Czechoslovak Mathematical Journal

Valter Šeda

On nonlinear differential systems with deviating arguments

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 3, 450–466

Persistent URL: http://dml.cz/dmlcz/102105

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

ON NONLINEAR DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

VALTER ŠEDA, Bratislava

(Received February 18, 1985)

In the last decade many papers have appeared which study the oscillatory properties of solutions of the differential equation with deviating argument and with quasi-derivatives. Such an equation is of the form

$$\tilde{L}_n\, y(t) + f(t, y[g(t)]) = 0 \tag{1}$$

where $n > 1$ is assumed, the expressions

$$\tilde{L}_0\, y(t) = \frac{1}{p_0(t)}\, y(t)\,, \quad \tilde{L}_i\, y(t) = \frac{1}{p_i(t)}\, [\tilde{L}_{i-1}\, y(t)]'\,, \quad i = 1, 2, \ldots, n\,, \tag{2}$$

are called the *quasiderivatives of order* 0 and *of order* i, respectively, *of the function* y *at the point* $t \in [a, \infty)$, and $p_0, p_1, \ldots, p_n$ are given positive continuous functions in the interval $[a, \infty)$. The equation (1) can be considered as a perturbed disconjugate linear differential equation where in general the nonlinear part involves deviation of the argument.

In connection with using quasiderivatives instead of derivatives many methods known in the theory of ordinary differential equations have been extended to the case of (1), such as Kiguradze lemmas ([6], [7], [1], [12], [3], [11]), Taylor's formula ([1], [4]), Hardy-Littlewood lemma ([8]) and fundamental lemma ([10]).

In [5] P. Marušiak directed his attention to the system

$$\begin{aligned} &y_i'(t) - f_i(t, y_{i+1}(t), y_{i+1}(h_{i+1}(t))) = 0\,, \quad i = 1, 2, \ldots, n-1 \\ &\{y_n'(t) + f_n(t, y_1(t), y_1(h_1(t)))\}\ \operatorname{sgn} y_1(h_1(t)) \leqq 0\,. \end{aligned} \tag{3}$$

Using the relations (2) one can prove that the equation (1) is a special case of (3) and thus the results obtained for the system (3) represent a further step in generalizing those obtained for the ordinary differential equation. In view of this, the question arises what is the most general form of a differential system for which a reasonable theory can be developed, e.g., for which the Kiguradze lemmas are true.

In this paper a differential system is investigated which in a certain sense generalizes (3) (the system consists only of equations) and two Kiguradze lemmas are generalized for it. Further sufficient conditions are established for the mentioned system to have the property A and the property B, respectively. The meaning of these properties will be given later on.

Consider the system

(S) $$y_i'(t) = f_i(t, y_{i+1}(t), y_{i+1}(h_{i+1}(t)), \ldots, y_n(t), y_n(h_n(t))) , \quad i = 1, \ldots, n-1 ,$$
$$y_n'(t) = f_n(t, y_1(t), y_1(h_1(t)), y_2(t), y_2(h_2(t)), \ldots, y_n(t), y_n(h_n(t))) ,$$

where $n > 1$.

Sometimes we will require the following conditions to be satisfied:

(4) $h_i \in C([a, \infty), [a, \infty)), \quad \lim_{t \to \infty} h_i(t) = \infty , \quad (i = 1, 2, \ldots, n)$;

(5) $f_i \in C([a, \infty) \times R^{2(n-i)}, R)$,
$v_{i+1} f_i(t, u_{i+1}, v_{i+1}, \ldots, u_n, v_n) \geqq 0$ for $u_{i+1} v_{i+1} > 0$
(f_i has the positive sign property), $i = 1, \ldots, n-1$;

(6) $f_n \in C([a, \infty) \times R^{2n}, R)$
$v_1 f_n(t, u_1, v_1, \ldots, u_n, v_n) \leqq 0$ for $u_1 v_1 > 0$
(f_n has the negative sign property);

(7) $f_n \in C([a, \infty) \times R^{2n}, R)$
$v_1 f_n(t, u_1, v_1, \ldots, u_n, v_n) \geqq 0$ for $u_1 v_1 > 0$
(f_n has the positive sign property);

(8) for any interval $[t_1, \infty)$ with $t_1 \geqq a$ and for any $2n$-tuple of continuous functions $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ in $[t_1, \infty)$ such that
$$a_{i+1}(t)\, b_{i+1}(t) > 0 \quad \text{in} \quad [t_1, \infty)$$
the identity
$$f_i(t, a_{i+1}(t), b_{i+1}(t), \ldots, a_n(t), b_n(t)) \equiv 0$$
cannot hold in $[t_1, \infty)$, $i = 1, \ldots, n-1$,
and
for any interval $[t_1, \infty)$ with $t_1 \geqq a$ and for any $2n$-tuple of continuous functions $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ in $[t_1, \infty)$ such that
$$a_1(t)\, b_1(t) > 0 \quad \text{in} \quad [t_1, \infty)$$
the identity
$$f_n(t, a_1(t), b_1(t), \ldots, a_n(t), b_n(t)) \equiv 0$$
cannot hold in $[t_1, \infty)$;

