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# On non-triangular sets in tensor algebras

by

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For an arbitrary regular symmetric Banach algebra $R(K)$ of continuous functions on a compact Hausdorff space $K$ and an arbitrary closed subset $E$ of $K$ we denote

$$I(E) = \{f; f\in R(K),\ f \text{ vanishes on } E\},$$
$$I_0(E) = \{f; f\in R(K),\ f \text{ vanishes on a neighbourhood of } E\}.$$

It is easy to see that $I(E)$ is a closed ideal of $R(K)$ and that $I_0(E)$ is an ideal in $R(K)$. The subset $E$ is said to be *of synthesis* if $\overline{I_0(E)} = I(E)$ (closure in $R(K)$) and $E$ is said to be a *strong Dytkin set* if there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \in I_0(E)$ $(n = 1, 2, \ldots)$ and for every $f\in I(E)$ we have $\tau_n f \to f$ as $n\to\infty$ for the norm of $R(K)$. Every strong Dytkin set is clearly a set of synthesis. Together the following conditions imply that $E$ is a strong Dytkin set:

1) $E$ is of synthesis;

2) there exist open sets $\Omega_n$ containing $E$ such that

$$\Omega_{n+1} \subseteq \Omega_n \quad \text{for} \quad n = 1, 2, \ldots \qquad \text{and} \qquad \bigcap_{n=1}^{\infty} \overline{\Omega}_n = E;$$

3) there exists a sequence $\{u_n\}_{n=1}^{\infty}$ with $1-u_n \in I_0(E)$, $n = 1, 2, \ldots$, satisfying the two conditions

$$(+)\qquad \begin{gathered} u_n(x) = 0 \quad \text{for all } x \notin \Omega_n, \\ \|u_n\|_{R(K)} \leqslant 1+\varepsilon_n, \end{gathered}$$

where $\{\varepsilon_n\}_{n=1}^{\infty}$ is a sequence decreasing to zero. We observe that these conditions tend to bear on the case $K$ metrizable.

To see this we take $\tau_n = 1-u_n$. Let $f\in I(E)$ and $\varepsilon > 0$ be arbitrary. By 1) there exists $g \in I_0(E)$ such that

$$\|f-g\|_R \leqslant \varepsilon.$$

By 2) there exists $N$ such that $g$ vanishes on $\Omega_n$ for $n \geqslant N$. We have

$$\tau_n f - f = \tau_n(f-g) - u_n g - (f-g)$$

and
$$\|\tau_n f - f\|_R \leqslant (1+\|\tau_n\|_R)\varepsilon \qquad (n \geqslant N)$$
since, by 3), $u_n g$ is identically zero $(n \geqslant N)$. Our claim follows since $\|\tau_n\|_R$ is bounded. It is for further aims that we stipulate $(+)$ in 3).

The following are examples of regular symmetric algebras:

A) All continuous functions $C(K)$ on a compact metrizable space $K$.

B) Absolutely convergent Fourier series $A(G)$ on a compact abelian metrizable group $G$. We denote by $\hat{G}$ the dual group of $G$ and by $G_d$ the group $G$ furnished with the discrete topology.

C) The tensor algebra $V(K_1\times K_2) = C(K_1)\hat{\otimes} C(K_2)$, where $K_1, K_2$ will always denote compact metrizable spaces. A theory of this algebra can be found in Varopoulos [2].

In this paper we shall be concerned with the examples B) and C). A closed subset $E$ of $K_1\times K_2$ is said to be *non-triangular* if for all $\Lambda_j \subseteq K_j$ such that $\operatorname{card}(\Lambda_j) = 2$ $(j = 1, 2)$ we have $\operatorname{card}\big(E \cap (\Lambda_1\times\Lambda_2)\big) \neq 3$. The set $\{0_G\}$ satisfies conditions 1), 2) and 3) for the algebra $A(G)$; it is the main object of this paper to use this result to show that every non-triangular set satisfies 1), 2) and 3) with respect to a tensor algebra and hence is a strong Dytkin set.

If $K$ is a compact Hausdorff space, we shall denote by $M(K)$ the space of bounded complex regular Borel measures on $K$ and by $M^+(K)$ the subset of such positive measures.

The reader should observe that a non-triangular subset $E$ of $D_1\times D_2$, the product of discrete spaces $D_1, D_2$, is the union of rectangles $X_\alpha\times Y_\alpha$ $(X_\alpha \subseteq D_1,\ Y_\alpha \subseteq D_2)$ with pairwise disjoint sides $(X_\alpha \cap X_\beta = Y_\alpha \cap Y_\beta = \emptyset,\ \alpha \neq \beta)$. We now prove the analogous result for compact metrizable spaces.

LEMMA 1. *Let $E \subseteq K_1\times K_2$ be a non-triangular closed subset. Then there exist a compact metrizable space $Q$ and continuous mappings $a_j\colon K_j \to Q$ $(j = 1, 2)$ such that*
$$E = \{(x_1, x_2);\ a_1(x_1) = a_2(x_2)\}.$$

Proof. We define an equivalence relation $\sim$ on $K_1 \cup K_2$ (the disjoint union of $K_1$ and $K_2$) as follows:

If $x \in K_1, y \in K_2$, then $x \sim y$ if and only if $(x, y) \in E$.

