

ON NONDEGENERACY OF CURVES

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ABSTRACT. We study the conditions under which an algebraic curve can be modelled by a Laurent polynomial that is nondegenerate with respect to its Newton polytope. We prove that every curve of genus $g \leq 4$ over an algebraically closed field is nondegenerate in the above sense. More generally, let $\mathcal{M}_g^{\text{nd}}$ be the locus of nondegenerate curves inside the moduli space of curves of genus $g \geq 2$. Then we show that $\dim \mathcal{M}_g^{\text{nd}} = \min(2g+1, 3g-3)$, except for $g = 7$ where $\dim \mathcal{M}_7^{\text{nd}} = 16$; thus, a generic curve of genus g is nondegenerate if and only if $g \leq 4$.

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Let k be a perfect field with algebraic closure \bar{k} . Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial, and write $f = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij} x^i y^j$. We denote by $\text{supp}(f) = \{(i,j) \in \mathbb{Z}^2 : c_{ij} \neq 0\}$ the support of f , and we associate to f its Newton polytope $\Delta = \Delta(f)$, the convex hull of $\text{supp}(f)$ in \mathbb{R}^2 . We assume throughout that Δ is 2-dimensional. For a face $\tau \subset \Delta$, let $f|_{\tau} = \sum_{(i,j) \in \tau} c_{ij} x^i y^j$. We say that f is *nondegenerate* if, for every face $\tau \subset \Delta$ (of any dimension), the system of equations

$$(1) \quad f|_{\tau} = x \frac{\partial f|_{\tau}}{\partial x} = y \frac{\partial f|_{\tau}}{\partial y} = 0$$

has no solutions in \bar{k}^{*2} .

From the perspective of toric varieties, the condition of nondegeneracy can be rephrased as follows. The Laurent polynomial f defines a curve $U(f)$ in the torus $\mathbb{T}_k^2 = \text{Spec } k[x^{\pm 1}, y^{\pm 1}]$, and \mathbb{T}_k^2 embeds canonically in the projective toric surface $X(\Delta)_k$ associated to Δ over k . Let $V(f)$ be the Zariski closure of the curve $U(f)$ inside $X(\Delta)_k$. Then f is nondegenerate if and only if for every face $\tau \subset \Delta$, we have that $V(f) \cap \mathbb{T}_{\tau}$ is smooth of codimension 1 in \mathbb{T}_{τ} , where \mathbb{T}_{τ} is the toric component of $X(\Delta)_k$ associated to τ . (See Proposition 1.2 for alternative characterizations.)

Nondegenerate polynomials have become popular objects in explicit algebraic geometry, owing to their connection with toric geometry [4]: a wealth of geometric information about $V(f)$ is contained in the combinatorics of the Newton polytope $\Delta(f)$. The notion was initially employed by Kouchnirenko [24], who studied nondegenerate polynomials in the context of singularity theory. Nondegenerate polynomials emerge naturally in the theory of sparse resultants [15] and admit a linear effective Nullstellensatz [8, Section 2.3]. They make an appearance in the study of real algebraic curves in maximal position [28] and in the problem of enumerating curves through a set of prescribed points [29]. In the case where k is a finite field, they arise in the construction of curves with many points [6, 25], in the p -adic cohomology theory of Adolphson and Sperber [2], and in explicit methods for computing zeta functions of varieties over k [8]. Despite their utility and seeming

ubiquity, the *intrinsic* property of nondegeneracy has not seen detailed study, with the exception of the Ph.D. thesis of Koelman [22] from 1991, otherwise unpublished (see Section 12 below).

We are therefore led to the central problem of this article: *Which curves are nondegenerate?* To the extent that toric varieties are generalizations of projective space, this question asks us to generalize the characterization of nonsingular plane curves amongst all curves. An immediate provocation for this question was to understand the locus of curves to which the point counting algorithm of Castryck–Denef–Vercauteren [8] actually applies. Our results are collected in two parts.

In the first part, comprising Sections 3–7, we investigate the nondegeneracy of some interesting classes of curves (hyperelliptic, C_{ab} , and low genus curves). Our conclusions can be summarized as follows.

Theorem. *Let V be a curve of genus g over a perfect field k . Suppose that one of the following conditions holds:*

- (i) $g = 0$;
- (ii) $g = 1$ and $V(k) \neq \emptyset$;
- (iii) $g = 2, 3$, and either $17 \leq \#k < \infty$, or $\#k = \infty$ and $V(k) \neq \emptyset$;
- (iv) $g = 4$ and $k = \bar{k}$.

Then V is nondegenerate.

Remark. The condition $\#k \geq 17$ in (iii) ensures that k is large enough to allow nontangency to the toric boundary of $X(\Delta)_k$, but is most likely superfluous; see Remark 7.2.

In the second part, consisting of Sections 8–12, we restrict to algebraically closed fields $k = \bar{k}$ and consider the locus $\mathcal{M}_g^{\text{nd}}$ of nondegenerate curves inside the coarse moduli space of all curves of genus $g \geq 2$. We prove the following theorem.

Theorem. *We have $\dim \mathcal{M}_g^{\text{nd}} = \min(2g + 1, 3g - 3)$, except for $g = 7$ where $\dim \mathcal{M}_7^{\text{nd}} = 16$. In particular, a generic curve of genus g is nondegenerate if and only if $g \leq 4$.*

Our methods combine ideas of Bruns–Gubeladze [7] and Haase–Schicho [17] and are purely combinatorial—only the universal property of the coarse moduli space is used.

Conventions and notations. Throughout, $\Delta \subset \mathbb{R}^2$ will denote a polytope with $\dim \Delta = 2$. The coordinate functions on the ambient space \mathbb{R}^2 will be denoted by X and Y . A *facet* or *edge* of a polytope is a face of dimension 1. A *lattice polytope* is a polytope with vertices in \mathbb{Z}^2 . Two lattice polytopes Δ and Δ' are *equivalent* if there is an affine map

$$\begin{aligned} \varphi : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ v &\mapsto Av + b \end{aligned}$$

such that $\varphi(\Delta) = \Delta'$ with $A \in GL_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$. Two Laurent polynomials f and f' are *equivalent* if f' can be obtained from f by applying such a map to the exponent vectors. Note that equivalence preserves nondegeneracy. For a polytope $\Delta \subset \mathbb{R}^2$, we let $\text{int}(\Delta)$ denote the interior of Δ . We denote the standard 2-simplex in \mathbb{R}^2 by $\Sigma = \text{conv}(\{(0, 0), (1, 0), (0, 1)\})$.

1. NONDEGENERATE LAURENT POLYNOMIALS

In this section, we review the geometry of nondegenerate Laurent polynomials. We retain the notation used in the introduction: in particular, k is a perfect field, $f = \sum c_{ij} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}]$ is an irreducible Laurent polynomial, and Δ is its Newton polytope. Our main implicit reference on toric varieties is Fulton [14].

Let $k[\Delta]$ denote the graded semigroup algebra over k generated in degree d by the monomials that are supported in $d\Delta$, i.e.

$$k[\Delta] = \bigoplus_{d=0}^{\infty} \langle x^i y^j t^d \mid (i, j) \in (d\Delta \cap \mathbb{Z}^2) \rangle_k.$$

Then $X = X(\Delta)_k = \text{Proj } k[\Delta]$ is the projective toric surface associated to Δ over k . This surface naturally decomposes into toric components as

$$X = \bigsqcup_{\tau \subset \Delta} \mathbb{T}_\tau,$$

where τ ranges over the faces of Δ and $\mathbb{T}_\tau \cong \mathbb{T}_k^{\dim \tau}$. The surface X is nonsingular except possibly at the zero-dimensional toric components associated to the vertices of Δ . The Laurent polynomial f defines a curve in $\mathbb{T}_k^2 \cong \mathbb{T}_\Delta \subset X$, and we denote by $V = V(f)$ its closure in X . Alternatively, if we denote $A = \Delta \cap \mathbb{Z}^2$, then X can be canonically embedded in $\mathbb{P}_k^{\#A-1} = \text{Proj } k[t_{ij}]_{(i,j) \in A}$, and V is the hyperplane section $\sum c_{ij} t_{ij} = 0$ of X .

We abbreviate $\partial_x = x \frac{\partial}{\partial x}$ and $\partial_y = y \frac{\partial}{\partial y}$.

Definition 1.1. *The Laurent polynomial f is nondegenerate if for each face $\tau \subset \Delta$, the system*

$$f|_\tau = \partial_x f|_\tau = \partial_y f|_\tau = 0$$

*has no solution in \overline{k}^{*2} .*

We will sometimes write that f is Δ -nondegenerate to emphasize that $\Delta(f) = \Delta$.

Proposition 1.2. *The following statements are equivalent.*

- (i) f is nondegenerate.
- (ii) For each face $\tau \subset \Delta$, the ideal of $k[x^{\pm 1}, y^{\pm 1}]$ generated by

$$f|_\tau, \partial_x f|_\tau, \partial_y f|_\tau$$

is the unit ideal.

- (iii) For each face $\tau \subset \Delta$, the intersection $V \cap \mathbb{T}_\tau$ is smooth of codimension 1 in the torus orbit \mathbb{T}_τ associated to τ .
- (iv) The sequence of elements $f, \partial_x f, \partial_y f$ (in degree one) forms a regular sequence in $k[\Delta]$.
- (v) The quotient of $k[\Delta]$ by the ideal generated by $f, \partial_x f, \partial_y f$ is finite of k -dimension equal to $2 \text{vol}(\Delta)$.

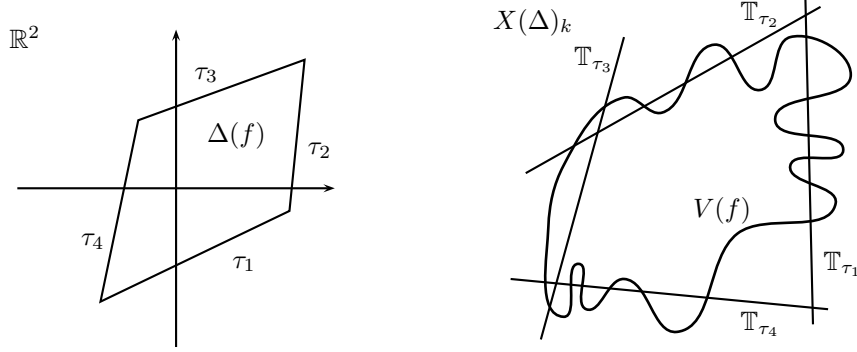
Remark 1.3. Condition (iii) can also be read as: V is smooth and intersects $X \setminus \mathbb{T}_k^2$ transversally and outside the zero-dimensional toric components associated to the vertices of Δ .

