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On Nondifferentiable and Nonconvex Vector Optimization Problems[1]

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Abstract. In this paper, we prove the equivalence among the Minty vector variational-like inequality, Stampacchia vector variational-like inequality, and a nondifferentiable and nonconvex vector optimization problem. By using a fixed-point theorem, we establish also an existence theorem for generalized weakly efficient solutions to the vector optimization problem for nondifferentiable and nonconvex functions.

Key Words. Variational-like inequalities, vector optimization problems, generalized solutions, subinvex functions, η-subdifferential, fixed points.

1. Introduction and Preliminaries

In the recent past, vector variational inequalities (VVI) and their generalizations have been used as a tool to solve vector optimization problems (VOP). For details on VVI and their generalizations, we refer to Ref. 1 and references therein. In Ref. 2, it is shown that a necessary condition for a point to be a so-called weakly efficient solution of a VOP for differentiable functions is that the point be a solution of a suitable VVI. Following the approaches of Ref. 3, Lee studied in Ref. 4 the equivalence between Minty VVI, Stampacchia VVI, both for multivalued maps, and nondifferentiable convex VOP. Inspired by the work of Ref. 3, Komlósi dealt in Ref. 5 with the relationships between Minty VVI, Stampacchia VVI, and their connections with VOP for differentiable functions, while Ref. 6 characterized

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generalized monotonicity for multivalued maps in terms of the properties of the associated Minty variational inequalities. In Ref. 7, the equivalence is established between a vector variational-like inequality (VVLI), a generalized form of the variational inequality studied also in Refs. 8–9, with a multiobjective programming problem for generalized invex functions. In Ref. 10, the equivalence is studied between a VVLI for η-subdifferential functions and multiobjective optimization problems. The VVLI approach was used in Refs. 11–12 to prove some existence theorems for generalized efficient solutions of nondifferentiable invex VOP. The results in Refs. 11–12 are generalizations of existence results established in Ref. 13 for differentiable and convex VOP and in Ref. 14 for differentiable preinvex VOP under the assumption that the constraint set is compact.

In this paper, we prove the equivalence among Minty VVLI, Stampacchia VVLI, both for η-subdifferentiable functions (Ref. 15), and nondifferentiable nonconvex VOP. We also establish an existence theorem for so-called generalized weakly efficient solutions of nondifferentiable nonconvex VOP by using a fixed-point theorem due to these authors (see Ref. 16).

Let X be a nonempty subset of $\mathbb{R}^l$, let $f_i: \mathbb{R}^l \to \mathbb{R}$, $i \in I := \{1, \ldots, l\}$ be real-valued functions, and let $\{C(x): x \in X\}$ be a family of convex cones of $\mathbb{R}^l$ each with apex at the origin, such that $\forall x \in X$, $\operatorname{int} C(x) \neq \emptyset$, $C(x) \neq \mathbb{R}^l$, and $\mathbb{R}^l_+ \subseteq \mathrm{C}(\mathrm{x})$. We consider the following VOP in the unknown[4] y:

$$\min_{x \in X} {}_{\operatorname{int} C(y)} f(x), \tag{1}$$

where $f(x) := (f_1(x), \ldots, f_l(x))$, $\min_{\operatorname{int} C(y)}$ marks the vector minimum[5] with respect to the cone $C(y)$; namely, $y \in X$ is a (global) vector minimum point (vmp) of (1) iff

$$f(y) \not\geq_{\operatorname{int} C(y)} f(x), \qquad \forall x \in X, \tag{2}$$

where the inequality means $f(y) - f(x) \notin \operatorname{int} C(y)$. Problem (1) is called the generalized weak vector Pareto problem and its vmp's are called generalized weakly efficient points; see Ref. 11. When $C(y)$ is independent of y and $C = \mathbb{R}^l_+$, then (1) collapses to the weak vector Pareto problem and its vmp's are called weakly efficient points; see Ref. 17. The term "weakly" comes from the following tradition: when $C(y) \equiv \mathbb{R}^l_+$, if in (1) int C is replaced by[6] $\mathbb{R}^l_+ \backslash \{\mathbb{0}\}$, then we have the classic Pareto problem, whose vmp's are called efficient; since the solution set of (1) contains that of Pareto problem, then such a relaxation is named weak.

[4]Note that the ordering cone depends on the unknown.
[5]int denotes interior.
[6]The symbol $\mathbb{0}$ denotes the origin of the nonnegative orthant, namely $\mathbb{R}^l_+$.

