}_{ab}$. The near-horizon geometry $\mathring{g}$ is the product of two-dimensional anti-de Sitter space with a compact Einstein space of negative curvature.

- In space-time dimension four, whatever $\Lambda$,

  - if $\psi$ is not constant, we obtain

$$|\mathring{D}\psi|^2 = \alpha + \frac{\beta}{\psi} - \frac{\Lambda}{3}\psi^2 =: F(\psi) \qquad (19)$$

    and, locally,

$$\mathring{h}_{ab}dx^a dx^b = \frac{d\psi^2}{\alpha + \frac{\beta}{\psi} - \frac{\Lambda}{3}\psi^2} + \Big(\alpha + \frac{\beta}{\psi} - \frac{\Lambda}{3}\psi^2\Big)d\varphi^2 \ . \qquad (20)$$

    The near-horizon geometry $\mathring{g}$ is an analytically continued version of the $\Lambda$-generalized Schwarzschild solution (equation (18) with $U(R) = k - 2M/R - \Lambda R^2/3$).
    Assuming compactness of the cross-section, the strictly positive function $\psi$ has at least one maximum and at least one distinct minimum so there exist $0 < a < b$ such that $F(a) = F(b) = 0$; for $\Lambda > 0$ this enforces $\beta < 0$. Regularity at the zeros of $F$ (see [6, end of Section 2] for a careful treatment of a similar problem, or the arguments around (3.6) in [9]) imposes

$$F'(a) + F'(b) = 0 \ .$$

    Elementary algebra leads to $a = b$, a contradiction. Therefore, in the case of positive $\Lambda$ as well, horizons with non-trivial $\psi$ and with compact cross-sections do not exist.
  - if instead $\mathring{h}_a \equiv 0$, then $\mathring{g}_{\mu\nu}$ is locally a product of two-dimensional de Sitter space with $S^2$ (if $\Lambda > 0$) or two-dimensional anti-de Sitter space with hyperbolic space (if $\Lambda < 0$).

We conclude that, whatever $\Lambda \in \mathbb{R}$, *static, vacuum, four dimensional solutions with a degenerate Killing horizon with compact cross-sections have vanishing rotation one-form $\mathring{h}_a dx^a$, and $\mathring{A} = \Lambda$, with $\mathring{h}_{ab}dx^a dx^b$ of constant scalar curvature $2\Lambda$.*

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