Proof. See Batyrev [3, Section 4] for a proof of these equivalences and further discussion. \square

Remark 1.4. Some authors refer to nondegenerate as Δ -regular, though we will not employ this term. The use of *nondegenerate* to indicate a projective variety which is not contained in a smaller projective space is unrelated to our present usage.

Example 1.5. Let $f(x, y) \in k[x, y]$ be a bivariate polynomial of degree $d \in \mathbb{Z}_{\geq 1}$ with Newton polytope $\Delta = d\Sigma = \text{conv}(\{(0, 0), (d, 0), (0, d)\})$. The toric variety $X(\Delta)_k$ is the d -uple Veronese embedding of \mathbb{P}_k^2 in $\mathbb{P}_k^{d(d+3)/2}$, and $V(f)$ is the projective curve in \mathbb{P}_k^2 defined by the homogenization $F(x, y, z)$ of f . We see that $f(x, y)$ is Δ -nondegenerate if and only if $V(f)$ is nonsingular, does not contain the coordinate points $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$, and is not tangent to any coordinate axis.

Example 1.6. The following picture illustrates nondegeneracy in case of a quadrilateral Newton polytope.



Proposition 1.7. *If $f \in k[x^{\pm 1}, y^{\pm 1}]$ is nondegenerate, then there exists a k -rational canonical divisor K_Δ on $V = V(f)$ such that $\{x^i y^j : (i, j) \in \text{int}(\Delta) \cap \mathbb{Z}^2\}$ is a k -basis for the Riemann-Roch space $\mathcal{L}(K_\Delta) \subset k(V)$. In particular, the genus of V is equal to $\#\text{int}(\Delta) \cap \mathbb{Z}^2$.*

Proof. See Khovanskii [21] or Castryck–Denef–Vercauteren [8, Section 2.2]. \square

Remark 1.8. In general, if f is irreducible (but not necessarily nondegenerate), one has that the geometric genus of $V(f)$ is bounded by $\#\text{int}(\Delta) \cap \mathbb{Z}^2$: this is also known as Baker’s inequality [6, Theorem 4.2].

We conclude this section with the following intrinsic definition of nondegeneracy.

Definition 1.9. *A curve V over k is Δ -nondegenerate if V is birational over k to a curve $U \subset \mathbb{T}_k^2$ defined by a nondegenerate Laurent polynomial f with Newton polytope Δ . The curve V is nondegenerate if it is Δ -nondegenerate for some Δ . The curve V is geometrically nondegenerate if $V \times_k \bar{k}$ is nondegenerate over \bar{k} .*

2. MODULI OF NONDEGENERATE CURVES

We now construct the moduli space of nondegenerate curves of given genus $g \geq 2$. Since in this article we will be concerned with dimension estimates only, we restrict to the case $k = \bar{k}$.

We denote by \mathcal{M}_g the coarse moduli space of curves of genus $g \geq 2$ over k , with the property that for any flat family $\mathcal{V} \rightarrow M$ of curves of genus g , there is a (unique) morphism $M \rightarrow \mathcal{M}_g$ which maps each closed point $f \in M$ to the isomorphism class of the fiber \mathcal{V}_f . (See e.g. Mumford [31, Theorem 5.11].)

Let $\Delta \subset \mathbb{R}^2$ be a lattice polytope with g interior lattice points. We will construct a flat family $\mathcal{V}(\Delta) \rightarrow M_\Delta$ which parametrizes all Δ -nondegenerate curves over k . The key ingredient is provided by the following result of Gel'fand–Kapranov–Zelevinsky. Let $A = \Delta \cap \mathbb{Z}^2$ and define the polynomial ring $R_\Delta = k[c_{ij}]_{(i,j) \in A}$.

Proposition 2.1 (Gel'fand–Kapranov–Zelevinsky [15]). *There exists a polynomial $E_A \in R_\Delta$ with the property that for any Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$ with $\text{supp}(f) \subset \Delta$, we have that f is Δ -nondegenerate if and only if $E_A(f) \neq 0$.*

Proof. The proof of Gel'fand–Kapranov–Zelevinsky [15, Chapter 10] is over \mathbb{C} ; however, the construction yields a polynomial over \mathbb{Z} which is easily seen to characterize nondegeneracy for an arbitrary field. \square

The polynomial E_A is known as the *principal A -determinant* and is given by the *A -resultant* $\text{res}_A(F, \partial_1 F, \partial_2 F)$. It is homogeneous in the variables c_{ij} of degree $6 \text{vol}(\Delta)$, and its irreducible factors are the *face discriminants* D_τ for faces $\tau \subset \Delta$.

Example 2.2. Consider the universal plane conic

$$F = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2,$$

associated to the Newton polytope 2Σ as in Example 1.5.

Then

$$E_A = c_{00}c_{02}c_{20}(c_{11}^2 - 4c_{02}c_{20})(c_{10}^2 - 4c_{00}c_{20})(c_{01}^2 - 4c_{00}c_{02})D_\Delta$$

where

$$D_\Delta = 4c_{00}c_{20}c_{02} - c_{00}c_{11}^2 - c_{10}^2c_{02} - c_{01}^2c_{20} + c_{10}c_{01}c_{11}.$$

The nonvanishing of the factor $c_{00}c_{02}c_{20}$ (corresponding to the discriminants of the zero-dimensional faces) ensures that the curve does not contain a coordinate point, and in particular does not have Newton polytope smaller than 2Σ ; the nonvanishing of the quadratic factors (corresponding to the one-dimensional faces) ensures that the curve intersects the coordinate lines in two distinct points; and the nonvanishing of D_Δ ensures that the curve is smooth.

Let M_Δ be the complement in $\mathbb{P}_k^{\#A-1} = \text{Proj } R_\Delta$ of the algebraic set defined by E_A . By the above, M_Δ parameterizes nondegenerate polynomials having Δ as Newton polytope. One can show that

$$(2) \quad \dim M_\Delta = \#A - 1,$$

which is a non-trivial statement if k is of finite characteristic (and false in general for an arbitrary number of variables), see [8, Section 2]. Let $\mathcal{V}(\Delta)$ be the closed subvariety of

$$X(\Delta)_k \times M_\Delta \subset \text{Proj } k[t_{ij}] \times \text{Proj } k[c_{ij}]$$

defined by the universal hyperplane section

$$\sum_{(i,j) \in A} c_{ij}t_{ij} = 0.$$

Then the universal family of Δ -nondegenerate curves is realized by the projection map $\varphi : \mathcal{V}(\Delta) \rightarrow M_\Delta$. The fiber $\mathcal{V}(\Delta)_f$ above a nondegenerate Laurent polynomial $f \in M_\Delta$ is precisely the corresponding curve $V(f)$, realized as the corresponding hyperplane section of $X(\Delta)_k \subset \text{Proj } k[t_{ij}]$. Note that φ is indeed flat [19, Theorem III.9.9], since the Hilbert polynomial of $\mathcal{V}(\Delta)_f$ is independent of f : its degree is equal to $\deg X(\Delta)_k$ and its genus is g by Proposition 1.7.

Thus by the universal property of \mathcal{M}_g , there is a morphism $h_\Delta : M_\Delta \rightarrow \mathcal{M}_g$, the image of which consists precisely of all isomorphism classes containing a Δ -nondegenerate curve. Let \mathcal{M}_Δ denote the Zariski closure of the image of h_Δ . Finally, let

$$\mathcal{M}_g^{\text{nd}} = \bigcup_{g(\Delta)=g} \mathcal{M}_\Delta,$$

where the union is taken over all polytopes Δ with g interior lattice points, of which there are finitely many up to equivalence (see Hensley [20]).

The aim of Sections 8–12 is to estimate $\dim \mathcal{M}_g^{\text{nd}}$. This is done by first refining the obvious upper bounds $\dim \mathcal{M}_\Delta \leq \dim M_\Delta = \#(\Delta \cap \mathbb{Z}^2) - 1$, taking into account the action of the automorphism group $\text{Aut}(X(\Delta)_k)$, and then estimating the outcome in terms of g .

Remark 2.3. It follows from the fact that \mathcal{M}_g is of general type for $g \geq 23$ (see e.g. [18]) that $\dim \mathcal{M}_g^{\text{nd}} < \dim \mathcal{M}_g = 3g - 3$ for $g \geq 23$, since each component of $\mathcal{M}_g^{\text{nd}}$ is unirational. Below, we obtain much sharper results which do not rely on this deep statement.

3. TRIANGULAR NONDEGENERACY

In Sections 4–6, we study the nondegeneracy of certain well-known classes, such as elliptic, hyperelliptic and C_{ab} curves. In many cases, classical constructions provide models for these curves that are supported on a triangular Newton polytope; the elementary observations in this section will allow us to prove that these models are nondegenerate when $\#k$ is not too small.

Lemma 3.1. *Let $f(x, y) \in k[x, y]$ define a smooth affine curve of genus g and suppose that $\#k > 2(g + \max(\deg_x f, \deg_y f) - 1) + \min(\deg_x f, \deg_y f)$. Then there exist $x_0, y_0 \in k$ such that the translated curve $f(x - x_0, y - y_0)$ does not contain $(0, 0)$ and is also nontangent to both the x - and the y -axis.*

Proof. Suppose $\deg_y f \leq \deg_x f$. Applying the Riemann-Hurwitz theorem to the projection map $(x, y) \mapsto x$, one verifies that there are at most $2(g + \deg_y f - 1)$ points with a vertical tangent. Therefore, we can find an $x_0 \in k$ such that $f(x - x_0, y)$ is nontangent to the y -axis. Subsequently, there are at most $2(g + \deg_x f - 1) + \deg_y f$ values of $y_0 \in k$ for which $f(x - x_0, y - y_0)$ is tangent to the x -axis and/or contains $(0, 0)$. \square

Lemma 3.2. *Let $a \leq b \in \mathbb{Z}_{\geq 2}$ be such that $\gcd(a, b) \in \{1, a\}$, and let Δ be the triangular lattice polytope $\text{conv}(\{(0, 0), (b, 0), (0, a)\})$. Let $f(x, y) \in k[x, y]$ be an irreducible polynomial such that:*

- *f is supported on Δ , and*
- *the genus of $V(f)$ equals $g = \#(\text{int}(\Delta) \cap \mathbb{Z}^2)$.*

Then if $\#k > 2(g + b - 1) + a$, we have that $V(f)$ is Δ -nondegenerate.