A differentiable function $g: \mathbb{R}^l \to \mathbb{R}$ is said to be invex (Ref. 18) with respect to (wrt) a given $\eta: \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}^l$ iff

$$g(x) - g(z) \geq \langle \nabla g(z), \eta(x, z) \rangle, \qquad \forall x, z \in \mathbb{R}^l, \tag{3}$$

where $\nabla g(z)$ is the gradient vector of g at z and $\langle \cdot , \cdot \rangle$ denotes the scalar product in $\mathbb{R}^l$.

It is observed in Ref. 15 that $\nabla g(z)$ is not the only element in $\mathbb{R}^l$ which satisfies (3) with the given η. For example (Ref. 15), if $g: \mathbb{R} \to \mathbb{R}$ is a constant function and

$$\eta(x, z) = (x - z)^2,$$

then any $u \in \mathbb{R}$ such that $u \leq 0$ satisfies the following inequality:

$$g(x) - g(z) \geq \langle u, \eta(x, z) \rangle, \qquad \forall x, z \in \mathbb{R}.$$

The above example also holds for

$$\eta(x, z) = (x + z)^2, \qquad \forall x, z \in \mathbb{R},$$

and for

$$\eta(x, z) = \begin{cases} a(x - z), & \text{if } x \geq z, \\ -a(x - z), & \text{if } x \leq z, \end{cases}$$

for $a > 0$ and $x, z \in \mathbb{R}$.

We note that, when

$$\eta(x, z) = (x - z)^2,$$

then

$$\eta(x, x) = 0 \quad \text{and} \quad \eta(x, z) + \eta(z, x) \neq 0, \text{ for } x, z \neq 0;$$

when

$$\eta(x, z) = (x + z)^2,$$

then

$$\eta(x, x) \neq 0, \qquad \text{for } x \neq 0,$$

and

$$\eta(x, z) + \eta(z, x) \neq 0, \qquad \text{for } x, z \neq 0;$$

when

$$\eta(x, z) = \begin{cases} a(x - z), & \text{if } x \geq z, \\ -a(x - z), & \text{if } x \leq z, \end{cases}$$

for $a>0$ and $x, z\in\mathbb{R}$, then

$$\eta(x,x)=0 \quad \text{and} \quad \eta(x,z)+\eta(z,x)=0.$$

This observation motivated us to introduce the following concepts.

A function $g: \mathbb{R}^l \to \mathbb{R}$ is said to be subinvex[7] at $z\in\mathbb{R}^l$ wrt a given map $\eta: \mathbb{R}^l\times\mathbb{R}^l\to\mathbb{R}^l$ iff $\exists\xi\in\mathbb{R}^l$ such that

$$g(x)-g(z)\geq\langle\xi, \eta(x,z)\rangle, \qquad \forall x\in\mathbb{R}^l.$$

Such ξ is called an η-subgradient of g at z, and the set

$$\partial^\eta g(z) := \{\xi\in\mathbb{R}^l: g(x)-g(z)\geq\langle\xi, \eta(x,z)\rangle, \forall x\in\mathbb{R}^l\}$$

is called the η-subdifferential of g at z. Therefore, a function is subinvex at a given point wrt η iff it has a nonempty η-subdifferential at that point. The function g is said to be subinvex iff it is subinvex at each $z\in\mathbb{R}^l$. It is also noted in Ref. 15 that the set $\partial^\eta g(z)$ is not necessarily a singleton, even if g is differentiable, and that it coincides with the subdifferential of convex analysis (Ref. 19) when g is a convex function and $\eta(x,z)=x-z$. Clearly, the class of subinvex functions includes the invex functions with $\xi=\nabla g(z)$.

We give an example of a function g which is subinvex wrt η and is discontinuous on an open set X.

Let $X=(0,1)$ and define

$$g(x)=\begin{cases} a, & \text{if } x\in(0,1/2),\\ b, & \text{if } x\in[1/2,1),\end{cases}$$

where $a, b>0$ and $a\neq b$ and

$$\eta(x,z)=\begin{cases} c(x-z), & \text{if } x\geq z,\\ -c(x-z), & \text{if } x\leq z,\end{cases}$$

where $c=|a-b|$. Then, the function g is subinvex wrt η and is discontinuous on an open set X.

Proposition 1.1. See Ref. 15. If $g: \mathbb{R}^l\to\mathbb{R}$ is subinvex wrt η: $\mathbb{R}^l\times\mathbb{R}^l\to\mathbb{R}^l$, then $\forall z\in\mathbb{R}^l$, $\partial^\eta g(z)$ is a nonempty closed convex subset of $\mathbb{R}^l$.