If $x_1, x_2 \in K_1$, then $x_1 \sim x_2$ if and only if either $x_1 = x_2$ or there exists $y \in K_2$ such that $(x_1, y)$ and $(x_2, y) \in E$.

If $y_1, y_2 \in K_2$, then $y_1 \sim y_2$ if and only if either $y_1 = y_2$ or there exists $x \in K_1$ such that $(x, y_1)$ and $(x, y_2) \in E$.

The relation $\sim$ is clearly reflexive and symmetric. We show that $\sim$ is transitive. There are essentially 3 cases.



A) $x_1, x_2, x_3 \in K_1$ (identical argument for $K_2$) and $x_1 \sim x_2$, $x_2 \sim x_3$. There exist $y_1, y_2 \in K_2$ such that $(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_3, y_2) \in E$. If $y_1 = y_2$, then $x_1 \sim x_3$. If $y_1 \neq y_2$, then $(x_3, y_1) \in E$ since already $(x_2, y_1)$, $(x_2, y_2), (x_3, y_2) \in E$ and $E$ is non-triangular. Hence $x_1 \sim x_3$.

B) $y \in K_2$, $x_1, x_2 \in K_1$ (or vice-versa) and $y \sim x_1$, $x_1 \sim x_2$. There exists $y_0 \in K_2$ such that $(x_1, y_0), (x_2, y_0) \in E$. If $y = y_0$, then clearly $y \sim x_2$. If $y \neq y_0$, then $(x_2, y) \in E$ since already $(x_2, y_0), (x_1, y), (x_1, y_0) \in E$. Hence $x_2 \sim y$.

C) $y \in K_2$, $x_1, x_2 \in K_1$ (or vice-versa) and $x_1 \sim y$, $y \sim x_2$. Clearly $(x_1, y), (x_2, y) \in E$. Hence $x_1 \sim x_2$.

Next we show that the $\sim$-saturation of any closed subset of $K_1 \cup K_2$ is closed. Let $\pi_j$ denote the projection of $K_1\times K_2$ onto $K_j$ for $j = 1, 2$. For $L \subseteq K_1$ we define
$$\sigma_1(L) = \pi_2\big((L\times K_2) \cap E\big) \subseteq K_2$$
and $\sigma_2$ is defined similarly. If $L$ is closed in $K_1$, then we observe that $\sigma_1(L)$ is closed in $K_2$. Let $M$ be an arbitrary closed subset of $K_1 \cup K_2$. Then $M = M_1 \cup M_2$, where $M_j \subseteq K_j$ $(j = 1, 2)$ is closed. We observe that saturation $(M) = M_1 \cup M_2 \cup \sigma_1(M_1) \cup \sigma_2(M_2) \cup \sigma_2\circ\sigma_1(M_1) \cup \sigma_1\circ\sigma_2(M_2)$ is closed. Let $g\colon K_1 \cup K_2 \to Q$ be the canonical projection associated with $\sim$. Since $\sim$-saturation preserves closedness and $K_1 \cup K_2$ is a normal space, we see that the projection $g$ is Hausdorff and that $Q$ is a compact metrizable space in the quotient topology. It is an immediate consequence of the definition of $\sim$ that
$$E = \{(x_1, x_2);\ a_1(x_1) = a_2(x_2)\}, \quad \text{where } a_j = g|_{K_j} \quad (j = 1, 2).$$

If we write $Q_j = a_j(K_j)$ $(j = 1, 2)$, $Q = Q_1 \cup Q_2$, $P = Q_1 \cap Q_2$, then we have $E = (a_1\times a_2)^{-1}(\Delta)$, where $\Delta$ denotes the diagonal of $P\times P$ considered as a subset of $Q_1\times Q_2$.

Now let us explain how a non-triangular set $E$ satisfies conditions 2) and 3). Since $Q$ is a compact metrizable space, it can be embedded in $T_\infty$ — the torus of countable infinite dimension. We consider the mapping $\varrho\colon K_1\times K_2 \to T_\infty$ given by
$$\varrho(x_1, x_2) = a_1(x_1) - a_2(x_2),$$
where the subtraction takes place relative to the group structure of $T_\infty$. There exist open sets $\Sigma_n \subseteq T_\infty$ such that $0 \in \Sigma_n$, $\Sigma_{n+1} \subseteq \Sigma_n$ for $n = 1, 2, \ldots$ and $\bigcap_{n=1}^{\infty} \bar{\Sigma}_n = \{0\}$ and also functions $v_n \in A(T_\infty)$ such that $1 - v_n \in I_0(\{0\})$ $(n = 1, 2, \ldots)$, $v_n(x) = 0$ for all $x \notin \Sigma_n$ $(n = 1, 2, \ldots)$ and $\|v_n\|_A \leqslant 1 + \varepsilon_n$, where $\varepsilon_n$ is a sequence of positive numbers decreasing to zero. We define $\Omega_n = \varrho^{-1}(\Sigma_n)$ $(n = 1, 2, \ldots)$ open sets in $K_1\times K_2$ satisfying condition 2) with respect to $E$. We also set $u_n = v_n \circ \varrho$ $(\in C(K_1\times K_2))$ functions

taking the value 1 in a neighbourhood of $E$ and vanishing outside $\Omega_n$ $(n = 1, 2, \ldots)$. To show that 3) is satisfied it remains to show that the mapping

$$v \to v \circ \varrho$$

is norm decreasing between the spaces $A(T_\infty)$ and $V(K_1 \times K_2)$. Let $\chi \in \hat{T}_\infty$. We observe that

$$\chi \circ \varrho = (\chi \circ a_1) \otimes (\bar{\chi} \circ a_2),$$

where $\chi \circ a_1$ and $\bar{\chi} \circ a_2$ are functions of unit modulus on $K_1, K_2$ respectively. Extending by linearity and continuity we have the result.