Proof. First suppose that $\gcd(a, b) = 1$. The coefficients of x^b and y^a must be nonzero, because else $\#(\text{int}(\Delta(f)) \cap \mathbb{Z}^2) < g$, which contradicts Baker's inequality. For the same reason, f must define a smooth affine curve: if (x_0, y_0) is a singular point (over \bar{k}), then $\#(\text{int}(\Delta(f(x - x_0, y - y_0)) \cap \mathbb{Z}^2)) < g$. The result now follows from Lemma 3.1. Note that the nonvanishing of the face discriminant D_τ , where τ

is the edge connecting $(b, 0)$ and $(0, a)$, follows automatically from the fact that τ has no interior lattice points.

Next, suppose that $\gcd(a, b) = a$. Then we may assume that the coefficients of x^b and y^a are nonzero. Indeed, if $a < b$ then the coefficient of y^a must be nonzero. Let $g(t) \in k[t]$ be the coefficient of x^b in $f(x, y + tx^{b/a})$. It is of degree a and therefore has a non-root $t_0 \in k$. Then substituting $y \leftarrow y + t_0 x^{b/a}$ ensures that the coefficient of x^b is nonzero as well. If $a = b$ then the coefficient of y^a might be zero, but f must contain at least one non-zero term of total degree a , and a similar argument proves the claim.

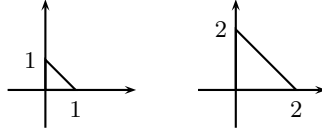
Then as above, we have that f defines a smooth affine curve. So by applying Lemma 3.1, we may assume that the face discriminants decomposing $E_{\Delta \cap \mathbb{Z}^2}$ are nonvanishing at f , with the possible exception of D_τ , where τ is the edge connecting $(b, 0)$ and $(0, a)$. However, under the equivalence

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2 : (X, Y) \mapsto (b - X - \frac{b}{a}Y, Y),$$

τ is interchanged with the edge connecting $(0, 0)$ and $(0, a)$. By applying Lemma 3.1 again, we obtain full nondegeneracy. \square

4. NONDEGENERACY OF CURVES OF GENUS AT MOST ONE

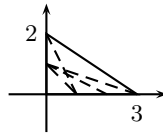
Curves of genus 0. Let V be a curve of genus 0 over k . The anticanonical divisor embeds $V \hookrightarrow \mathbb{P}_k^2$ as a smooth conic. If $\#k = \infty$, then by Lemma 3.2 and Proposition 1.2, we see that V is nondegenerate. If $\#k < \infty$ then $V(k) \neq \emptyset$ by Wedderburn, hence $V \cong \mathbb{P}_k^1$ can be embedded as a nondegenerate line in \mathbb{P}^2 . Therefore, any curve V of genus 0 is Δ -nondegenerate, where Δ is one of the following:



Curves of genus 1. Let V be a curve of genus 1 over k . First suppose that $V(k) \neq \emptyset$. Then V is an elliptic curve and hence can be defined by a nonsingular Weierstrass equation

$$(3) \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_i \in k$. The corresponding Newton polytope Δ is



where one of the dashed lines appears as a facet if $a_6 = 0$. By Lemma 3.2, we have that V is nondegenerate if $\#k \geq 9$. With some extra work we can get rid of this condition.

For $A = \Delta \cap \mathbb{Z}^2$, the principal A -determinant has 7 or 9 face discriminants D_τ as irreducible factors. The nonvanishing of D_Δ corresponds to the fact that our curve is smooth in \mathbb{T}_k^2 . In case τ is a vertex or a facet containing no interior lattice points, the nonvanishing of D_τ is automatic. Thus it suffices to consider the discriminants D_τ for τ a facet supported on the X -axis (denoted τ_X) or the

Y -axis (denoted τ_Y). First, suppose that $\text{char } k \neq 2$. After completing the square, we have $a_1 = a_3 = 0$ and the nonvanishing of D_{τ_X} follows from the fact that the polynomial $p(x) = x^3 + a_2x^2 + a_4x + a_6$ is squarefree. The nonvanishing of D_{τ_Y} (if τ_Y exists) is clear. Now suppose $\text{char } k = 2$. Let δ be the number of distinct roots (over \bar{k}) of $p(x) = x^3 + a_2x^2 + a_4x + a_6$. If $\delta = 3$ then D_{τ_X} is non-vanishing. For the nonvanishing of D_{τ_Y} , it then suffices to substitute $x \leftarrow x + 1$ if necessary, so that a_3 is nonzero (note that not both a_1 and a_3 can be zero). If $\delta < 3$ then $p(x)$ has a root x_0 of multiplicity at least 2. Since k is perfect, this root is k -rational and after substituting $x \mapsto x + x_0$ we have $p(x) = x^3 + a_2x^2$. In particular, D_{τ_X} (if τ_X exists) and D_{τ_Y} do not vanish.

In conclusion, we have shown that every genus 1 curve V over a field k is non-degenerate, given that $V(k) \neq \emptyset$. This condition is automatically satisfied if k is a finite field (by Hasse–Weil) or if k is algebraically closed. In particular, every genus 1 curve is geometrically nondegenerate. More generally, we define the *index* of a curve V over a field k to be the least degree of an effective non-zero k -rational divisor on V (equivalently, the least extension degree of a field $L \supset k$ for which $V(L) \neq \emptyset$). We then have the following criterion.

Lemma 4.1. *A curve V of genus 1 is nondegenerate if and only if V has index at most 3.*

Proof. First, assume that V is nondegenerate. There are exactly 16 equivalence classes of polytopes with only 1 interior lattice point; see [34, Figure 2] or the appendix at the end of this article. So we may assume that V is Δ -nondegenerate with Δ in this list. Now for every facet $\tau \subset \Delta$, the toric component \mathbb{T}_τ of $X(\Delta)_k$ cuts out an effective k -rational divisor of degree $\ell(\tau)$ on V , where $\ell(\tau) + 1$ is the number of lattice points on τ . The result then follows, since one easily verifies that every polytope in the list contains a facet τ with $\ell(\tau) \leq 3$.

Conversely, suppose that V has index $\iota \leq 3$. If $\iota = 1$, we have shown above that V is nondegenerate. If $\iota = 2$ (resp. $\iota = 3$), using Riemann-Roch one can construct a plane model $f \in k[x, y]$ with $\Delta(f) \subset \text{conv}(\{(0, 0), (4, 0), (0, 2)\})$ (resp. $\Delta(f) \subset 3\Sigma$); see e.g. Fisher [13, Section 3] for details. Then since $V(k) = \emptyset$ and hence $\#k = \infty$, an application of Lemma 3.2 concludes the proof. \square

Remark 4.2. There exist genus 1 curves of arbitrarily large index over every number field; see Clark [9]. Hence there exist infinitely many genus 1 curves which are not nondegenerate.

5. NONDEGENERACY OF HYPERELLIPTIC CURVES AND C_{ab} CURVES

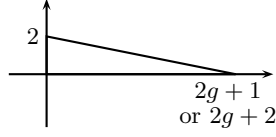
Hyperelliptic curves. A curve V over k of genus $g \geq 2$ is *hyperelliptic* if there exists a nonconstant morphism $V \rightarrow \mathbb{P}_k^1$ of degree 2. The morphism is automatically separable [19, Proposition IV.2.5] and the curve can be defined by a Weierstrass equation

$$(4) \quad y^2 + q(x)y = p(x).$$

Here $p(x), q(x) \in k[x]$ satisfy $2 \deg q(x) \leq \deg p(x)$ and $\deg p(x) \in \{2g + 1, 2g + 2\}$, as long as $k \neq \mathbb{F}_2$: see Enge [12, Theorem 7]. (This condition will fail for any hyperelliptic curve C over $k = \mathbb{F}_2$ for which the degree 2 morphism $\pi : C \rightarrow \mathbb{P}^1$ splits completely over k , meaning that above each point $0, 1, \infty \in \mathbb{P}^1(k)$ there are

two distinct k -rational points of C .) For the rest of this subsection, we suppose $k \neq \mathbb{F}_2$, and we leave the small modifications in this case to the reader.

The universal such curve has Newton polytope as follows:



By Lemma 3.2, if $\#k \geq 6g + 5$ then V is nondegenerate. In particular, if $\#k \geq 17$ then every curve of genus 2 is nondegenerate.

If $\text{char } k \neq 2$, we can drop the condition on $\#k$ by completing the square, as in the elliptic curve case. This observation immediately weakens the condition to $\#k \geq 2^{\lfloor \log_2(6g+5) \rfloor} + 1$. As a consequence, $\#k \geq 17$ is also sufficient for every hyperelliptic curve of genus 3 or 4 to be nondegenerate.

Conversely, any curve defined by a nondegenerate polynomial as in (4) is hyperelliptic. We conclude that $\dim \mathcal{M}_\Delta = \dim \mathcal{H}_g = 2g - 1$ [19, Example IV.5.5.5].

One can decide if a nondegenerate polynomial f defines a hyperelliptic curve according to the following criterion, which also appears in Koelman [22, Lemma 3.2.9] with a more complicated proof.

Lemma 5.1. *Let $f \in k[x^\pm, y^\pm]$ be nondegenerate and suppose $\# \text{int}(\Delta(f) \cap \mathbb{Z}^2) \geq 2$. Then $V(f)$ is hyperelliptic if and only if the interior lattice points of $\Delta(f)$ are collinear.*

Proof. We may assume that $\Delta = \Delta(f)$ has $g \geq 3$ interior lattice points, since all curves of genus 2 are hyperelliptic and any two points are collinear.

Let $L \subset k(V)$ be the subfield generated by all quotients of functions in $\mathcal{L}(K)$, where K is a canonical divisor on V . Then L does not depend on the choice of K , and L is isomorphic to the rational function field $k(\mathbb{P}_k^1)$ if and only if V is hyperelliptic.

We now show that $L \cong k(\mathbb{P}_k^1)$ if and only if the interior lattice points of Δ are collinear. We may assume that $(0, 0)$ is in the interior of Δ . Then from Proposition 1.7, we see that L contains all monomials of the form $x^i y^j$ for $(i, j) \in \text{int}(\Delta) \cap \mathbb{Z}^2$. In particular, if the interior lattice points of Δ are not collinear then after a transformation we may assume further that $(0, 1), (1, 0) \in \text{int}(\Delta)$, whence $L \supset k(x, y) = k(V)$; and if they are collinear, then clearly $L \cong k(\mathbb{P}_k^1)$. The result then follows. \square

For this reason, we call a lattice polytope *hyperelliptic* if its interior lattice points are collinear.