(9) the functions $f_i(t, u_{i+1}, v_{i+1}, \ldots, u_n, v_n)$, $i = 1, \ldots, n-1$, are nondecreasing in all variables $u_{i+1}, v_{i+1}, \ldots, u_n, v_n$ for each fixed $t \in [a, \infty)$;

(10) the function $f_n(t, u_1, v_1, \ldots, u_n, v_n)$ is nonincreasing in all variables $u_1, v_1, \ldots$ $\ldots, u_n, v_n$ for each fixed $t \in [a, \infty)$;

(11) the function $f_n(t, u_1, v_1, \ldots, u_n, v_n)$ is nondecreasing in all variables $u_1, v_1, \ldots$ $\ldots, u_n, v_n$ for each fixed $t \in [a, \infty)$.

Remark 1. Under assumptions (4), (5) and (6) or (7) the function $y(t) \equiv 0$ is a solution of the system (S) in $[a, \infty)$.

Denote by W the set of all solutions $y(t) = (y_1(t), \ldots, y_n(t))$ of the system (S) which exist on some ray $[T_y, \infty) \subset [a, \infty)$ and satisfy the condition

$$\sup \{ \sum_{i=1}^{n} |y_i(t)| : t_1 \leqq t < \infty \} > 0 \quad \text{for any} \quad t_1 \in [T_y, \infty) .$$

Similarly as in [5, p. 73] we shall use the following definitions.

Definition 1. A solution $y \in W$ is called *oscillatory* (*weakly oscillatory*) if each component (at least one component, respectively) has arbitrarily large zeros.

A solution $y \in W$ is called *nonoscillatory* (*weakly nonoscillatory*) if each component (at least one component, respectively) is eventually of a constant sign.

Definition 2. We shall say that the system (S) has the property A if every solution $y \in W$ is oscillatory for n even, while for n odd it is either oscillatory or y_i ($i = 1, 2, \ldots$ $\ldots, n$) tend monotonically to zero as $t \to \infty$.

Further, in accordance with [1, p. 94] we introduce the following definition.

Definition 3. We shall say that the system (S) has the property B if for n even every solution $y \in W$ is either oscillatory or y_i ($i = 1, 2, \ldots, n$) tend monotonically to zero as $t \to \infty$, or $|y_i|$ ($i = 1, 2, \ldots, n$) tend monotonically to ∞ as $t \to \infty$, while for n odd every solution $y \in W$ is either oscillatory or $|y_i|$ ($i = 1, 2, \ldots, n$) tend monotonically to ∞ as $t \to \infty$.

In both definitions the monotonicity of the components y_i is understood in a neighbourhood of ∞ and not necessarily on the whole interval of definition of y.

Now we shall give a condition under which each weakly nonoscillatory solution is nonoscillatory and each weakly oscillatory solution is oscillatory. Then each solution $y \in W$ is either oscillatory or nonoscillatory.

Lemma 1. *Suppose that* (4), (5), (6) *or* (7), *and* (8) *are satisfied. Let* $y = (y_1, \ldots, y_n) \in$ $\in W$ *and let there exist a* k, $1 \leqq k \leqq n$ *and a* $t_0 \geqq a$ *such that*

$$y_k(t) \neq 0 \quad in \quad [t_0, \infty) .$$

Then there exists a $T \geqq t_0$ *such that each component* y_i *of the solution* y *is in* $[T, \infty)$ *different from* 0, *monotone and there exists finite or infinite* $\lim_{t \to \infty} y_i(t) = L_i$.

Proof. By (4), there exists a $t_1 \geqq t_0$ such that $y_k(t) \neq 0$, $y_k(h_k(t)) \neq 0$ and $y_k(t)\, y_k(h_k(t)) > 0$ in $[t_1, \infty)$. Two cases may occur.

i) $1 < k \leqq n$. With the help of (5), the system (S) implies that either $y'_{k-1}(t) \geqq 0$ in $[t_1, \infty)$ or $y'_{k-1}(t) \leqq 0$ in $[t_1, \infty)$. At the same time, if $y_{k-1}(t) \equiv 0$ were true on an interval $[t_2, \infty) \subset [t_1, \infty)$, then $y'_{k-1} \equiv 0$ would hold, which contradicts assumption (8) for $i + 1 = k$. Hence y_{k-1} is monotone in $[t_1, \infty)$ and $y_{k-1}(t) \equiv 0$ does not hold on any interval $[t_2, \infty) \subset [t_1, \infty)$. Therefore there is a $t_3 > t_1$ such that

