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ON NONAMENABLE GROUPS

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ABSTRACT. A sufficient condition is given for a countable discrete group G to contain a free subgroup of two generators.

KEY WORDS AND PHRASES. Nonamenable group, free group.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 22D05.

Given a topological group G, we denote by L the Banach algebra of all real valued bounded left uniformly continuous functions on G with the supremem norm. A mean m on L is a continuous, positive, linear functional such that $m(1) = 1$. A mean is called invariant if $m(f^g) = m(f)$ for every $f \in L$ and $g \in G$, where f^g is the translate of f by g.

G is called amenable if there exists an invariant mean on L. G has the fixed point property if whenever G acts on a compact convex set Q affinely in a locally convex topological vector space E, then G has a fixed point in Q [2].

It is well known that G is amenable if and only if G has the fixed point property for any topological group G.

In [4], von Neumann proved that if G has a free subgroup of two generators then G is not amenable and conjectured that the converse is true. In this paper, we shall give a sufficient condition for a discrete group G to contain a free subgroup of two generators. This result may be interesting to the investigation of von Neumann's conjecture.

Let ϕ be an affine transformation of a compact convex set Q in a locally convex topological vector space E. Then ϕ has a fixed point in Q by the famous Tychonoff fixed point theorem. Furthermore, one can prove easily that the fixed point set F_ϕ of an affine transformation ϕ of Q is a compact convex subset of Q.

Let us consider a discrete group G acting affinely on Q. The fixed point set F_ϕ of each element ϕ of G coincides with the fixed point set $F_{\phi^{-1}}$ of the inverse ϕ^{-1}. An element ϕ of G is said to be attractive if for each weak neighborhood U_ϕ of the fixed point set F_ϕ of ϕ, the orbit $\{\phi^n(S) | n \in \mathbb{Z}\}$ of any compact convex subset S in $Q - U_\phi$ converges to the fixed point set F_ϕ of ϕ, that is, there is a positive integer N such that for all $|n| > N, \phi^n(S) \subset U_\phi$. An element ϕ of G is said to be weakly attractive if, for each weak neighborhood U_ϕ of the fixed point set F_ϕ of ϕ, there is a positive integer N' such that for all $n \in \mathbb{Z}^{*}$ (1), $\phi^{nN'}(S) \subset U_\phi$. It is obvious that an attractive element ϕ of G is weakly attractive. [Note: (1) $\mathbb{Z}^* = \mathbb{Z} - \{0\}$]

<u>THEOREM</u>. If a discrete group G acts on a compact convex set of Q of a locally convex topological vector space E affinely such that G contains at least two weakly attractive elements without common fixed points, then G contains a free subgroup of two generators.

<u>PROOF</u>. Let ϕ and ψ be two weakly attractive elements of G. Then the fixed point sets F_ϕ and F_ψ are disjoint. By the seperation theorem [6], there exist

a linear functional L on E and real numbers c_1 and c_2 such that $L x < c_1 < c_2 < Ly$ for every x in F_ϕ and every y in F_ψ. Without loss of generality, we may assume that $c_1 < 0 < c_2$.

Thus $K_1 = \{x \in Q | Lx < 0\}$ is a weak convex neighborhood of F_ϕ and $K_2 = \{x \in Q | Lx > 0\}$ is a weak convex neighborhood of F_ψ. The complements K_1^c and K_2^c of K_1 and K_2 respectively are compact and convex sets in $Q - K_1$ and $Q - K_2$. By the definition of weak attractiveness, there exist positive integers N' and N'' such that $\phi^{nN'}(K_1^c) \subset K_1$ and $\psi^{nN''}(K_2^c) \subset K_2$ for all $n \in \mathbb{Z}^*$. Let $s = \phi^{N'}$ and $t = \psi^{N''}$. Then the group F generated by s and t is a free group. In fact, for any relation $s^p t^q \ldots = id$, we have $s^p t^q \ldots (z) = z$ for each z in the hyperplane section $K_1^c \cap K_2^c = \{z \in Q | Lz = 0\}$ of Q. But clearly $Ls^p t^q \ldots (z) \neq 0$, while $Lz = 0$. We have a contradiction.

COROLLARY. If a nonamenable discrete group G acts on a compact convex set Q of a locally convex topological vector space E affinely such that G contains all weakly attractive elements then G contains a free subgroup of two generators.

PROOF. This follows from the theorem and the non-fixed point property of nonamenable groups.

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