A curve V over k of genus $g \geq 2$ is called *geometrically hyperelliptic* if $V_{\bar{k}} = V \times_k \bar{k}$ is hyperelliptic. Every hyperelliptic curve is geometrically hyperelliptic, but not conversely: if $V \rightarrow C \subset \mathbb{P}_k^{g-1}$ is the canonical morphism, then V is hyperelliptic if and only if $C \cong \mathbb{P}_k^1$. This latter condition is satisfied if and only if $C(k) \neq \emptyset$, which is guaranteed when k is finite, when $V(k) \neq \emptyset$, and when g is even.

Lemma 5.2. *Let V be a geometrically hyperelliptic curve which is nonhyperelliptic. Then V is not nondegenerate.*

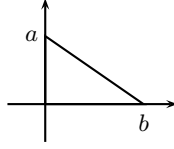
Proof. Suppose that V is geometrically hyperelliptic and Δ -nondegenerate for some lattice polytope Δ . Then applying Lemma 5.1 to $V_{\bar{k}}$, we see that the interior lattice

points of Δ are collinear. But then again by Lemma 5.1 (now applied to V itself), V must be hyperelliptic. \square

C_{ab} curves. Let $a, b \in \mathbb{Z}_{\geq 2}$ be coprime. A C_{ab} curve is a curve having a place with Weierstrass semigroup $a\mathbb{Z}_{\geq 0} + b\mathbb{Z}_{\geq 0}$ (see Miura [30]). Any C_{ab} curve is defined by a Weierstrass equation

$$(5) \quad f(x, y) = \sum_{\substack{i, j \in \mathbb{N} \\ ai + bj \leq ab}} c_{ij} x^i y^j = 0.$$

with $c_{0a}, c_{b0} \neq 0$. By Lemma 3.2, if $\#k \geq 2(g + a + b - 2)$ then we may assume that this polynomial is nondegenerate with respect to its Newton polytope Δ_{ab} :



Conversely, every curve given by a Δ_{ab} -nondegenerate polynomial is C_{ab} , and the unique place dominating the point at projective infinity has Weierstrass semigroup $a\mathbb{Z}_{\geq 2} + b\mathbb{Z}_{\geq 2}$ (see Matsumoto [27]). Note that if k is algebraically closed, the class of hyperelliptic curves of genus g coincides with the class of $C_{2,2g+1}$ curves.

The moduli space of all C_{ab} curves (for varying a and b) of fixed genus g is then a finite union of moduli spaces $\mathcal{M}_{\Delta_{ab}}$. One can show that its dimension equals $2g - 1$ by an analysis of the Weierstrass semigroup, which has been done in Rim–Vitulli [35, Corollary 6.3]. This dimension equals $\dim \mathcal{H}_g = \dim \mathcal{M}_{\Delta_{2,2g+1}}$ and in fact this is the dominating part: in Example 8.7 we will show that $\dim \mathcal{M}_{\Delta_{ab}} < 2g - 1$ if $a, b \geq 3$ and $g \geq 6$.

6. NONDEGENERACY OF CURVES OF GENUS THREE AND FOUR

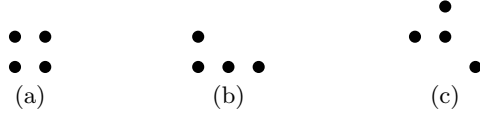
Curves of genus 3. A genus 3 curve V over k is either geometrically hyperelliptic or it canonically embeds in \mathbb{P}_k^2 as a plane quartic.

If V is geometrically hyperelliptic, then V may not be hyperelliptic and hence (by Lemma 5.2) not nondegenerate. For example, over \mathbb{Q} there exist degree 2 covers of the imaginary circle having genus 3. However, if k is finite or $V(k) \neq \emptyset$ then every geometrically hyperelliptic curve is hyperelliptic. If moreover $\#k \geq 17$ we can conclude that V is nondegenerate. See Section 5 for more details.

If V is embedded as a plane quartic, then assuming $\#k \geq 17$, we can apply Lemma 3.2 and see that V is defined by a 4Σ -nondegenerate Laurent polynomial.

Curves of genus 4. Let V be a curve of genus 4 over k . If V is a geometrically hyperelliptic curve then it is hyperelliptic, since the genus is even; thus if $\#k \geq 17$ then V is nondegenerate (see Section 5). Assume therefore that V is nonhyperelliptic. Then it canonically embeds as a curve of degree 6 in \mathbb{P}_k^3 which is the complete intersection of a unique quadric surface Q and a (non-unique) cubic surface C [19, Example IV.5.2.2].

First, we note that if V is Δ -nondegenerate for some nonhyperelliptic lattice polytope $\Delta \subset \mathbb{R}^2$, then Q or C must have combinatorial origins as follows. Let $\Delta^{(1)} = \text{conv}(\text{int}(\Delta) \cap \mathbb{Z}^2)$. Up to equivalence, there are three possible arrangements for these interior lattice points:

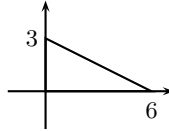


By Proposition 1.7, one verifies that V canonically maps to $X^{(1)} = X(\Delta^{(1)})_k \subset \mathbb{P}_k^3$. In (a), $X^{(1)}$ is nothing else but the Segre product $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ defined by the equation $xz = yw$ in \mathbb{P}_k^3 , and by uniqueness it must equal Q . For (b), $X^{(1)}$ is the singular quadric cone $yz = w^2$, which again must equal Q . For (c), $X^{(1)}$ is the singular cubic $xyz = w^3$, which must be an instance of C . Note that a curve V can be Δ -nondegenerate with $\Delta^{(1)}$ as in (a) or (b), but not both: whether Q is smooth or not is intrinsic, since Q is unique. The third type (c) is special, and we leave it as an exercise to show that the locus of curves of genus 4 which canonically lie on such a singular cubic surface is a codimension ≥ 2 subspace of \mathcal{M}_4 (use the dimension bounds from Section 8).

With these observations in mind, we work towards conditions under which our given nonhyperelliptic genus 4 curve V is nondegenerate. Suppose first that the quadric Q has a (necessarily k -rational) singular point T ; then V is called *conical*. This corresponds to the case where $V_{\bar{k}} = V \times_k \bar{k}$ has a unique g_3^1 , and represents a codimension 1 subscheme of \mathcal{M}_4 [19, Exercise IV.5.3]. If $Q(k) = \{T\}$ then V cannot be nondegenerate with respect to any polytope with $\Delta^{(1)}$ as in (a) or (b), since then Q is not isomorphic to either of the corresponding canonical quadric surfaces $X^{(1)}$. If $Q(k) \supsetneq \{T\}$, which is guaranteed if k is finite or if $V(k) \neq \emptyset$, then after a choice of coordinates we can identify Q with the weighted projective space $\mathbb{P}(1, 2, 1)$. Our degree 6 curve V then has an equation of the form

$$f(x, y, z) = y^3 + a_2(x, z)y^2 + a_4(x, z)y + a_6(x, z)$$

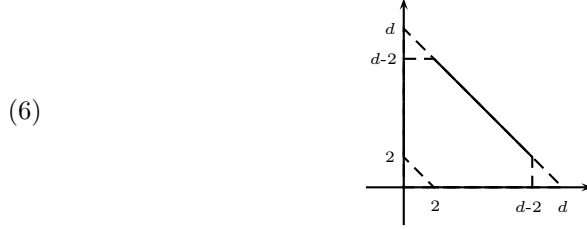
with $\deg a_i = i$; the equation is monic in y because $T \notin V$. By Lemma 3.2, if $\#k \geq 23$ then we may assume that $f(x, y, 1)$ is nondegenerate with respect to its Newton polytope Δ as follows:



Remark 6.1. This argument shows that every conical genus 4 curve is *potentially nondegenerate*, i.e., becomes nondegenerate after a finite extension of k . In fact, we need only take a degree 2 extension which splits the quadric Q : after a k -rational linear change of variable, Q is the cone over a conic C over k , so we may take any field over which C acquires a rational point. This argument works even when $\text{char } k = 2$.

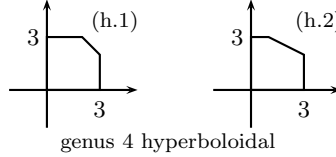
Next, suppose that Q is smooth; then V is called *hyperboloidal*. This corresponds to the case where $V_{\bar{k}}$ has two g_3^1 's, and represents a dense subscheme of \mathcal{M}_4 [19, Exercise IV.5.3]. If $Q \not\cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$ (e.g. this will be the case whenever the discriminant of Q is nonsquare), then again V cannot be nondegenerate with respect to Δ with $\Delta^{(1)}$ as in (a) or (b). Therefore suppose that k is algebraically closed. Then $Q \cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$ and V can be projected to a plane quintic with 2 nodes [19, Exercise IV.5.4].

Consider the line connecting these nodes. Generically, it will intersect the nodes with multiplicity 2, i.e. it will intersect all branches transversally. By Bezout, the line will then intersect the curve transversally in one other point. This observation fits within the following general phenomenon. Let $d \in \mathbb{Z}_{\geq 4}$, and consider the polytope $\Delta = d\Sigma$ with up to three of its angles pruned as follows:



Let $f \in k[x, y]$ be a nondegenerate polynomial with Newton polytope Δ . If we prune no angle of $d\Sigma$, then $X(\Delta)_k \cong \mathbb{P}_k^2$ (it is the image of the d -uple embedding) and $V(f)$ is a smooth plane curve of degree d . Pruning an angle has the effect of blowing up $X(\Delta)_k$ at a coordinate point; the image of $V(f)$ under the natural projection $X(\Delta)_k \rightarrow \mathbb{P}_k^2$ has a node at that point. If we prune $m = 2$ (resp. $m = 3$) angles, then we likewise obtain the blow-up of \mathbb{P}_k^2 at m points and the image of $V(f)$ in \mathbb{P}_k^2 has m nodes. Since f is nondegenerate, the line connecting any two of these nodes intersects the curve transversally elsewhere, and due to the shape of Δ the intersection multiplicity at the nodes will be 2. Conversely, every projective plane curve having at most 3 nodes such that the line connecting any two nodes intersects the curve transversally (also at the nodes themselves), is nondegenerate. Indeed, after an appropriate projective transformation, it will have a Newton polytope as in (6).

Exceptionally, the line connecting the two nodes of our quintic may be tangent to one of the branches at a node. Using a similar reasoning, we conclude that V is Δ -nondegenerate, with Δ equal to polytope (h.2) from Section 7 below.