A real-valued function $g: \mathbb{R}^l\to\mathbb{R}$ is said to be locally Lipschitz with respect to $\eta: \mathbb{R}^l\times\mathbb{R}^l\to\mathbb{R}^l$ iff, $\forall\tilde{x}\in\mathbb{R}^l$, there exist a neighborhood $N(\tilde{x})$ of $\tilde{x}$

[7]The term subinvex has been suggested by Prof. F. Giannessi to replace the term semi-invex used in Ref. 15. Indeed, the term semi-invex would lead one to think that there exist upper and lower semi-invex functions. Instead, what merely happens is that the gradient is replaced by a subgradient.

and a constant $k>0$ such that

$$|g(x)-g(z)|\leq k\|\eta(x,z)\|, \qquad \forall x,z\in N(\tilde{x}).$$

If $\eta(x,z)=x-z$, then the concept of locally Lipschitz functions wrt η reduces to the definition of locally Lipschitz functions (Ref. 20).

We note that the well-known Rademacher theorem states that a function which is Lipschitz on an open subset of $\mathbb{R}^l$ is differentiable almost everywhere (in the sense of the Lebesgue measure) on that subset; see for example Ref. 20. It will be interesting to prove the above result for the class of locally Lipschitz functions with respect to some (fixed) function η.

Let $T:\mathbb{R}^l \rightrightarrows \mathbb{R}^l$ be a multivalued map. Then, T is called locally bounded at $z_0\in\mathbb{R}^l$ iff there exist a neighborhood $N(z_0)$ of z_0 and a constant $k>0$ such that the inequality $\|w\|\leq k$ holds $\forall z\in N(z_0)$ and $\forall w\in T(z)$.

T is locally bounded on $\mathbb{R}^l$ iff it is locally bounded at each $z\in\mathbb{R}^l$.

Proposition 1.2. See Ref. 15. Let $g:\mathbb{R}^l\to\mathbb{R}$ be subinvex wrt $\eta:\mathbb{R}^l\times\mathbb{R}^l\to\mathbb{R}^l$, such that η is an open map and $\eta(z,z)=\mathbb{O}$, $\forall z\in\mathbb{R}^l$. If g is locally Lipschitz wrt η, then the multivalued map $\partial^\eta g$ is locally bounded on $\mathbb{R}^l$.

Proposition 1.3. Let $g:\mathbb{R}^l\to\mathbb{R}$ be subinvex wrt $\eta:\mathbb{R}^l\times\mathbb{R}^l\to\mathbb{R}^l$, such that $\eta(z',z'')+\eta(z'',z')=\mathbb{O}$, $\forall z',z''\in\mathbb{R}^l$. Then, $\partial^\eta g$ is an η-monotone multivalued map; that is, $\forall z',z''\in\mathbb{R}^l$,

$$\langle\zeta-\xi,\eta(z',z'')\rangle\geq 0, \qquad \forall\zeta\in\partial^\eta g(z') \text{ and } \forall\xi\in\partial^\eta g(z'').$$

Proof. Let $\zeta\in\partial^\eta g(z')$ and $\xi\in\partial^\eta g(z'')$. Then, we have

$$g(x)-g(z')\geq\langle\zeta,\eta(x,z')\rangle, \qquad \forall x\in\mathbb{R}^l, \tag{4}$$

$$g(x)-g(z'')\geq\langle\xi,\eta(x,z'')\rangle, \qquad \forall x\in\mathbb{R}^l. \tag{5}$$

Put $x=z''$ in (4) and $x=z'$ in (5), and then add (4) and (5); we get

$$\langle\zeta,\eta(z'',z')\rangle+\langle\xi,\eta(z',z'')\rangle\leq 0.$$

Since

$$\eta(z',z'')+\eta(z'',z')=\mathbb{O}, \qquad \forall z',z''\in\mathbb{R}^l,$$

we have

$$\langle\zeta-\xi,\eta(z',z'')\rangle\geq 0, \qquad \forall\zeta\in\partial^\eta g(z') \text{ and } \forall\xi\in\partial^\eta g(z''). \qquad \square$$

We remark that, if

$$\eta(z', z'') + \eta(z'', z') \neq \mathbb{0},$$

then $\partial^\eta g$ is not necessarily an η-monotone multivalued map. For example, if g is a constant function and

$$\eta(z', z'') = (z' + z'')^2, \qquad \forall z', z'' \in \mathbb{R}\backslash\{0\},$$

then it is easy to see that $\partial^\eta g$ is not η-monotone.