THEOREM 1. *Every non-triangular set satisfies conditions 2) and 3) of the introduction with respect to the tensor algebra.*

Suppose now that $G$ is a compact abelian group and that $K_1, K_2$ are two disjoint compact metrizable subsets of $G$ such that $K_1 \cup K_2$ is a Kronecker set. It is well known (Varopoulos [2], 4, § 2) that the restriction algebra $A(K_1 + K_2)$ can be identified isometrically with $V(K_1 \times K_2)$ by means of the dual of the multiplication mapping $\sigma$: $K_1 \times K_2 \to K_1 + K_2$. Let $E$ be a closed subset of $K_1 \times K_2$ and let $\tilde{E} = \sigma(E)$ be the corresponding set in $K_1 + K_2$.

THEOREM 2. *The set $E$ is non-triangular if and only if $\tilde{E} = (K_1 + K_2) \cap (g + H)$ for some $g \in G$ and some algebraic subgroup $H$ of $G$.*

Proof. Suppose the latter statement holds. Let $x_1, x_2 \in K_1$, $y_1, y_2 \in K_2$ such that $x_1 \neq x_2$, $y_1 \neq y_2$ and $(x_1, y_1), (x_1, y_2), (x_2, y_1) \in E$. Since $x_1 + y_1$, $x_1 + y_2$, $x_2 + y_1$ belong to $\tilde{E} = (K_1 + K_2) \cap (g + H)$, we can write $x_1 + y_1 = g + h_1$, $x_1 + y_2 = g + h_2$, $x_2 + y_1 = g + h_3$ where $h_1, h_2, h_3 \in H$. Hence $x_2 + y_2 = g + (h_2 + h_3 - h_1)$ and it follows that $x_2 + y_2 \in (K_1 + K_2) \cap (g + H)$ and that $(x_2, y_2) \in E$. This shows that $E$ is non-triangular.

Suppose now that $E$ is non-triangular and that $\{u_n\}_{n=1}^{\infty}$ is the sequence constructed in Theorem 1. Regarding $u_n$ as elements of $A(K_1 + K_2)$ we choose extensions $\tilde{\omega}_n$ to $A(G)$ such that

$$\|\tilde{\omega}_n\|_{A(G)} \leqslant 1 + 2\varepsilon_n, \quad \tilde{\omega}_n|_{K_1 + K_2} = u_n.$$

On account of the fact that there exists $g \in G$ such that $\tilde{\omega}_n(g) = 1$ the Fourier coefficients of $\tilde{\omega}_n$ are very well aligned; we shall perturb the $\tilde{\omega}_n$ very slightly so as to make the alignment perfect. Towards this let $\omega \in A(G)$ be such that $\|\omega\|_{A(G)} \leqslant 1 + 2\varepsilon$ and $\omega(0) = 1$. We have

$$\sum_{\chi \in \hat{G}} \hat{\omega}(\chi) = 1 \quad \text{and} \quad \sum_{\chi \in \hat{G}} |\hat{\omega}(\chi)| \leqslant 1 + 2\varepsilon.$$



Let $\hat{\lambda}(\chi) = |\hat{\omega}(\chi)|$ and define $\theta_\chi = \arg[\hat{\omega}(\chi)]$. Then

$$\|\lambda - \omega\|_{A(G)} = 2^{1/2} \sum_{\chi \in \hat{G}} \hat{\lambda}(\chi)(1 - \cos\theta_\chi)^{1/2}$$
$$\leqslant 2^{1/2} \Big(\sum_{\chi \in \hat{G}} \hat{\lambda}(\chi)\Big)^{1/2} \Big(\sum_{\chi \in \hat{G}} \hat{\lambda}(\chi)(1 - \cos\theta_\chi)\Big)^{1/2} \leqslant 2\varepsilon^{1/2}(1 + 2\varepsilon)^{1/2}.$$