Remark 6.2. Again, this argument can be used to show that any hyperboloidal curve of genus 4 is potentially nondegenerate. Standard results in the theory of quadratic forms over fields k with $\text{char } k \neq 2$ imply that Q splits, so that $Q \cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$, if and only if $Q(k) \neq \emptyset$ and the discriminant of Q is a square in k : if $Q(k) \neq \emptyset$ then Q splits a hyperbolic plane; by scaling, the orthogonal complement is of the form $x^2 - dy^2$, so if $d \in k^{\times 2}$ then Q splits, and conversely. It follows that any quadric over k splits over an at most quadratic extension. To proceed, we then project V to a plane quintic, which requires $\#k$ to be sufficiently large: one could make this explicit, one could use the Bertini theorem over finite fields due to Poonen [33] and analyze explicitly the finitely many exceptions. Assuming that V has been so projected (extending k further, if necessary), the rest of the argument holds.

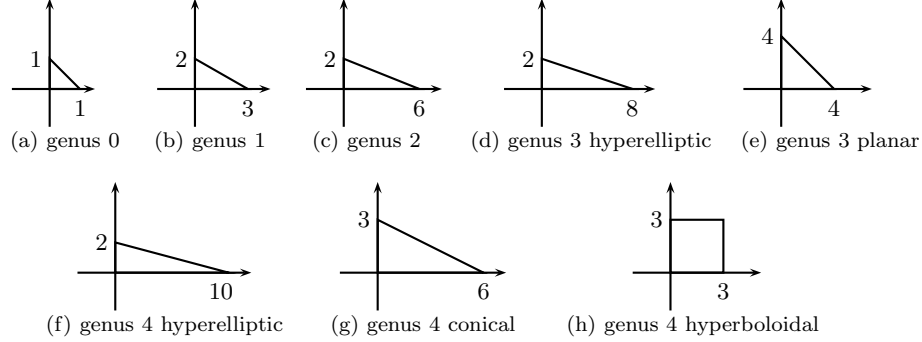
Now, however, we can make a further change of variables to bring all genus 4 hyperboloidal curves under a single polytope. Indeed, if $f(x, y)$ has a Newton polytope of type (h.1) or (h.2), then applying a change of variables to $x^3y^3f(x^{-1}, y^{-1})$

of the form $(x, y) \mapsto (x+a, y+b)$ for $a, b \in k$ yields a square 3×3 Newton polytope. So replacing the two polytopes of class (h) by the single polytope in the summary below.

Remark 6.3. As in Remark 4.2, an argument based on the index shows that there exist genus 4 curves which are not nondegenerate. A result by Clark [10] states that for every $g \geq 2$, there exists a number field k and a genus g curve V over k , such that the index of V is equal to $2g - 2$, the degree of the canonical divisor. In particular, there exists a genus 4 curve V of index 6. Such a curve cannot be nondegenerate. Indeed, for each of the above arrangements (a)–(c), $X^{(1)}$ contains the line $z = w = 0$, which cuts out an effective divisor on V of degree 3 in cases (a) and (b) and degree 2 in case (c).

7. NONDEGENERACY OF LOW GENUS CURVES: SUMMARY

We now summarize the results of the preceding sections. If k is an algebraically closed field, then every curve V of genus at most 4 over k can be modeled by a nondegenerate polynomial having one of the following as Newton polytope:



Moreover, each of these classes are disjoint. For the polytopes (c)–(h), we have $\dim \mathcal{M}_\Delta = 3, 5, 6, 7, 8, 9$, respectively. All hyperelliptic curves and C_{ab} curves are nondegenerate.

For an arbitrary perfect field k , if V is not hyperboloidal and has genus at most 4, then V is nondegenerate whenever k is a sufficiently large finite field, or when k is infinite and $V(k) \neq \emptyset$; for the former, the condition $\#k \geq 23$ is sufficient but most likely superfluous (see Remark 7.2).

Remark 7.1. We can situate the nonhyperelliptic C_{ab} curves that lie in this classification. In genus 3, we have $C_{3,4}$ curves, which have a smooth model in \mathbb{P}_k^2 , since $\Delta_{3,4}$ is nonhyperelliptic. In genus 4, we have $C_{3,5}$ curves, which are conical: this can be seen by analyzing the interior lattice points of $\Delta_{3,5}$, as in Section 6.

Remark 7.2. In case $\#k < \infty$, we proved (without further condition on $\#k$) that if V is not hyperboloidal then it can be modeled by a polynomial $f \in k[x, y]$ with Newton polytope contained in one of the polytopes (a)–(g). The condition on $\#k$ then came along with an application of Lemma 3.2 to deduce nondegeneracy. In the $g = 1$ case, we got rid of this condition by using non-linear transformations (completing the square) and allowing smaller polytopes. Similar techniques can be used to improve (and probably even remove) the bounds on $\#k$ in genera $2 \leq g \leq 4$.

For example, using naive brute force computation we have verified that in genus 2, all curves are nondegenerate whenever $\#k = 2, 4, 8$.

8. AN UPPER BOUND FOR $\dim \mathcal{M}_g^{\text{nd}}$

From now on, we assume $k = \bar{k}$. In this section, we prepare for a proof of Theorem 11.1, which gives an upper bound for $\dim \mathcal{M}_g^{\text{nd}}$ in terms of g .

For a lattice polytope $\Delta \subset \mathbb{Z}^2$ with $g \geq 2$ interior lattice points, we sharpen the obvious upper bound $\dim \mathcal{M}_\Delta \leq \dim M_\Delta = \#(\Delta \cap \mathbb{Z}^2) - 1$ (see (2)) by incorporating the action of the automorphism group of $X(\Delta)_k$, which has been explicitly described by Bruns and Gubeladze [7, Section 5]. In Sections 9–11 we then work towards a bound in terms of g , following ideas of Haase and Schicho [17].

The automorphisms of $X(\Delta)_k = \text{Proj } k[\Delta] \hookrightarrow \mathbb{P}^{\#(\Delta \cap \mathbb{Z}^2) - 1}$ correspond to the graded k -algebra automorphisms of $k[\Delta]$, and admit a combinatorial description as follows.

Definition 8.1. A nonzero vector $v \in \mathbb{Z}^2$ is a column vector of Δ if there exists a facet $\tau \subset \Delta$ (the base facet) such that

$$v + ((\Delta \setminus \tau) \cap \mathbb{Z}^2) \subset \Delta.$$

We denote by $c(\Delta)$ the number of column vectors of Δ .

Example 8.2. Any multiple of the standard 2-simplex Σ has 6 column vectors. The octagonal polytope below shows that a polytope may have no column vectors.

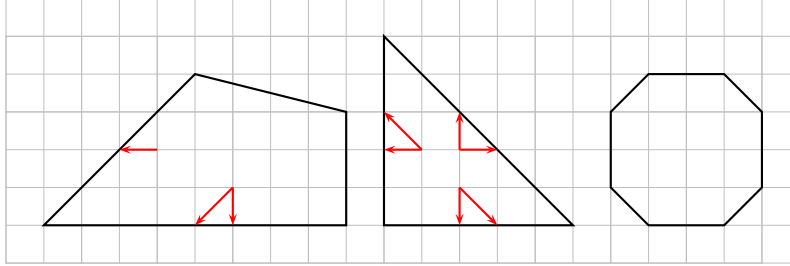


Figure 8.2: Column vectors of some lattice polytopes

The dimension of the automorphism group $\text{Aut}(X(\Delta)_k)$ is then determined as follows.

Proposition 8.3 (Bruns–Gubeladze [7, Theorem 5.3.2]). *We have*

$$\dim \text{Aut}(X(\Delta)_k) = c(\Delta) + 2.$$

Proof sketch. One begins with the 2-dimensional subgroup of $\text{Aut}(X(\Delta)_k)$ induced by the inclusion $\text{Aut}(\mathbb{T}^2) \hookrightarrow \text{Aut}(X(\Delta)_k)$. On the $k[\Delta]$ -side, this corresponds to the graded automorphisms induced by $(x, y) \mapsto (\lambda x, \mu y)$ for $\lambda, \mu \in k^{*2}$.

Next, column vectors of Δ correspond to automorphisms of $X(\Delta)_k$ in the following way. If v is a column vector, modulo equivalence we may assume that $v = (0, -1)$, that the base facet is supported on the X -axis, and that Δ is contained in the positive quadrant $\mathbb{R}_{\geq 0}^2$. Let $f(x, y) \in k[x, y]$ be supported on Δ . Since the vector $v = (0, -1)$ is a column vector, the polynomial $f(x, y + \lambda)$ will again be

supported on Δ , for any $\lambda \in k$. Hence v induces a family of graded automorphisms $k[\Delta] \rightarrow k[\Delta]$, corresponding to a one-dimensional subgroup of $\text{Aut}(X(\Delta)_k)$.

It then remains to show that these subgroups are algebraically independent from each other and from $\text{Aut}(\mathbb{T}^2)$, and that together they generate $\text{Aut}(X(\Delta)_k)$ (after including the finitely many automorphisms coming from \mathbb{Z} -affine transformations mapping Δ to itself). \square

Using the fact that a curve of genus $g \geq 2$ has finitely many automorphisms we obtain the following corollary. We leave the details as an exercise.

Corollary 8.4. *We have $\dim \mathcal{M}_\Delta \leq m(\Delta) := \#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3$.*

Remark 8.5. In order to have equality, it is sufficient that Δ is a so-called *maximal polytope* (see Section 10 for the definition). This is the main result of Koelman's thesis [22, Theorem 2.5.12].

Example 8.6. Let $\Delta = \text{conv}(\{(0,0), (2g+2,0), (0,2)\})$ as in section 5, so that $\dim \mathcal{M}_\Delta = 2g - 1$. One verifies that $c(\Delta) = g + 4$, so the upper bound in Corollary 8.4 reads $m(\Delta) = (3g+6) - (g+4) - 3 = 2g - 1$; so in this case, the bound is sharp. It is easy to verify that the bound is also sharp if $\Delta = d\Sigma$, $d \in \mathbb{Z}_{\geq 4}$; then $\dim \mathcal{M}_\Delta$ reads $(d+1)(d+2)/2 - 9 = g + 3d - 9 \leq 2g$. The latter are examples of maximal polytopes. Opposed to this, let $(d\Sigma)_0$ be obtained from $d\Sigma$ by pruning off $(0,0)$. This reduces the number of lattice points by 1 and the number of column vectors by 2. Hence our bound increases, although $d\Sigma$ and $(d\Sigma)_0$ give rise to the same moduli space. Indeed, pruning off $(0,0)$ only forces our curves in $X(d\Sigma)_k \cong \mathbb{P}^2$ to pass through $(0,0,1)$.