If A and K are nonempty subsets of a topological vector space E with $A \subseteq K$, we shall denote by $\text{int}_K A$ the interior of A in K. If A is a subset of a vector space, $\text{conv}\, A$ denotes the convex hull of A. Let $T: X \rightrightarrows Y$ be a multivalued map from a space X to another space Y. The graph of T, denoted by $G(T)$, is

$$G(T) = \{(x, u) \in X \times Y: x \in X, u \in T(x)\}.$$

The inverse T^{-1} of T is a multivalued map from the range of T to X, defined by

$$x \in T^{-1}(u), \qquad \text{iff } u \in T(x).$$

We shall use the following fixed-point theorem, which generalizes known results in the literature (see Ref. 16), to prove the existence of generalized weakly efficient solutions to the VOP for nondifferentiable and nonconvex functions.

Theorem 1.1. See Ref. 16. Let X be a nonempty and convex subset of a Hausdorff topological vector space E, and let $S, T: X \rightrightarrows X$ be two multivalued maps such that $\text{conv}\, S(x) \subseteq T(x)$, $S(x) \neq \emptyset$, $\forall x \in X$, $X = \bigcup \{\text{int}_X S^{-1}(z): z \in X\}$. If X is not compact, we assume that there exist a nonempty compact convex subset B of X and a nonempty compact subset D of X such that, $\forall z \in X\backslash D$, $\exists \tilde{x} \in B$ such that $z \in \text{int}_X S^{-1}(\tilde{x})$. Then, there exists $\bar{z} \in X$ such that $\bar{z} \in T(\bar{z})$.

2. Vector Variational-Like Inequalities

Let X be a nonempty subset of $\mathbb{R}^l$, and let $f_i: \mathbb{R}^l \to \mathbb{R}$, $i \in I \quad := \{1, \ldots, l\}$, be real-valued functions; let $\eta: X \times X \to \mathbb{R}^l$ be a given map. Then, we consider the following Minty vector variational-like inequality (MVVLI) and Stampacchia vector variational-like inequality (SVVLI):

(MVVLI) find $y \in X$ such that, $\forall x \in X$ and $\forall \zeta_i \in \partial^\eta f_i(x)$, $i \in I$,

$$(\langle \zeta_1, \eta(y, x)\rangle, \ldots, \langle \zeta_l, \eta(y, x)\rangle) \not\geq_{\text{int}\, C(y)} \mathbb{0};$$

$(\text{SVVLI})_w$ find $y\in X$ such that, $\forall x\in X, \exists \xi_i\in\partial^\eta f_i(y), i\in I$,

$$(\langle \xi_1, \eta(y,x)\rangle, \ldots, \langle \xi_l, \eta(y,x)\rangle) \not\geq_{\text{int}\, C(y)} \mathbb{0};$$

$(\text{SVVLI})_s$ find $y\in X$ such that $\exists \xi_i\in\partial^\eta f_i(y)$, $i\in I$, and $\forall x\in X$,

$$(\langle \xi_1, \eta(y,x)\rangle, \ldots, \langle \xi_l, \eta(y,x)\rangle) \not\geq_{\text{int}\, C(y)} \mathbb{0}.$$

A solution $y\in X$ of $(\text{SVVLI})_w$ is called a weak solution of $(\text{SVVLI})_s$; a solution of $(\text{SVVLI})_s$ is called a strong solution of $(\text{SVVLI})_w$. We remark that, in $(\text{SVVLI})_w$, $\xi_i\in\partial^\eta f_i(y)$, $i\in I$, depend on $x\in X$, but that this is not the case in $(\text{SVVLI})_s$. Therefore, every strong solution is a weak solution, but the converse assertion is not true in general. It is sufficient to discuss the weak solution, which is the only need of this paper. For further details on strong and weak solutions of generalized vector variational inequalities, we refer to Refs. 21–22.

Proposition 2.1. Let X be a nonempty subset of $\mathbb{R}^l$, and let $f_i: \mathbb{R}^l \to \mathbb{R}$, $i\in I$, be subinvex wrt $\eta: X\times X\to\mathbb{R}^l$, such that $\eta(x,z)+\eta(z,x)=\mathbb{0}$, $\forall x,z\in X$. Then, any solution of $(\text{SVVLI})_w$ is also a solution of (MVVLI).

Proof. Suppose that $y\in X$ is a solution of $(\text{SVVLI})_w$, but not a solution of (MVVLI). Then, $\exists \hat{x}\in X$ and $\hat{\zeta}_i\in\partial^\eta f_i(\hat{x})$, $i\in I$, such that

$$(\langle \hat{\zeta}_1, \eta(y,\hat{x})\rangle, \ldots, \langle \hat{\zeta}_l, \eta(y,\hat{x})\rangle)\in \text{int}\, C(y).$$