Let $\omega_n(x) = \tilde{\omega}_n(x + g)$ and define $\lambda_n$ by the method indicated above. We can regard $\hat{\lambda}_n$ as belonging to $M^+([G_d]^\wedge)$ with $\|\hat{\lambda}_n\|_M \leqslant 1 + 2\varepsilon_n$ which bound decreases to 1 as $n \to \infty$. By the weak compactness of the unit ball of $M([G_d]^\wedge)$ the sequence $\{\hat{\lambda}_n\}_{n=1}^{\infty}$ has a weak limit point $\hat{\lambda} \in M^+([G_d]^\wedge)$ such that $\|\hat{\lambda}\|_M \leqslant 1$. Since $\|\omega_n - \lambda_n\|_{A(G)} \leqslant 2\varepsilon_n^{1/2}(1 + 2\varepsilon_n)^{1/2}$ tends to zero as $n \to \infty$, we see that $\hat{\lambda}$ is also a weak limit point of the sequence $\{\hat{\omega}_n\}_{n=1}^{\infty}$. The Fourier transform $\lambda$ of $\hat{\lambda}$ can be identified with a bounded function on $G_d$. We claim that $H = \{h; h \in G, \lambda(h) = 1\}$ is an algebraic subgroup of $G$ on account of the implications

$$\lambda(h) = 1 \Leftrightarrow \int_{[G_d]^\wedge} \langle h, \chi\rangle d\hat{\lambda}(\chi) = 1 \Leftrightarrow \langle h, \chi\rangle = 1 \quad \hat{\lambda}\text{-a.e.}$$

But $\lambda(k)$ is a limit point of $\{\omega_n(k)\}_{n=1}^{\infty}$ and hence also of $\{u_n(g + k)\}_{n=1}^{\infty}$. Given that $g + k \in K_1 + K_2$ we shall have $g + k \in E$ if and only if $k \in H$. Hence $\tilde{E} = (K_1 + K_2) \cap (g + H)$. This completes the proof.

COROLLARY. *Conditions 2) and 3) characterize non-triangular sets.*

This follows from the proof of Theorem 2.

In the remainder of the paper we discuss condition 1) that is the synthesis of non-triangular sets. We denote by $\mathrm{BM}(K_1 \times K_2) = [V(K_1 \times K_2)]'$ the dual space of $V(K_1 \times K_2)$ whose elements are called *bimeasures*. For $E$ a closed subset of $K_1 \times K_2$ we define the space $\mathrm{BM}(E)$ of bimeasures supported on $E$ as the annihilator $[I_0(E)]^0$ of the ideal $I_0(E)$. The set $E$ has the *unit bounded synthesis property* if for every $S \in \mathrm{BM}(E)$ there exists a sequence $\{\mu_n\}_{n=1}^{\infty}$

$$\mu_n \in M(E), \quad \|\mu_n\|_{\mathrm{BM}} \leqslant \|S\|_{\mathrm{BM}} \quad (n \geqslant 1)$$

with $\mu_n \to S$ for the weak topology $\sigma(\mathrm{BM}, V)$. Such a set is evidently a set of synthesis. We aim to show that non-triangular sets have the unit bounded synthesis property. We shall need the following standard lemma:

LEMMA 2. *Let $L_1$ be closed in $K_1$ and $E$ be closed in $L_1 \times K_2$. The two spaces $\mathrm{BM}_{L_1 \times K_2}(E)$ and $\mathrm{BM}_{K_1 \times K_2}(E)$ of bimeasures supported on $E$ defined with reference to the two tensor algebras $V(L_1 \times K_2)$ and $V(K_1 \times K_2)$ are isometrically identified and the two corresponding weak topologies on them coincide.*

We start by considering those non-triangular sets for which the projection $a_2$: $K_2 \to Q_2$ is identical. This is the case in which each ordinate $\{k_1\} \times K_2$ $(k_1 \in K_1)$ cuts the set $E$ in at most one point. Such a non-triangular set will be called a *graph*.

THEOREM 3. *For every graph $E$ the inclusion $M(E) \subsetneq \mathrm{BM}(E)$ is an isometric identification.*

Proof. We embed $K_2$ into a compact abelian metrizable group $G$. We write $K_1' = \alpha_1^{-1}(Q_1 \cap Q_2)$. Hence $E \subseteq K_1' \times K_2$ and by Lemma 2 it suffices to prove the result with respect to the algebra $V(K_1' \times G)$. The set $E$ is given by

$$E = \mathrm{graph}(\alpha) = \{(k_1, \alpha(k_1));\ k_1 \in K_1'\},$$

where $\alpha\colon K_1' \to G$ is the restriction of $\alpha_1$ to $K_1'$. We shall need the following mappings:

$$\pi\colon K_1' \times G \to K_1', \qquad \pi(k, g) = k,$$
$$i\colon K_1' \to E, \qquad i(k) = (k, \alpha(k)),$$
$$\sigma\colon K_1' \times G \to G, \qquad \sigma(k, g) = g - \alpha(k),$$

where $-$ is taken in the group $G$. The significance of $\sigma$ is that the dual mapping

$$\sigma^*\colon A(G) \to V(K_1' \times G)$$

is norm decreasing. By extension by linearity and continuity it suffices to check this on an arbitrary character $\chi \in \hat{G}$:

$$[\sigma^*(\chi)](k, g) = \chi(g - \alpha(k)) = \chi(g) \cdot \overline{\chi \circ \alpha(k)} = [\overline{\chi \circ \alpha} \otimes \chi](k, g).$$

For an arbitrary $S \in \mathrm{BM}(E)$ we have $\check{\pi}(S) \in M(K_1')$ and $\mu = \check{i} \circ \check{\pi}(S) \in M(E)$ where $\check{\pi}$ and $\check{i}$ are the norm decreasing bidual mappings of $\pi$ and $i$:

$$\check{\pi}\colon \mathrm{BM}(K_1' \times G) \to M(K_1'), \qquad \check{i}\colon M(K_1') \to M(E).$$