$y_{k-1}(t) \neq 0$ in $[t_3, \infty)$, y_{k-1} is monotone in that interval and there exists $\lim_{t\to\infty} y_{k-1}(t) = L_{k-1}$. By finite induction we prove analogous statements for $y_{k-2}, \ldots, y_1$. The statements for $y_n, \ldots, y_{k+1}$ will follow from the case ii).

ii) $k = 1$. From the n-th equation of (S) we get that $y_n'(t) \geqq 0$ in $[t_1, \infty)$ or $y_n'(t) \leqq 0$ in $[t_1, \infty)$ and similarly as in the previous case $y_n(t) \neq 0$ can be proved in an interval $[t_4, \infty) \subset [t_1, \infty)$. Hence the statement of the lemma is valid for the n-th component of y and now we consider the case i) for $k = n$ and obtain that the statement is true for all components of y.

Remark 2. From the proof of the lemma it is clear that the lemma is valid also for a system of the form

$$y_i'(t) = f_i(t, y_1(t), y_1(h_1(t)), y_2(t), y_2(h_2(t)), \ldots, y_n(t), y_n(h_n(t))),$$
$$i = 1, 2, \ldots, n,$$

when properly modifying conditions (5), (8). (f_i are functions of $2n + 1$ variables). Here it suffices to assume that each function f_i has either the positive or the negative sign property ($f_1, \ldots, f_{n-1}$ need not have the same sign property).

The relation between the limits L_i, $i = 1, 2, \ldots, n$, of components of a nonoscillatory solution $y \in W$ is determined by Lemma 2.

Lemma 2. *Suppose that conditions* (4), (5), (6) *or* (7), (8), (9) *as well as condition*

(12) *there is an* $\varepsilon > 0$ *such that for* $i = 1, \ldots, n - 1$ *and each* $c_{i+1} > 0$,

$$\lim_{t\to\infty} \int_a^t f_i(s, c_{i+1}, c_{i+1}, -\varepsilon, -\varepsilon, \ldots, -\varepsilon, -\varepsilon)\, ds\,,$$
$$= \infty\,,$$

and for each $c_{i+1} < 0$,

$$\lim_{t\to\infty} \int_a^t f_i(s, c_{i+1}, c_{i+1}, \varepsilon, \varepsilon, \ldots, \varepsilon, \varepsilon)\, ds = -\infty$$

are satisfied. Then the following statement is true:

If $y = (y_1, \ldots, y_n) \in W$ *is a nonoscillatory solution of* (S), $y_i(t) \neq 0$ *in* $[T, \infty)$ *and* $L_i = \lim_{t\to\infty} y_i(t)$, $i = 1, \ldots, n$, *then:*

i) *If* $1 \leqq k < n$, $-\infty < L_k < \infty$ *implies that* $L_{k+1} = \ldots = L_n = 0$;

(ii) *If* $1 < k \leqq n$, $0 < L_k \leqq \infty$ $(0 > L_k \geqq -\infty)$ *implies that*

$$L_{k-1} = \ldots = L_1 = \infty \quad (L_{k-1} = \ldots = L_1 = -\infty);$$

iii) *If* $1 < k \leqq n$, $y_k(t) > 0$, $y_{k-1}(t) > 0$ $(y_k(t) < 0,\ y_{k-1}(t) < 0)$ *in* $[T, \infty)$ *imply* $y_k(t) > 0$, $y_{k-1}(t) > 0, \ldots, y_1(t) > 0$, $(y_k(t) < 0,\ y_{k-1}(t) < 0, \ldots, y_1(t) < 0)$ *in* $[T, \infty)$.

Proof. i) Suppose that $-\infty < L_k < \infty$, $1 \leqq k < n$. If $L_n > 0$ were true, then on the basis of the $(n - 1)$-st equation of (S), as well as of assumptions (4), (9), (12) there are a c_n, $0 < c_n < L_n$, and a $T_{n-1} > T$ such that

$$y_{n-1}(t) = y_{n-1}(T_{n-1}) + \int_{T_{n-1}}^{t} f_{n-1}(s, y_n(s), y_n(h_n(s)))\, ds \geqq$$

$$\geqq \int_{T_{n-1}}^{t} f_{n-1}(s, c_n, c_n)\, ds \to \infty \quad \text{as} \quad t \to \infty$$

which gives $L_{n-1} = \infty$. Repeating this consideration with the $(n - 2)$-nd equation of the system (S), we get that