Example 8.7. We now use Corollary 8.4 to show that the dimension of the moduli space of nonhyperelliptic C_{ab} curves of genus g (where a and b may vary) has dimension strictly smaller than $2g - 1 = \dim \mathcal{H}_g$ whenever $g \geq 6$. Consider $\Delta_{ab} = \text{Conv}\{(0,a), (b,0), (0,0)\}$ with $a, b \in \mathbb{Z}_{\geq 3}$ coprime. Then we have

$$g = (a-1)(b-1)/2, \quad \#(\Delta \cap \mathbb{Z}^2) = g + a + b + 1,$$

and the set of column vectors is given by

$$\{(n, -1) : n = 0, \dots, \lfloor b/a \rfloor\} \cup \{(-1, m) : m = 0, \dots, \lfloor a/b \rfloor\}.$$

Suppose without loss of generality that $a < b$. Then a is bounded by $\sqrt{2g} + 1$. Corollary 8.4 yields

$$\dim \mathcal{M}_\Delta \leq m(\Delta) = g + a + b + 1 - \left(\left\lfloor \frac{b}{a} \right\rfloor + 2 \right) - 3 < a + \frac{2g-1}{a} + g - 2.$$

As a (real) function of a , this upper bound has a unique minimum at $a = \sqrt{2g-1}$. Therefore, to deduce that it is strictly smaller than $2g - 1$ for all $a \in [3, \sqrt{2g} + 1]$, it suffices to verify so for the boundary values $a = 3$ and $a = \sqrt{2g} + 1$, which is indeed the case if $g \geq 6$.

9. A BOUND IN TERMS OF THE GENUS

Throughout the rest of this article, we will employ the following notation. Let $\Delta^{(1)}$ be the convex hull of the interior lattice points of Δ . Let r (resp. $r^{(1)}$) denote the number of lattice points on the boundary of Δ (resp. $\Delta^{(1)}$), and let $g^{(1)}$ denote the number of interior lattice points in $\Delta^{(1)}$, so that $g = g^{(1)} + r^{(1)}$.

We now prove the following preliminary bound.

Proposition 9.1. *If Δ has at least $g \geq 2$ interior lattice points, then $\dim \mathcal{M}_\Delta \leq 2g + 3$.*

Proof. We may assume that Δ is nonhyperelliptic, because otherwise $\dim \mathcal{M}_\Delta \leq 2g - 1$ by Lemma 5.1. We may also assume that $\Delta^{(1)}$ is not a multiple of Σ , since otherwise Δ -nondegenerate curves are canonically embedded in $X(\Delta^{(1)})_k \cong \mathbb{P}_k^2$ using Proposition 1.7; then from Example 8.6 it follows that $\dim \mathcal{M}_\Delta \leq 2g$.

An upper bound for $\dim \mathcal{M}_\Delta$ in terms of g then follows from a lemma by Haase and Schicho [17, Lemma 12], who proved that $r \leq r^{(1)} + 9$, in which equality holds if and only if $\Delta = d\Sigma$ for some $d \in \mathbb{Z}_{\geq 4}$ (a case which we have excluded). Hence

$$(7) \quad \#(\Delta \cap \mathbb{Z}^2) = g + r \leq g + r^{(1)} + 8 = 2g + 8 - g^{(1)},$$

and thus

$$(8) \quad \dim \mathcal{M}_\Delta \leq m(\Delta) = \#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 \leq 2g + 5 - c(\Delta) - g^{(1)} \leq 2g + 5.$$

This bound improves to $2g + 3$ if $g^{(1)} \geq 2$, so we remain with two cases: $g^{(1)} = 0$ and $g^{(1)} = 1$.

Suppose first that $g^{(1)} = 0$. Then by Lemma 9.2 below, any Δ -nondegenerate curve is either a smooth plane quintic (excluded), or a trigonal curve. Since the moduli space of trigonal curves has dimension $2g + 1$ (a classical result, see also Section 12 below), the bound holds.

Next, suppose that $g^{(1)} = 1$. Then, up to equivalence, there are only 16 possibilities for $\Delta^{(1)}$, which are listed in [34, Figure 2] or in the appendix below. Hence, there are only finitely many possibilities for Δ , and for each of these polytopes we find that $\#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 \leq 2g + 2$. \square

In fact, for all but the 5 polytopes in Figure 9 (up to equivalence), we find that the stronger bound $\#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 \leq 2g + 1$ holds.

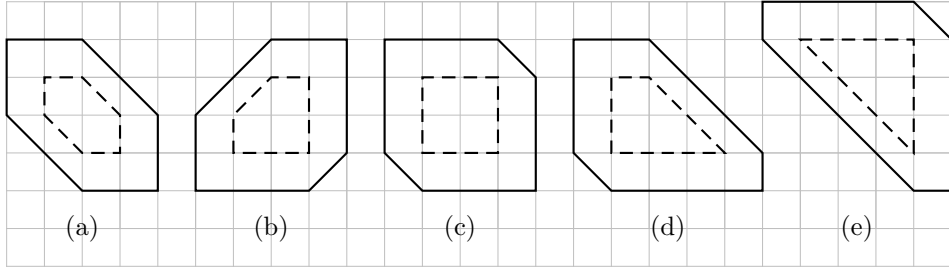


Figure 9: Polytopes with $g^{(1)} = 1$ and $\#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 = 2g + 2$

Lemma 9.2. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be nondegenerate and suppose that the interior lattice points of $\Delta(f)$ are not collinear. Let $\Delta^{(1)}$ be the convex hull of these interior lattice points.*

- (a) *If $\Delta^{(1)}$ has no interior lattice points, then $V(f)$ is either trigonal or isomorphic to a smooth plane quintic.*
- (b) *If $V(f)$ is trigonal or isomorphic to a smooth plane quintic, and $\Delta^{(1)}$ has at least 4 lattice points on the boundary, then $\Delta^{(1)}$ has no interior lattice points.*

Proof. Koelman gives a proof of this in his Ph.D. thesis [22, Lemma 3.2.13], based on Petri's theorem. A more combinatorial argument uses the fact that lattice polytopes of genus 0 are equivalent with either 2Σ , or with a polytope that is caught between two horizontal lines of distance 1. This was proved independently by Arkinstall, Khovanskii, Koelman, and Schicho (see the generalized statement by Batyrev-Nill [5, Theorem 2.5]).

In the first case, Δ -nondegenerate curves are canonically embedded in $X(2\Sigma)_k \cong \mathbb{P}_k^2$, hence they are isomorphic to smooth plane quintics.

In the second case, it follows that Δ is caught between two horizontal lines of distance 3. This may actually fail if $\Delta^{(1)} = \Sigma$, which corresponds to smooth plane quartics. But in both situations, Δ -nondegenerate curves are trigonal.

For (b), using the canonical divisor K_Δ from Proposition 1.7, one sees that the canonical embedding of $V(f)$ in \mathbb{P}_k^{g-1} is contained in $X(\Delta^{(1)})_k$. According to a theorem of Koelman [23], the condition of having at least 4 lattice points on the boundary ensures that $X(\Delta^{(1)})$ is generated by quadrics. Now since $V(f)$ is trigonal or isomorphic to a smooth plane quintic, by Petri's theorem the intersection of *all* quadrics containing $V(f)$ is a surface of sectional genus 0. Hence this surface must be $X(\Delta^{(1)})_k$ and $\Delta^{(1)}$ must have genus 0. \square

Remark 9.3. The condition that $\Delta^{(1)}$ should have at least 4 lattice points on the boundary is necessary in Lemma 9.2. For example, let k be algebraically closed and let $\Delta = \text{conv}\{(2, 0), (0, 2), (-2, -2)\}$. Then Δ is a lattice polytope of genus 4, hence all Δ -nondegenerate curves are trigonal. However, $\Delta^{(1)}$ contains $(0, 0)$ in its interior. Note that $X(\Delta^{(1)})_k \subset \mathbb{P}_k^3$ is the cubic $xyz = w^3$.

10. REFINING THE UPPER BOUND: MAXIMAL POLYTOPES

We further refine the bound in Proposition 9.1 by adapting the proof of the Haase–Schicho bound $r \leq r^{(1)} + 9$ in order to obtain an estimate for $r - c(\Delta)$ instead of just r . We first do this for *maximal* polytopes, and treat nonmaximal polytopes in the next section.

Definition 10.1. A lattice polytope $\Delta \subset \mathbb{Z}^n$ is maximal if Δ is not properly contained in another lattice polytope with the same interior lattice points, i.e., for all lattice polytopes $\Delta' \supsetneq \Delta$, we have

$$\text{int}(\Delta') \cap \mathbb{Z}^n \neq \text{int}(\Delta) \cap \mathbb{Z}^n.$$

We define the *relaxed polytope* $\Delta^{(-1)}$ of a lattice polytope $\Delta \subset \mathbb{Z}^2$ as follows. Assume that $0 \in \Delta$. To each facet $\tau \subset \Delta$ given by an inequality of the form $a_1X + a_2Y \leq b$ with $a_i \in \mathbb{Z}$ coprime, we define the *relaxed inequality* $a_1X + a_2Y \leq b + 1$ and let $\Delta^{(-1)}$ be the intersection of these relaxed inequalities. If p is a vertex of Δ given by the intersection of two such facets, we define the *relaxed vertex* $p^{(-1)}$ to be the intersection of the boundaries of the corresponding relaxed inequalities.

Lemma 10.2 (Haase–Schicho [17, Lemmas 9–10], Koelman [22, Section 2.2]). *Let $\Delta \subset \mathbb{Z}^2$ be a 2-dimensional lattice polytope. Then $\Delta^{(-1)}$ is a lattice polytope if and only if $\Delta = \Delta'^{(1)}$ for some lattice polytope Δ' . Furthermore, if Δ is nonhyperelliptic, then Δ is maximal if and only if $\Delta = (\Delta^{(1)})^{(-1)}$.*

The proof of the Haase–Schicho bound $r \leq r^{(1)} + 9$ utilizes a theorem of Poonen and Rodriguez-Villegas [34], which we now introduce.

A *legal move* is a pair (v, w) with $v, w \in \mathbb{Z}^2$ such that $\text{conv}(\{0, v, w\})$ is a 2-dimensional triangle whose only nonzero lattice points lie on $e(v, w)$, the edge between v and w . The *length* of a legal move (v, w) is

$$\ell(v, w) = \det \begin{pmatrix} v \\ w \end{pmatrix},$$

which is of absolute value $r - 1$, where $r = \#(e(v, w) \cap \mathbb{Z}^2)$ is the number of lattice points on the edge between v and w . Note that the length can be negative.