It suffices to show that $S = \mu$. We observe first that $\check{\pi}(S) = \check{\pi}(\mu)$ since $\pi \circ i = 1_{K_1'}$. Let $f \in C(K_1')$ and $\chi \in \hat{G}$ be arbitrary elements. We have

$$\begin{aligned}[f \otimes \chi - (f \cdot (\chi \circ \alpha) \otimes 1_G)](k, g) &= f(k)[\chi(g) - \chi \circ \alpha(k)] \\ &= f(k) \cdot \chi \circ \alpha(k) \cdot [\chi(g - \alpha(k)) - 1] \\ &= [(f \cdot (\chi \circ \alpha) \otimes 1_G) \cdot (\sigma^*(\chi - 1_G))](k, g).\end{aligned}$$

Now $\chi - 1_G$ vanishes on $\{0_G\}$ a set of synthesis for $A(G)$. Hence we can find functions $\varphi_n \in A(G)$ vanishing on a neighbourhood of $0_G$ and with $\varphi_n \to \chi - 1_G$ in $A(G)$. The functions $\sigma^*(\varphi_n)$ vanish on a neighbourhood of $E$ and tend to $\sigma^*(\chi - 1_G)$ in $V(K_1' \times G)$. Hence

$$\langle S - \mu, [(f \cdot (\chi \circ \alpha) \otimes 1_G) \cdot (\sigma^*(\chi - 1_G))] \rangle = 0.$$

Also we have

$$\langle S - \mu, (f \cdot (\chi \circ \alpha) \otimes 1_G) \rangle = \langle \check{\pi}(S - \mu), f \cdot (\chi \circ \alpha) \rangle = 0.$$

Therefore $\langle S - \mu, f \otimes \chi \rangle = 0$. Extending by linearity and continuity and using the fact that trigonometric polynomials are uniformly dense in $C(G)$ we see that $S \doteq \mu$.



Let $K$ be a compact metrizable space. We shall denote by $\check{K}$ the space of continuous mappings of $K$ into $T$. $\check{K}$ is a group under pointwise multiplication on $K$ and with the discrete topology. The dual group of $\check{K}$ is denoted by $K^f$. There is a natural topological embedding $i_K$ of $K$ in $K^f$. A continuous surjection $\alpha\colon K \to Q$ between compact metrizable spaces $K$ and $Q$ defines a dual mapping $\alpha^*\colon \check{Q} \to \check{K}$ a group monomorphism (an embedding) and a bidual mapping $\alpha^f\colon K^f \to Q^f$ a continuous surjective group homomorphism with the property $\alpha^f \circ i_K = i_Q \circ \alpha$.

LEMMA 3. *Let $\alpha\colon K \to Q$ be a continuous surjection between compact metrizable spaces $K$ and $Q$. There exists $\pi\colon G \to H$ a continuous surjective group homomorphism between compact abelian metrizable groups $G$ and $H$ and embeddings $\varepsilon_K\colon K \to G$, $\varepsilon_Q\colon Q \to H$ such that $\pi \circ \varepsilon_K = \varepsilon_Q \circ \alpha$.*

Proof. There exists a countable subset $B$ of $\check{K}$ which separates the points of $K$. To see this we embed $K$ in $T_\infty$ and project $T_\infty$ onto its coordinate spaces. Let $A$ be a similar subset of $\check{Q}$. We define the countable groups $\hat{H}$ and $\hat{G}$ to be the groups generated by $A$ and $\alpha^*(A) \cup B$ in $\check{Q}$ and $\check{K}$ respectively. Since $\alpha^*$ identifies $\hat{H}$ to $\alpha^*(\hat{H})$, the inclusions $\hat{H} \subset \check{Q}$, $\hat{G} \subset \check{K}$ and $\alpha^*(\hat{H}) \subset \hat{G}$ dualize to continuous surjective group homomorphisms $p_Q\colon Q^f \to H$, $p_K\colon K^f \to G$ and $\pi\colon G \to H$ respectively such that $\pi \circ p_K = p_Q \circ \alpha^f$, where $G$ and $H$ are compact abelian metrizable groups. The continuous mappings $\varepsilon_K = p_K \circ i_K\colon K \to G$ and $\varepsilon_Q = p_Q \circ i_Q\colon Q \to H$ are embeddings since $\hat{G}$ and $\hat{H}$ separate the points of $K$ and $Q$ respectively. Evidently, $\pi \circ \varepsilon_K = \varepsilon_Q \circ \alpha$. This completes the proof.

In the situation of Lemma 3 we define $\Lambda = \pi^{-1}(0_H)$ a closed subgroup of $G$ and $L = \pi^{-1}(Q) = K + \Lambda$ a closed subset of $G$. When we come to apply Lemma 3 we shall regularize on $K$ by the action of $\Lambda$. To compensate for the fact that $K$ is not $\Lambda$-stable we shall need a well behaved Borel mapping $\beta\colon L \to K$.

Since $G$ is compact metrizable, we may choose a translation invariant metric $d$ on $G$ of total distance 1 giving the topology of $G$.