From Proposition 1.3, $\partial^\eta f_i$, $i\in I$, are η-monotone multivalued maps, so that we have

$$\langle \xi_i-\zeta_i, \eta(y,\hat{x})\rangle\geq 0, \qquad \forall \zeta_i\in\partial^\eta f_i(\hat{x}) \text{ and } \forall \xi_i\in\partial^\eta f_i(y),\ i\in I\ .$$

Consequently,

$$\langle \xi_i, \eta(y,\hat{x})\rangle-\langle \zeta_i, \eta(y,\hat{x})\rangle\in\mathbb{R}_+, \qquad \forall \zeta_i\in\partial^\eta f_i(\hat{x}) \text{ and } \forall \xi_i\in\partial^\eta f_i(y),\ i\in I\ .$$

Since

$$\mathbb{R}^l_+\subseteq C(y), \qquad \forall \xi_i\in\partial^\eta f_i(y),\ i\in I\ ,$$

we have

$$\begin{aligned}(\langle \xi_1, \eta(y,\hat{x})\rangle, \ldots, \langle \xi_l, \eta(y,\hat{x})\rangle)&\in \text{int}\, C(y)+\mathbb{R}^l_+\\ &\subseteq \text{int}\, C(y)+C(y)\\ &\subseteq \text{int}\, C(y),\end{aligned}$$

which contradicts the assumption. This completes the proof. □

Proposition 2.2. Let X be a nonempty convex subset of $\mathbb{R}^l$, and let η: $X \times X \to \mathbb{R}^l$ be an open map such that it is continuous in the first argument, affine in the second argument, and $\forall x \in X$, $\eta(x, x) = \mathbb{0}$. Let $f_i : \mathbb{R}^l \to \mathbb{R}$, $i \in I$, be subinvex and locally Lipschitz wrt η. Then, every solution of (MVVLI) is also a solution of $(\text{SVVLI})_w$.

Proof. Let $y \in X$ be a solution of (MVVLI). Consider any $x \in X$ and any sequence $\{\alpha_m\} \searrow 0$ with $\alpha_m \in (0,1]$. Since X is convex,

$$x_m := (1 - \alpha_m)y + \alpha_m x \in X.$$

Since $y \in X$ is a solution of (MVVLI), $\exists \zeta_i^m \in \partial^\eta f_i(x_m)$, $i \in I$, such that

$$(\langle \zeta_1^m, \eta(y, x_m)\rangle, \ldots, \langle \zeta_l^m, \eta(y, x_m)\rangle) \not\geq_{\text{int } C(y)} \mathbb{0}.$$

Since $\eta(\cdot, \cdot)$ is affine in the second argument and $\eta(x, x) = \mathbb{0}$, we obtain

$$\alpha_m(\langle \zeta_1^m, \eta(y, x)\rangle, \ldots, \langle \zeta_l^m, \eta(y, x)\rangle) \not\geq_{\text{int } C(y)} 0.$$

We note that

$$D(y) = \mathbb{R}^l \backslash \text{int } C(y)$$

is a closed cone (see e.g. Ref. 5, pp. 240); therefore, we have

$$(\langle \zeta_1^m, \eta(y, x)\rangle, \ldots, \langle \zeta_l^m, \eta(y, x)\rangle) \not\geq_{\text{int } C(y)} \mathbb{0}.$$

From Proposition 1.2, $\partial^\eta f_i$, $i \in I$, are locally bounded at y; hence, there exist a neighborhood $N(y)$ of y and a constant $k > 0$ such that, $\forall z \in N(y)$ and $\forall \xi_i \in \partial^\eta f_i(z)$, we have

$$\|\xi_i\| \leq k, \qquad i = 1, \ldots, l. \tag{6}$$

Since $x_m \to y$ as $m \to \infty$, $\exists m_0$ such that $x_m \in N(y)$, $\forall m \geq m_0$. Consequently, by (6), we get

$$\|\zeta_i^m\| \leq k, \qquad \text{for } m \geq m_0.$$

Without loss of generality, we may assume that the sequence $\{\zeta_i^m\}$ converges to ξ_i. Since f_i, $i \in I$, are locally Lipschitz wrt η, and since η is continuous in the second variable, it is easy to check that the multivalued map $z \mapsto \partial^\eta f_i(z)$ has a closed graph, from which it follows that

$$\xi_i \in \partial^\eta f_i(y), \qquad \forall i \in I.$$

Thus, $\forall x \in X$, $\exists \xi_i \in \partial^\eta f_i(y)$, $i \in I$, such that

$$(\langle \xi_1, \eta(y, x) \rangle, \ldots, \langle \xi_l, \eta(y, x) \rangle) \not\leq_{\mathrm{int}\, C(y)} \mathbb{0}.$$

Hence, $y \in X$ is a solution of $(\mathrm{SVVLI})_w$. □

Combining Propositions 2.1 and 2.2, we have the following result.