A *legal loop* \mathcal{P} is a sequence of vectors $v_1, v_2, \dots, v_n \in \mathbb{Z}^2$ such that for all $i = 1, \dots, n$ and indices taken modulo n , we have:

- (v_i, v_{i+1}) is a legal move, and
- v_{i-1}, v_i, v_{i+1} are not contained in a line.

The *length* $\ell(\mathcal{P})$ of a legal loop \mathcal{P} is the sum of the lengths of its legal moves.

The *winding number* of a legal loop is its winding number around 0 in the sense of algebraic topology. The *dual loop* \mathcal{P}^\vee is given by w_1, \dots, w_n , where $w_i = \ell(v_i, v_{i+1})^{-1} \cdot (v_{i+1} - v_i)$ for $i = 1, \dots, n$. One can check that this is again a legal loop with the same winding number as \mathcal{P} and that $\mathcal{P}^{\vee\vee} = \mathcal{P}$ after a 180° rotation.

Theorem 10.3 (Poonen–Rodriguez-Villegas [34, Section 9.1]). *Let \mathcal{P} be a legal loop with winding number w . Then $\ell(\mathcal{P}) + \ell(\mathcal{P}^\vee) = 12w$.*

Now let $\Delta \subset \mathbb{Z}^2$ be a maximal polytope with 2-dimensional interior $\Delta^{(1)}$. We associate to Δ a legal loop $\mathcal{P}(\Delta)$ as follows. By Lemma 10.2, Δ is obtained from $\Delta^{(1)}$ by relaxing the edges. Let p_1, \dots, p_n be the vertices of $\Delta^{(1)}$, enumerated counterclockwise; then $\mathcal{P}(\Delta)$ is given by the sequence $q_i = p_i^{(-1)} - p_i$ where $p_i^{(-1)}$ is the relaxed vertex of p_i .

Example 10.4. The following picture, inspired by Haase–Schicho [17, Figure 20], is illustrative: it shows a polytope Δ with 2-dimensional interior $\Delta^{(1)}$, the associated legal loop $\mathcal{P}(\Delta)$, and its dual $\mathcal{P}(\Delta)^\vee$. In this example, $\ell(\mathcal{P}(\Delta)) = \ell(\mathcal{P}(\Delta)^\vee) = 6$.

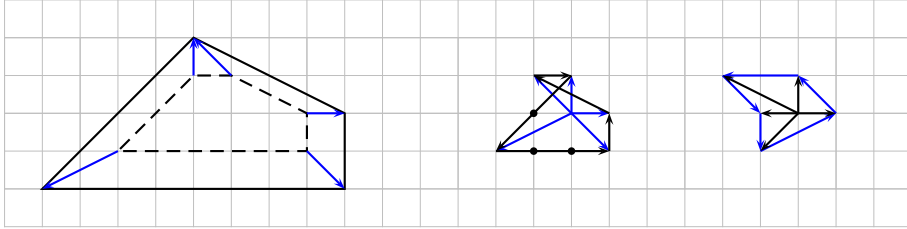


Figure 10.4: The legal loop $\mathcal{P}(\Delta)$ associated to a lattice polytope Δ

A crucial observation is that the bold-marked lattice points of $\mathcal{P}(\Delta)$ are column vectors of Δ . This holds in general and lies at the core of our following refinement of the Haase–Schicho bound.

Lemma 10.5. *If Δ is maximal and nonhyperelliptic, then:*

- $r - r^{(1)} = \ell(\mathcal{P}(\Delta)) \leq 9$.
- $r - r^{(1)} - c(\Delta) \leq \min(\ell(\mathcal{P}(\Delta)), \ell(\mathcal{P}(\Delta)^\vee)) \leq 6$.

Proof. We abbreviate $\mathcal{P} = \mathcal{P}(\Delta)$.

Inequality (a) is by Haase–Schicho [17, Lemma 11] and works as follows. The length of the legal move (q_i, q_{i+1}) measures the difference between the number of

lattice points on the facet of Δ connecting $p_i^{(-1)}$ and $p_{i+1}^{(-1)}$, and the number of lattice points on the edge of $\Delta^{(1)}$ connecting p_i and p_{i+1} . Therefore $r - r^{(1)} = \ell(\mathcal{P})$. The dual loop \mathcal{P}^\vee walks (in a consistent and counterclockwise-oriented way) through the direction vectors of the edges of $\Delta^{(1)}$, therefore each move has positive length and we have $\ell(\mathcal{P}(\Delta)^\vee) \geq 3$. Since \mathcal{P}^\vee has winding number 1, the statement follows from Theorem 10.3. (One can further show that equality holds if and only if Δ is a multiple of the standard 2-simplex Σ .)

To prove inequality (b), we first claim: there is a bijection between lattice points v which lie properly on a counterclockwise-oriented (positive length) legal move $q_i q_{i+1}$ of \mathcal{P} , and column vectors of Δ with base facet $p_i^{(-1)} p_{i+1}^{(-1)}$. Indeed, after an appropriate transformation, we may assume as in Proposition 8.3 that $v = (0, -1)$, that $p_i^{(-1)}$ and $p_{i+1}^{(-1)}$ lie on the X -axis, and that Δ is contained in the positive quadrant $\mathbb{R}_{\geq 0}^2$; after these normalizations, the claim is straightforward.

Now, since the dual loop \mathcal{P}^\vee consists of counterclockwise-oriented legal moves only, it has at most $\ell(\mathcal{P}^\vee)$ vertices. Since $\mathcal{P} = \mathcal{P}^{\vee\vee}$ (after 180° rotation), \mathcal{P} has at most $\ell(\mathcal{P}^\vee)$ vertices. By the claim, we have $\ell(\mathcal{P}) \leq \ell(\mathcal{P}^\vee) + c$, and the result follows by combining this with part (a) and Theorem 10.3. \square

Corollary 10.6. *If Δ is maximal, then $\dim \mathcal{M}_\Delta \leq 2g + 3 - g^{(1)}$. In particular, if $g^{(1)} \geq 2$ then $\dim \mathcal{M}_\Delta \leq 2g + 1$.*

Proof. By Lemma 10.5, we have $m(\Delta) = g + r - 3 - c(\Delta) \leq g + r^{(1)} + 3 \leq 2g + 3 - g^{(1)}$. \square

Remark 10.7. Note that Lemma 10.5(a) immediately extends to nonmaximal polytopes ($r - r^{(1)}$ can only decrease), so the Haase–Schicho bound holds for arbitrary nonhyperelliptic polytopes. This we cannot conclude for part (b): if r decreases, $c(\Delta)$ may decrease more quickly so that the bound no longer holds. An example of such behaviour can be found in Figure 9(c).

11. REFINING THE UPPER BOUND: GENERAL POLYTOPES

We are now ready to prove the main result of Sections 8–11.

Theorem 11.1. *If $g \geq 2$, then $\dim \mathcal{M}_g^{\text{nd}} \leq 2g + 1$ except for $g = 7$ where we have $\dim \mathcal{M}_7^{\text{nd}} \leq 16$.*

Proof. It suffices to show that the claimed bounds hold for all polytopes Δ with g interior lattice points. By the proof of Proposition 9.1, we may assume that $\Delta^{(1)}$ is two-dimensional, that it is not a multiple of Σ , and that it has $g^{(1)} \geq 1$ interior lattice points.

Let us first assume that $g^{(1)} \geq 2$. We will show that $\dim \mathcal{M}_\Delta \leq 2g + 1$. From Corollary 10.6, we know that this is true if Δ is maximal. Therefore, suppose that Δ is nonmaximal; then it is obtained from a maximal polytope $\tilde{\Delta}$ by taking away points on the boundary (keeping the interior lattice points intact). If two or more boundary points are taken away, then as in (8) we have

$$m(\Delta) \leq \#(\Delta \cap \mathbb{Z}^2) - 3 \leq \#(\tilde{\Delta} \cap \mathbb{Z}^2) - 2 - 3 \leq 2g + 5 - g^{(1)} - 2 \leq 2g + 1.$$

So we may assume that $\Delta = \text{conv}(\tilde{\Delta} \cap \mathbb{Z}^2 \setminus \{p\})$ for a vertex $p \in \tilde{\Delta}$. Similarly, we may assume that $c(\Delta) < c(\tilde{\Delta})$, for else

$$m(\Delta) = \#(\Delta \cap \mathbb{Z}^2) - c(\Delta) - 3 \leq \#(\tilde{\Delta} \cap \mathbb{Z}^2) - c(\tilde{\Delta}) - 3 = m(\tilde{\Delta}) \leq 2g + 1.$$

Let v be a column vector of $\tilde{\Delta}$ that is no longer a column vector of $\Delta = \text{conv}(\tilde{\Delta} \cap \mathbb{Z}^2 \setminus \{p\})$. Then p must lie on the base facet τ of v . After an appropriate transformation, we may assume that $p = (0, 0)$, that $v = (0, -1)$, that τ lies along the X -axis, and that $\tilde{\Delta}$ lies in the positive quadrant, as follows.

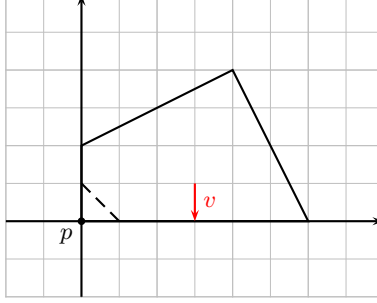


Figure 11.2: An almost maximal polytope.

Note that $(1, 1) \in \text{int}(\tilde{\Delta})$ since otherwise v would still be a column vector of Δ . But then the other facet of $\tilde{\Delta}$ which contains p must be supported on the Y -axis, for else $(1, 1)$ would no longer be in $\text{int}(\Delta)$. One can now verify that if $f(x, y)$ is Δ -nondegenerate, then for all but finitely many $\lambda \in k$, the polynomial $f(x, y + \lambda)$ will have Newton polytope $\tilde{\Delta}$ and all but finitely of those will be $\tilde{\Delta}$ -nondegenerate. Therefore, we have $\mathcal{M}_{\Delta} \subset \mathcal{M}_{\tilde{\Delta}}$, and the dimension estimate follows.