Let $I = [0, 1]$ be the unit interval and let $X$ be a closed subspace of $L \times I$ such that the coordinate projection $X \to L$ is onto. We define the mapping $\theta\colon L \to I$ by

$$\theta(l) = \inf\{t;\ (l, t) \in X\}.$$

We denote

$$X' = \mathrm{graph}(\theta) = \{(l, \theta(l));\ l \in L\}$$

the unique subset of $X$ with the properties:

B) $(l, t_1) \in X \Rightarrow \exists\, t_2 \in I$ such that $(l, t_2) \in X'$;

C) $(l, t_1), (l, t_2) \in X' \Rightarrow t_1 = t_2$;

D) $(l, t_1) \in X', (l, t_2) \in X \Rightarrow t_1 \leqslant t_2$.

LEMMA 4. *In addition we have*

A) $X'$ *is a* $G_\delta$ (*intersection of a sequence of open sets*).

Proof. The mapping $\theta$ is lower semicontinuous and therefore has a $G_\delta$ graph. We leave the details to the reader.

LEMMA 5. *There exists a Borel mapping* $\beta\colon L\to K$ *such that*:

E) $a\circ\beta(l) = \pi(l)$, $\forall l\in L$;

F) $k\in K$, $l\in L$, $a(k) = \pi(l) \Rightarrow d\big(l, \beta(l)\big)\leqslant d(l, k)$.

Proof. We consider the continuous mapping

$$\gamma\colon L\times K\to L\times I$$

given by $\gamma(l, k) = \big(l, d(k, l)\big)$ and the closed subset $Y = \{(l, k);\ l\in L, k\in K, a(k) = \pi(l)\}$ of $L\times K$. We set $X = \gamma(Y)$ a closed subset of $L\times I$ and denote by $X'$ the subset of $X$ in Lemma 4. The subset $Y' = Y\cap \gamma^{-1}(X')$ of $L\times K$ has the following properties:

A′) $Y'$ is $G_\delta$.

B′) For all $l\in L$ $\exists k\in K$ such that $(l, k)\in Y'$.

D′) $(l, k_1)\in Y'$, $(l, k_2)\in Y \Rightarrow d(l, k_1)\leqslant d(l, k_2)$.

E′) $(l, k)\in Y' \Rightarrow \pi(l) = a(k)$.

Let $p_L\colon Y'\to L$ and $p_K\colon Y'\to K$ be the continuous mappings defined by the inclusion of $Y'$ into $L\times K$ followed by projection on the coordinate spaces. On account of B′) $p_L$ is onto. On account of A′) and Bourbaki [1] $Y'$ is an "espace polonais". The projection $p_L\colon Y'\to L$ satisfies the conditions of the Borel section theorem (Bourbaki [1]). It follows there exists a Borel mapping $\bar\beta\colon L\to Y'$ which is injective and satisfies $p_L\circ\bar\beta = 1_L$. We set $\beta = p_K\circ\bar\beta\colon L\to K$ a Borel mapping which satisfies E) on account of E′) and F) on account of D′). This completes the proof.

LEMMA 6. *Let* $\beta\colon L\to K_2$ *be Borel. Then the bidual mapping*

$$(1_{K_1}\times\beta)^\vee\colon M(K_1\times L)\to M(K_1\times K_2)$$

*is norm decreasing for the bimeasure norm.*

Proof. Let $\mu\in M(K_1\times L)$ with $\|\mu\|_{\mathrm{BM}}\leqslant 1$ and $f\in C(K_1)$ with $\|f\|_\infty\leqslant 1$ be fixed. The mapping

$$g\to\langle\mu, f\otimes g\rangle$$

is norm decreasing and linear from $C(L)$ to $\mathbf{C}$. Hence there exists a measure $\nu\in M(L)$ with $\|\nu\|_M\leqslant 1$ such that

$$(*)\qquad \langle\mu, f\otimes g\rangle = \langle\nu, g\rangle,\quad \forall g\in C(L).$$

It follows that $(*)$ is true for every bounded Borel function $g$. Let $h\in C(K_2)$ with $\|h\|_\infty\leqslant 1$. Then

$$|\langle(1_{K_1}\times\beta)^\vee(\mu), f\otimes h\rangle| = |\langle\mu, f\otimes h\circ\beta\rangle| = |\langle\nu, h\circ\beta\rangle|\leqslant 1.$$

This gives the result.



Let $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence of positive reals decreasing to zero fixed for the remainder of the paper. In the situation of Lemma 5 we define

$$U_n = \{\lambda;\ d(0_G, \lambda)\leqslant\varepsilon_n, \lambda\in\Lambda\}$$

and also $L_n = K+U_n \subseteq G$. We denote by $\beta_n$ the Borel mapping $\beta_n\colon L_n\to K$ obtained by restricting the $\beta$ of Lemma 5. On account of F) we have

$$d\big(l, \beta_n(l)\big)\leqslant\varepsilon_n,\qquad \forall l\in L_n.$$

LEMMA 7. *Let* $\mu_n$ *be a sequence of measures with* $\mu_n\in M(L_n)$, $\|\mu_n\|_M\leqslant 1$ *and* $\mu_n\to\mu$ *weakly, where* $\mu\in M(K)$. *Then* $\check\beta_n(\mu_n)\to\mu$ *weakly also.*

Proof. Let $f\in C(L)$ with $\|f\|_\infty\leqslant 1$ and $\varepsilon>0$. There exists $n$ such that

(i) $|\langle\mu-\mu_m, f\rangle|\leqslant\varepsilon/2$, $\forall m\geqslant n$,

(ii) $d(l_1, l_2)\leqslant\varepsilon_n \Rightarrow |f(l_1)-f(l_2)|\leqslant\varepsilon/2$.