Theorem 2.1. Let X be a nonempty and convex subset of $\mathbb{R}^l$, and let $\eta: X \times X \to \mathbb{R}^l$ be an open map such that it is continuous in the first argument, affine in the second argument, and $\eta(x, z) + \eta(z, x) = \mathbb{0}$, $\forall x, z \in X$. Let $f_i: \mathbb{R}^l \to \mathbb{R}$, $i \in I$, be subinvex and locally Lipschitz wrt η. Then, $y \in X$ is a solution of (MVVLI) iff it is a solution of $(\mathrm{SVVLI})_w$.

We give an example of a function which is subinvex and locally Lipschitz wrt η.

Let $X = (0, 1)$ and define, for each $i \in I$,

$$f_i(x) = \begin{cases} a_i, & \text{if } x \in (0, 1/2), \\ b_i, & \text{if } x \in [1/2, 1), \end{cases}$$

where $a_i, b_i \geq 0$ and $a_i \neq b_i$ and

$$\eta(x, z) = \begin{cases} c(x - z), & \text{if } x \geq z, \\ -c(x - z), & \text{if } x \leq z, \end{cases}$$

where

$$c = \max_{i \in I} |\mathrm{a}_i - \mathrm{b}_i|.$$

Then f_i, $i \in I$, is subinvex and locally Lipschitz wrt η.

Now, we present the equivalence between (MVVLI) and VOP.

Proposition 2.3. Let X be a nonempty subset of $\mathbb{R}^l$, and let $f_i: \mathbb{R}^l \to \mathbb{R}$, $i \in I$, be subinvex wrt $\eta: X \times X \to \mathbb{R}^l$ such that $\eta(x, z) + \eta(z, x) = \mathbb{0}$, $\forall x, z \in X$. If $y \in X$ is a solution of $(\mathrm{SVVLI})_w$, then it is a generalized weakly efficient solution of VOP.

Proof. Suppose that y is a solution of $(\mathrm{SVVLI})_w$, but not a generalized weakly efficient solution of VOP. Then, $\exists \hat{x} \in X$ such that

$$f(y) - f(\hat{x}) \geq_{\mathrm{int}\, C(y)} \mathbb{0}, \qquad \text{that is, } f(y) - f(\hat{x}) \in \mathrm{int}\, C(y).$$

Since $\mathbb{R}^l_+ \subseteq C(y)$ and f_i, $i \in I$, are subinvex wrt η, $\forall \xi_i \in \partial^\eta f_i(y)$, $i \in I$, we have

$$(\langle \xi_1, \eta(y, \hat{x}) \rangle, \ldots, \langle \xi_l, \eta(y, \hat{x}) \rangle) \in f(y) - f(\hat{x}) + \mathbb{R}^l_+$$
$$\subseteq \operatorname{int} C(y) + C(y)$$
$$\subseteq \operatorname{int} C(y),$$

which contradicts our supposition. This completes the proof. □

Proposition 2.4. Let X be a nonempty subset of $\mathbb{R}^l$, and let $\eta: X \times X \to \mathbb{R}^l$ be a map. Let $f_i: \mathbb{R}^l \to \mathbb{R}$, $i \in I$, be subinvex wrt η. Then, any generalized weakly efficient solution of VOP is also a solution of (MVVLI).

Proof. Assume that $y \in X$ is a generalized weakly efficient solution of VOP, but not a solution of (MVVLI). Then, $\exists \hat{x} \in X$ and $\zeta_i \in \partial^\eta f_i(\hat{x})$, $i \in I$, such that

$$(\langle \zeta_1, \eta(y, \hat{x}) \rangle, \ldots, \langle \zeta_l, \eta(y, \hat{x}) \rangle) \geq_{\operatorname{int} C(y)} \mathbb{O}. \tag{7}$$

By subinvexity of f_i, $i \in I$, wrt η, we have

$$f_i(y) - f_i(\hat{x}) - \langle \zeta_i, \eta(y, \hat{x}) \rangle \in \mathbb{R}_+,$$

and thus,

$$\begin{aligned} &(f_1(y) - f_1(\hat{x}), \ldots, f_l(y) - f_l(\hat{x})) \\ &- (\langle \zeta_1, \eta(y, \hat{x}) \rangle, \ldots, \langle \zeta_l, \eta(y, \hat{x}) \rangle) \in \mathbb{R}^l_+. \end{aligned} \tag{8}$$

From (7)–(8), we then have

$$(f_1(y) - f_1(\hat{x}), \ldots, f_l(y) - f_l(\hat{x})) \geq_{\operatorname{int} C(y)} \mathbb{O},$$

which contradicts our assumption. Hence, the result is proved. □

From Theorem 2.1 and Propositions 2.3 and 2.4, we have the following result.