Now suppose that $g^{(1)} = 1$. From the finite computation in the proof of Proposition 9.1, we know that the bound $\dim \mathcal{M}_{\Delta} \leq 2g + 1$ holds if Δ is not among the polytopes listed in Figure 9. Now in this list, the polytopes (b)–(e) are not maximal, and for these polytopes the same trick as in the $g^{(1)} \geq 2$ case applies. However, polytope (a) is maximal and contains 7 interior lattice points: therefore, we can only prove $\dim \mathcal{M}_7^{\text{nd}} \leq 16$. \square

Let Δ be a nonmaximal nonhyperelliptic lattice polytope, and let $\tilde{\Delta} = (\Delta^{(1)})^{(-1)}$ be the smallest maximal polytope containing Δ . Let $f \in k[x^{\pm}, y^{\pm}]$ be a Δ -nondegenerate Laurent polynomial. Since $\Delta \subset \tilde{\Delta}$, we can consider the (degree 1) locus \tilde{V} of $f = 0$ in $X(\tilde{\Delta})_k = \text{Proj } k[\tilde{\Delta}]$. Then one can wonder whether the observation we made in the proof of Theorem 11.1 holds in general: is there always a $\sigma \in \text{Aut}(X(\tilde{\Delta})_k)$ such that $\sigma(\tilde{V}) \cap \mathbb{T}_k^2$ is defined by a $\tilde{\Delta}$ -nondegenerate polynomial? The answer is no, because it is easy to construct examples where the only automorphisms of $X(\tilde{\Delta})_k$ are those induced by $\text{Aut}(\mathbb{T}_k^2)$. Then $\sigma(\tilde{V}) \cap \mathbb{T}_k^2$ is always defined by $f(\lambda x, \mu y)$ (for some $\lambda, \mu \in k^*$), which does not have $\tilde{\Delta}$ as its Newton polytope and hence cannot be $\tilde{\Delta}$ -nondegenerate.

However, f is very close to being $\tilde{\Delta}$ -nondegenerate, and this line of thinking leads to the following observation. Let p be a vertex of $\tilde{\Delta}$ that is not a vertex of Δ , and let q_1, q_2 be the closest lattice points to p on the respective facets of $\tilde{\Delta}$ containing p . The triangle spanned by p, q_1, q_2 cannot contain any other lattice points, because otherwise removing p would affect the interior of $\tilde{\Delta}$. Thus the volume of this triangle is equal to $1/2$ by Pick's theorem, and the affine chart of $X(\tilde{\Delta})_k$ attached to the cone at p is isomorphic to \mathbb{A}_k^2 . In particular, $X(\tilde{\Delta})_k$ is nonsingular in the zero-dimensional torus \mathbb{T}_p corresponding to p . Then f fails to be $\tilde{\Delta}$ -nondegenerate

only because \tilde{V} passes through \mathbb{T}_p (i.e. passes through $(0,0) \in \mathbb{A}_k^2$), elsewhere it fulfils the conditions of nondegeneracy: \tilde{V} is smooth, intersects the 1-dimensional tori associated to the facets of $\tilde{\Delta}$ transversally, and does not contain the singular points of $X(\tilde{\Delta})_k$. Now following the methods of Section 2, one could construct the bigger moduli space of curves satisfying this weaker nondegeneracy condition. Its dimension would still be bounded by $\#(\tilde{\Delta} \cap \mathbb{Z}^2) - c(\tilde{\Delta}) - 3$, which by Lemma 10.5 is at most $2g + 3 - g^{(1)}$ because $\tilde{\Delta}$ is maximal. Therefore $\dim \mathcal{M}_\Delta \leq 2g + 3 - g^{(1)}$ for nonmaximal Δ , and this yields an alternative proof of Theorem 11.1. Related observations have been made by Koelman [22, Section 2.6].

12. TRIGONAL CURVES, TRINODAL SEXTICS, AND SHARPNESS OF OUR BOUNDS

For $g \geq 2$, we implicitly proved in Section 5 that $\dim \mathcal{M}_g^{\text{nd}} \geq 2g - 1$. But already in genera 3 and 4, by the results in Section 6 we have $\dim \mathcal{M}_3^{\text{nd}} = 6$ and $\dim \mathcal{M}_4^{\text{nd}} = 9$, so this lower bound is an underestimation. For higher genera, we prove in this last section that the bounds given in Theorem 11.1 are sharp, mainly by investigating spaces of trigonal curves. Our main result is the following.

Theorem 12.1. *If $g \geq 4$, then $\dim \mathcal{M}_g^{\text{nd}} = 2g + 1$ except for $g \neq 7$ where $\dim \mathcal{M}_7^{\text{nd}} = 16$.*

Proof. It suffices to find for every genus $g \geq 5$ a lattice polytope Δ with g interior lattice points, for which $\dim \mathcal{M}_\Delta = 2g + 1$ if $g \neq 7$, and $\dim \mathcal{M}_\Delta = 16$ if $g = 7$. If $g = 2h$ is even, let Δ be the rectangle

$$(9) \quad \text{conv}(\{(0,0), (0,3), (h+1,3), (h+1,0)\}).$$

Note that then $\#(\Delta \cap \mathbb{Z}^2) = 2g + 8$ and $c(\Delta) = 4$. If $g = 2h + 1$ is odd but different from 7, let Δ be the trapezium

$$(10) \quad \text{conv}(\{(0,0), (0,3), (h,3), (h+3,0)\}).$$

Again, $\#(\Delta \cap \mathbb{Z}^2) = 2g + 8$ and $c(\Delta) = 4$. Finally, if $g = 7$ then let Δ be

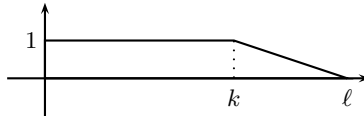
$$(11) \quad \text{conv}\{(2,0), (0,2), (-2,2), (-2,0), (0,-2), (2,-2)\}$$

(i.e. the polytope given in Figure 9(a)). Here, $\#(\Delta \cap \mathbb{Z}^2) = 19$ and $c(\Delta) = 0$.

We first prove that

$$(12) \quad \dim \mathcal{M}_\Delta = \#(\Delta \cap \mathbb{Z}^2) - 1 - \dim \text{Aut}(X(\Delta)_k),$$

holds for the families of polytopes (9) and (10), for which the result then follows from Proposition 8.3. This can be achieved using the well-known theory of trigonal curves [11, 26]. More generally, let $k, \ell \in \mathbb{Z}_{\geq 2}$ satisfy $k \leq \ell$, let $\Delta^{(1)}$ be the trapezium



and let $\Delta = \Delta^{(1)(-1)}$. In general, $\Delta^{(1)(-1)}$ need not be a lattice polygon: it may take some of its vertices outside \mathbb{Z}^2 ; but when $k = \ell$ and $k = \ell - 1$, corresponding to (9) and (10), respectively, the polytope $\Delta^{(1)(-1)}$ takes its vertices in \mathbb{Z}^2 .

Remark 12.2. In fact, using the combinatorial criterion from Lemma 10.2, one can verify that $\Delta^{(1)(-1)}$ is a lattice polygon if and only if $\ell \leq (2g - 2)/3$, where $g = k + \ell + 2$. This confirms a well-known inequality on the Maroni invariants of a trigonal curve (where the inequality is proven using the Riemann-Roch theorem).

Then in these cases, if a curve V is Δ -nondegenerate, it is trigonal of genus $g = k + \ell + 2$. By Proposition 1.7, it can be canonically embedded in $X(\Delta^{(1)})_k$, which is the rational surface scroll $S_{k,\ell} \subset \mathbb{P}_k^{g-1}$. By Petri's theorem [1], this scroll is the intersection of all quadrics containing the canonical embedding. As a consequence, two different such canonical embeddings must differ by an automorphism of \mathbb{P}_k^{g-1} that maps $X(\Delta^{(1)})_k$ to itself; in other words, any two canonical embeddings of V must differ by an automorphism of $X(\Delta^{(1)})_k$.

Now let $f_1, f_2 \in k[x^\pm, y^\pm]$ be Δ -nondegenerate polynomials such that $V(f_1)$ and $V(f_2)$ are isomorphic as abstract curves. Since the fans associated to Δ and $\Delta^{(1)}$ are the same, we have $X(\Delta)_k = X(\Delta^{(1)})_k$. Under this identification, $V(f_1)$ and $V(f_2)$ become canonical curves that must differ by an automorphism of $X(\Delta)_k$. Thus we can conclude (12). (We note that although any trigonal curve is canonically embedded in some rational normal scroll $S_{k,\ell}$ and hence in some $X(\Delta)_k$, it might fail to be nondegenerate because it can be impossible to avoid tangency to $X(\Delta)_k \setminus \mathbb{T}_k^2$.)

To conclude, suppose that Δ is as in (11). We refer to the pruned simplex (6) and the accompanying discussion; here we have $d = 6$. It follows that if f is a Δ -nondegenerate polynomial, then f gives rise to a plane sextic V with three nodes (at the coordinate points) and no other singularities. Conversely, any trinodal sextic where any line connecting two nodes intersects the curve transversally elsewhere, is Δ -nondegenerate. Since the latter is an open condition, \mathcal{M}_Δ is the Zariski closure of the moduli space $\mathcal{V}_{3,6}$ of trinodal plane sextics. The variety $\mathcal{V}_{3,6}$ is in its turn the image of a Severi variety [37], and it is classical that $\dim \mathcal{V}_{3,6} = 16$ —for a modern treatment, see Sernesi [36]. \square

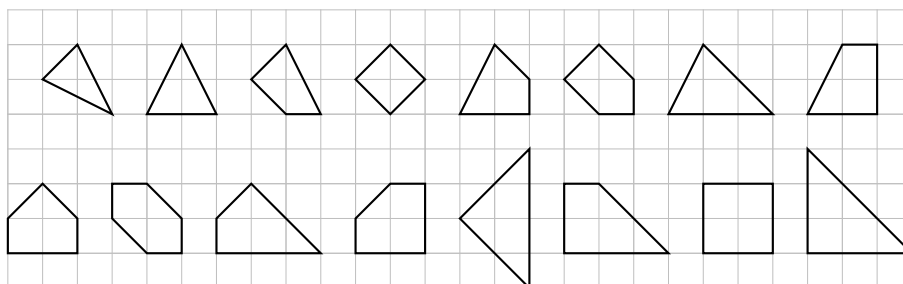
Remark 12.3. In his Ph.D. thesis, Koelman [22, Theorem 2.5.12] proves that Equation (12) holds for any polytope $\Delta \subset \mathbb{R}^2$ which is maximal and nonhyperelliptic. In fact, Koelman assumes $k = \mathbb{C}$, but his methods extend to an arbitrary algebraically closed field $k = \overline{k}$. This provides another proof of Theorem 12.1, but we are content to prove our results in the above more elementary (and classical) way.

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APPENDIX: LATTICE POLYTOPES OF GENUS ONE

There are 16 equivalence classes of lattice polytopes having one interior lattice point. Polytopes representing these are drawn below. This is a copy of [34, Figure 2], we include the list here for sake of self-containedness. It is an essential ingredient in the proofs of Lemma 4.1 and Proposition 9.1.



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