For $m\geqslant n$

$$|\langle\mu_m-\check\beta_m(\mu_m), f\rangle| = |\langle\mu_m, f-f\circ\beta_m\rangle|\leqslant\varepsilon/2.$$

Hence $|\langle\mu-\check\beta_m(\mu_m), f\rangle|\leqslant\varepsilon$ as required.

THEOREM 4. *Every non-triangular closed subset* $E\subseteq K_1\times K_2$ ($K_j$ *metrizable*) *has the unit bounded synthesis property. In particular, for tensor algebras conditions* 2) *and* 3) *of the introduction imply condition* 1).

Proof. The mappings $a_j\colon K_j\to Q_j$ $(j = 1, 2)$ are given by Lemma 1. We apply Lemma 3 to the mapping $a_2\colon K_2\to Q_2$ and define $G, H, L, L_n, \beta, \beta_n, U_n$ and $\Lambda$ with respect to $a_2$ as in Lemmas 3–7. By Lemma 2 it suffices to prove the result with respect to the tensor algebra $V(K_1\times G)$. We define the closed subset $E^*$ of $K_1\times G$:

$$E^* = \{(k, g);\ a_1(k) = \pi(g)\}.$$

For $f\in V(K_1\times G)$ and $\lambda\in\Lambda$ we define the translate $f_\lambda$ by

$$f_\lambda(k, g) = f(k, g-\lambda).$$

Evidently, $f_\lambda\to f$ in $V$ norm as $\lambda\to 0_\Lambda$. For $S\in\mathrm{BM}(K_1\times G)$ we define the translate $S_\lambda$ by

$$\langle S_\lambda, f\rangle = \langle S, f_\lambda\rangle.$$

Since $\|f_\lambda\|_V = \|f\|_V$, we have $\|S_\lambda\|_{\mathrm{BM}} = \|S\|_{\mathrm{BM}}$ and since $f_\lambda\to f$ as $\lambda\to 0_\Lambda$, we see that $S_\lambda\to S$ in $\sigma(\mathrm{BM}, V)$ as $\lambda\to 0_\Lambda$. We say that $S\in\mathrm{BM}(K_1\times G)$ is *invariant* if $S_\lambda = S$ for all $\lambda\in\Lambda$. Suppose that $S$ is invariant and is supported on $E^*$. We act on $S$ by the norm decreasing mapping

$$(1\times\pi)^\vee\colon \mathrm{BM}(K_1\times G)\to\mathrm{BM}(K_1\times H)$$

and observe that

$$\operatorname{supp}\big((1\times\pi)^\vee(S)\big)\subseteq\big\{\big(k_1, a_1(k_1)\big);\ k_1\in K_1,\ a_1(k_1)\in Q_2\big\},$$

where the right-hand side is a graph in $K_1\times H$. By Theorem 3 we have $(1\times\pi)^\vee(S)\in M(K_1\times H)$ and

$$\|(1\times\pi)^\vee(S)\|_M\leqslant\|S\|_{\mathrm{BM}}.$$

Let $f\in V(K_1\times G)$. Then

$$\langle S,f\rangle=\Big\langle S,\int f_\lambda d\eta_\Lambda(\lambda)\Big\rangle,$$

where $\eta_\Lambda$ is the Haar measure of $\Lambda$. The function $\int f_\lambda d\eta_\Lambda(\lambda)$ respects $(1\times\pi)$ and can be written

$$\int f_\lambda d\eta_\Lambda(\lambda)=F\circ(1\times\pi),$$

where $F\in C(K_1\times H)$ by virtue of the fact that $(1\times\pi)$ is a closed mapping. Hence

$$|\langle S,f\rangle|=|\langle(\pi\times 1)^\vee(S),F\rangle|\leqslant\|S\|_{\mathrm{BM}}\|F\|_\infty\leqslant\|S\|_{\mathrm{BM}}\|f\|_\infty.$$

Since $V(K_1\times G)$ is dense in $C(K_1\times G)$, we see that $S$ is a measure and that $\|S\|_M\leqslant\|S\|_{\mathrm{BM}}$.