Theorem 2.2. Let X be a nonempty and convex subset of $\mathbb{R}^l$, and let $\eta: X \times X \to \mathbb{R}^l$ be an open map such that it is continuous in the first argument, affine in the second argument, and $\eta(x, z) + \eta(z, x) = \mathbb{O}$, $\forall x, z \in X$. Let $f_i: \mathbb{R}^l \to \mathbb{R}$, $i \in I$, be subinvex and locally Lipschitz wrt η. Then, $y \in X$ is a generalized weakly efficient solution of VOP iff it is a solution of (MVVLI).

3. Existence Result for Vector Optimization Problems

In this section, we prove the existence of generalized weakly efficient solutions to the nondifferentiable and nonconvex VOP by using a fixed-point theorem due to these authors (Ref. 16).

Theorem 3.1. Let X be a nonempty and convex subset of $\mathbb{R}^l$, and let $\eta : X \times X \to \mathbb{R}^l$ be an open map such that it is continuous in the first argument, affine in the second argument, and $\eta(x, z) + \eta(z, x) = \mathbb{0}$, $\forall x, z \in X$. Let $f_i : \mathbb{R}^l \to \mathbb{R}$, $i \in I$, be subinvex wrt η, and let $W : K \rightrightarrows \mathbb{R}^l$ be a multivalued map defined by $W(x) = \mathbb{R}^l \setminus \operatorname{int} C(x)$, $\forall x \in X$, such that its graph is closed. Assume that there exist a nonempty compact convex subset B of X and a nonempty compact subset D of X such that, $\forall z \in X \setminus D$, $\exists \tilde{x} \in B$ and $\zeta_i \in \partial^\eta f_i(\tilde{x})$, $i \in I$, such that

$$(\langle \zeta_1, \eta(z, \tilde{x})\rangle, \ldots, \langle \zeta_l, \eta(z, \tilde{x})\rangle) \geq_{\operatorname{int} C(z)} \mathbb{0}.$$

Then, VOP has a generalized weakly efficient solution.

Proof. For the sake of simplicity, we denote $\partial^\eta f_1(x) \times \cdots \times \partial^\eta f_l(x)$ by $\partial^\eta f(x)$ and $(\langle \xi_1, \eta(z, x)\rangle, \ldots, \langle \xi_l, \eta(z, x)\rangle)$ by $\langle \xi, \eta(z, x)\rangle$, where $\xi = (\xi_1, \ldots, \xi_l) \in \mathbb{R}^{l \times l}$. We define a multivalued map $G : X \rightrightarrows X$ by

$$G(x) = \{z \in X : \forall \zeta \in \partial^\eta f(x) \text{ s.t. } \langle \zeta, \eta(z, x)\rangle \not\geq_{\operatorname{int} C(z)} \mathbb{0}\}, \qquad \forall x \in X.$$

Then, $\forall x \in X$, $G(x)$ is closed in X. Indeed, let $\{z_m\}$ be a sequence in $G(x)$ such that $z_m \to z \in X$. Then, $\forall \zeta \in \partial^\eta f(x)$, we have

$$\langle \zeta, \eta(z_m, x)\rangle \not\geq_{\operatorname{int} C(z_m)} \mathbb{0}, \qquad \text{i.e., } \langle \zeta, \eta(z_m, x)\rangle \in W(z_m).$$

Since $\eta(\cdot, x)$ and the pairing $\langle \cdot, \cdot \rangle$ are continuous, we have

$$\langle \zeta, \eta(z_m, x)\rangle \to \langle \zeta, \eta(z, x)\rangle \in W(z),$$

because the graph of W is closed. Hence,

$$\langle \zeta, \eta(z, x)\rangle \not\geq_{\operatorname{int} C(z)} \mathbb{0}, \qquad \forall \zeta \in \partial^\eta f(x).$$

Hence $z \in G(x)$, and thus $G(x)$ is closed in X.