Let $\Sigma\in\mathrm{BM}(E)$. We aim to synthesize $\Sigma$. Let $\chi\in[\Lambda]^\wedge$. We choose an extension $X$ of $\chi$ to $G$ with $X\in\hat{G}$. We observe that the bimeasure

$$S=(1_{K_1}\otimes\chi)\int\Sigma_\lambda\chi(\lambda)\,d\eta_\Lambda(\lambda)\in\mathrm{BM}(E^*)$$

(the product of the function $(1_{K_1}\otimes X)$ with the bimeasure $\int\Sigma_\lambda\chi(\lambda)\,d\eta_\Lambda(\lambda)$) is invariant. For $\varrho\in\Lambda$ we have

$$\begin{aligned}\langle S_\varrho,f\rangle&=\langle S,f_\varrho\rangle\\&=\Big\langle\int\Sigma_\lambda\chi(\lambda)\,d\eta_\Lambda(\lambda),\,(1_{K_1}\otimes X)f_\varrho\Big\rangle\\&=\int\langle\Sigma_\lambda,(1_{K_1}\otimes X)f_\varrho\rangle\chi(\lambda)\,d\eta_\Lambda(\lambda)\\&=\int\langle\Sigma,(1_{K_1}\otimes X_\lambda)f_{\varrho+\lambda}\rangle\chi(\lambda)\,d\eta_\Lambda(\lambda)\\&=\int\langle\Sigma,(1_{K_1}\otimes X_{\lambda+\varrho})f_{\varrho+\lambda}\rangle\chi(\lambda+\varrho)\,d\eta_\Lambda(\lambda)\end{aligned}$$

(since $X_\lambda(x)\chi(\lambda)=X(x-\lambda)\chi(\lambda)=X(x)\doteq X_{\lambda+\varrho}(x)\chi(\lambda+\varrho)$)

$$=\int\langle\Sigma,(1_{K_1}\otimes X_{\lambda'})f_{\lambda'}\rangle\chi(\lambda')\,d\eta_\Lambda(\lambda')$$

(by the substitution $\lambda'=\lambda+\varrho$ and by the translation invariance of $\eta_\Lambda$).

We write

$$\Sigma^{(\varphi)}=\int\Sigma_\lambda\varphi(\lambda)\,d\eta_\Lambda(\lambda)\quad\text{ for }\quad\varphi\in C(\Lambda).$$

We see that $\Sigma^{(\chi)}$ is a measure and $\|\Sigma^{(\chi)}\|_M\leqslant\|\Sigma\|_{\mathrm{BM}}$. Hence for $\varphi\in A(\Lambda)$ the two inequalities

$$\|\Sigma^{(\varphi)}\|_{\mathrm{BM}}\leqslant\|\varphi\|_{L^1(\Lambda)}\|\Sigma\|_{\mathrm{BM}},$$
$$\|\Sigma^{(\varphi)}\|_M\leqslant\|\varphi\|_{A(\Lambda)}\|\Sigma\|_{\mathrm{BM}}$$



hold. Let $\varphi_n$ be a sequence of functions in $A(\Lambda)$ which are positive, such that $\int\varphi_n(\lambda)\,d\eta_\Lambda(\lambda)=1$ and with $\mathrm{supp}(\varphi_n)\subseteq U_n$. It is easy to see that

(i) $\|\Sigma^{(\varphi_n)}\|_{\mathrm{BM}}\leqslant\|\Sigma\|_{\mathrm{BM}}$,

(ii) $\Sigma^{(\varphi_n)}$ is a measure,

(iii) $\Sigma^{(\varphi_n)}\to\Sigma$ in $\sigma(\mathrm{BM},V)$,

(iv) $\mathrm{supp}(\Sigma^{(\varphi_n)})\subseteq(K_1\times L_n)\cap E^*$.

We define $\nu_n=(1_{K_1}\times\beta_n)^\vee(\Sigma^{(\varphi_n)})\in M(K_1\times K_2)$. By Lemma 6 we have

$$(**)\qquad\|\nu_n\|_{\mathrm{BM}}\leqslant\|\Sigma\|_{\mathrm{BM}}$$

and by condition E) on $\beta$ we see that

$$\mathrm{supp}(\nu_n)\subseteq E.$$

We aim to show that $\nu_n\to\Sigma$ in $\sigma(BM,V)$ and by virtue of (**) it suffices to check the convergence on an arbitrary atom $\psi_1\otimes\psi_2$, $\psi_1\in C(K_1)$, $\psi_2\in C(G)$ with $\|\psi_j\|_\infty\leqslant 1$ $(j=1,2)$. We regard $\psi_1$ as fixed and let $\psi_2$ vary. Arguing as in Lemma 6 we have measures $\mu$, $\{\mu_n\}_{n=1}^\infty$ and $\{\omega_n\}_{n=1}^\infty$ in $M(G)$ bounded in the measure norm by $\|\Sigma\|_{\mathrm{BM}}$ and such that

$$\langle\Sigma,\psi_1\otimes\psi_2\rangle=\langle\mu,\psi_2\rangle,$$
$$\langle\Sigma^{(\varphi_n)},\psi_1\otimes\psi_2\rangle=\langle\mu_n,\psi_2\rangle,$$
$$\langle\nu_n,\psi_1\otimes\psi_2\rangle=\langle\omega_n,\psi_2\rangle,$$

where $\mathrm{supp}(\mu_n)\subseteq L_n$. By virtue of (iii) $\mu_n\to\mu$ weakly and evidently $\omega_n=\check{\beta}_n(\mu_n)$. We conclude from Lemma 7 that $\omega_n\to\mu$ weakly. Hence

$$\langle\nu_n,\psi_1\otimes\psi_2\rangle\to\langle\Sigma,\psi_1\otimes\psi_2\rangle.$$

This completes the proof.

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