In view of Theorem 2.2, it is sufficient to prove the existence of a solution of (MVVLI). Suppose that (MVVLI) does not have any solution. Then, $\forall z \in X$, the set

$$\{x \in X : \exists \zeta \in \partial^\eta f(x) \text{ s.t. } \langle \zeta, \eta(z, x)\rangle \geq_{\operatorname{int} C(z)} \mathbb{0}\} = \{x \in X : z \notin G(x)\} \neq \emptyset.$$

Now, we define two multivalued maps S, $T: X \rightrightarrows X$ by

$$S(z) = \{x \in X: \exists \zeta \in \partial^\eta f(x) \text{ s.t. } \langle \zeta, \eta(z, x) \rangle \geq_{\text{int } C(z)} \mathbb{O}\},$$

$$T(z) = \{x \in X: \forall \xi \in \partial^\eta f(z) \text{ s.t. } \langle \xi, \eta(z, x) \rangle \geq_{\text{int } C(z)} \mathbb{O}\}, \qquad \forall z \in X.$$

Then clearly, $\forall z \in X$,

$$S(z) \neq \emptyset.$$

Let $\{x_1, \ldots, x_m\}$ be a finite subset of $S(z)$. Then, $\exists \zeta_i \in \partial^\eta f(x_i)$, $i = 1, \ldots, m$, s.t.

$$\langle \zeta_i, \eta(z, x_i) \rangle \geq_{\text{int } C(z)} \mathbb{O}, \qquad \forall i = 1, \ldots, m. \tag{9}$$

Since $\partial^\eta f$ is an η-monotone multivalued map, we have, $\forall i = 1, \ldots, m$,

$$\langle \zeta_i, \eta(x_i, z) \rangle - \langle \xi, \eta(x_i, z) \rangle \in \mathbb{R}^l_+ \subseteq C(z), \qquad \forall \zeta_i \in \partial^\eta f(x_i) \text{ and } \forall \xi \in \partial^\eta f(z).$$

Since

$$\eta(x, z) + \eta(z, x) = \mathbb{O}, \qquad \forall x, z \in X,$$

we have

$$\langle \xi, \eta(z, x_i) \rangle - \langle \zeta_i, \eta(z, x_i) \rangle \subseteq C(z), \qquad \forall \zeta_i \in \partial^\eta f(x_i) \text{ and } \forall \xi \in \partial^\eta f(z). \tag{10}$$

From (9)–(10), we get, $\forall i = 1, \ldots, m$,

$$\begin{aligned} \langle \xi, \eta(z, x_i) \rangle &\in \langle \zeta_i, \eta(z, x_i) \rangle + C(z) \\ &\subseteq \text{int } C(z) + C(z) \\ &\subseteq \text{int } C(z). \end{aligned}$$

Let $\alpha_i \geq 0$ such that $\sum_{i=1}^m \alpha_i = 1$. Since $C(z)$ is a convex cone, we have

$$\langle \xi, \alpha_1 \eta(z, x_1) \rangle + \cdots + \langle \xi, \alpha_m \eta(z, x_m) \rangle \in \text{int } C(z).$$

Since $\eta(x, \cdot)$ is affine, we get

$$\langle \xi, \eta(z, \hat{x}) \rangle \in \text{int } C(z),$$

where $\hat{x} = \sum_{i=1}^m \alpha_i x_i$. Hence, $\hat{x} \in T(z)$; therefore,

$$\text{conv } S(z) \subseteq T(z), \qquad \forall z \in X.$$

Since $G(x)$ is closed in X, $S^{-1}(x) = [G(x)]^c$ [the complement ^{c} of $G(x)$ in X] is open in X, and hence

$$\text{int}_X S^{-1}(x) = S^{-1}(x).$$

Since, $\forall z \in X$, $S(z) \neq \emptyset$, we have

$$X = \bigcup_{x \in X} S^{-1}(x) = \bigcup_{x \in X} \text{int}_X S^{-1}(x).$$

Now, for each $z \in X \backslash D$, $\exists \tilde{x} \in B$ and $\zeta \in \partial^\eta f(\tilde{x})$ such that

$$\langle \zeta, \eta(\tilde{x}, z) \rangle \geq_{\text{int}\, C(z)} \mathbb{O},$$

and hence,

$$z \in S^{-1}(\tilde{x}) = \text{int}_X S^{-1}(\tilde{x}).$$

Thus, S and T satisfy all the conditions of Theorem 1.1. Therefore from Theorem 1.1, there exists $\bar{z} \in X$ such that $\bar{z} \in T(\bar{z})$, that is, $\forall \xi \in \partial^\eta f(\bar{z})$ s.t. $\langle \xi, \eta(\bar{z}, \bar{z}) \rangle \geq_{\text{int}\, C(\bar{z})} \mathbb{O}$. Since

$$\eta(z, x) + \eta(x, z) = 0, \qquad \forall x, z \in K,$$

we have $\eta(\bar{z}, \bar{z}) = \mathbb{O}$, and thus

$$\langle \xi, \eta(\bar{z}, \bar{z}) \rangle = \mathbb{O} \in \text{int}\, C(\bar{z}),$$

which contradicts the fact that $\mathbb{O} \notin \text{int}\, C(z)$. This completes the proof. □

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