$$y_{n-2}(t) = y_{n-2}(T_{n-2}) + \int_{T_{n-2}}^{t} f_{n-2}(s, y_{n-1}(s), y_{n-1}(h_{n-1}(s)), y_n(s), y_n(h_n(s)))\, ds \geqq$$

$$\geqq \int_{T_{n-2}}^{t} f_{n-2}(s, c_{n-1}, c_{n-1}, -\varepsilon, -\varepsilon)\, ds$$

where T_{n-2} is a sufficiently great number. The last inequality together with (12) imply that $L_{n-2} = \infty$. In a similar way we obtain that $L_{n-2} = \ldots = L_1 = \infty$ which contradicts the assumption $-\infty < L_k < \infty$. Therefore $L_n = 0$. Similarly $L_n < 0$ leads to the relation $L_{n-1} = -\infty$ as well as to the equalities $L_{n-2} = \ldots = L_1 = -\infty$. Therefore $L_n = 0$.

In the same way we can prove that $L_{n-1} = \ldots = L_{k+1} = 0$ and the statement i) is proved.

ii) If $0 < L_k \leqq \infty$ for a k, $1 < k \leqq n$, then arguing in a similar way as in the preceding case we get that $L_{k-1} = \infty$ and repeating this process we obtain that $L_{k-1} = \ldots = L_1 = \infty$. Similarly $0 > L_k \geqq -\infty$ leads to the equalities $L_{k-1} = \ldots \ldots = L_1 = -\infty$.

iii) If $y_k(t) > 0$, $y_{k-1}(t) > 0$ in $[T, \infty)$, then by (5) for $i + 1 = k$, $y'_{k-1}(t) \geqq 0$ and hence $0 < L_{k-1} \leqq \infty$, This implies that $L_{k-2} = \ldots = L_1 = \infty$, and in view of the fact that all $y_i(t) \neq 0$ in $[T, \infty)$, $y_{k-2}(t) > 0, \ldots, y_1(t) > 0$ in the same interval. Similarly we proceed in the case $y_k(t) < 0$, $y_{k-1}(t) < 0$ in $[T, \infty)$.

Now we prove a generalization of the first Kiguradze lemma.

Lemma 3. *Suppose that conditions* (4), (5), (6), (8), (9), (12) *are satisfied. Let, further,* $y = (y_1, \ldots, y_n) \in W$ *be a nonoscillatory solution of* (S) *in* $[t_0, \infty)$, $a \leqq t_0$, *with* $L_i = \lim_{t\to\infty} y_i(t)$, $i = 1, 2, \ldots, n$. *Then there exist an integer* $l \in \{1, 2, \ldots, n\}$ *with* $n + l$ *even and a* $T \geqq t_0$ *such that*

(13) $$y_i(t)\, y_1(t) > 0 \quad on \quad [T, \infty) \quad for \quad i = 1, \ldots, l\,,$$

and if $l < n$,

(14) $$(-1)^{l+i}\, y_i(t)\, y_1(t) > 0 \quad on \quad [T, \infty) \quad for \quad i = l + 1, \ldots, n\,.$$

If we denote $\delta = \operatorname{sgn} y_1(t)$ *on* $[T, \infty)$, *then*

(15) $$0 \leqq \delta L_l < \infty$$

and if $l < n$ *then*

(16) $$L_{l+1} = \ldots = L_n = 0\,,$$

while for $l > 1$

(17) $$0 < \delta L_{l-1} \leqq \infty$$

and for $l > 2$

(18) $$\delta L_1 = \ldots = \delta L_{l-2} = \infty\,.$$

Proof. Since y is nonoscillatory and in view of condition (4) there is an interval $[T, \infty)$ such that $y_i(t) \neq 0$, $y_i(h_i(t)) \neq 0$ and $y_i(t)\, y_i(h_i(t)) > 0$, $i = 1, \ldots, n$, on that interval. We shall consider the case $y_1(t) > 0$ in $[T, \infty)$. Similar arguments hold if $y_1(t) < 0$ for $t \geqq T$. As $y_1(t) > 0$, the n-th equation of (S), by (6), gives that $y_n'(t) \leqq 0$ and y_n is nonincreasing in $[T, \infty)$. If $y_n(t) < 0$ were true on the interval mentioned, then $L_n < 0$ would hold and, by Lemma 2, $L_{n-1} = \ldots = L_1 = -\infty$ which contradicts the positiveness of $y_1(t)$ in $[T, \infty)$. Hence $y_1(t) > 0$ implies $y_n(t) > 0$ in this interval and $0 \leqq L_n < \infty$.

With the help of the $(n-1)$-st equation of (S) we obtain that $y_{n-1}(t)$ is a nondecreasing function in $[T, \infty)$. If it is positive in that interval, by Lemma 2 all components $y_1(t) > 0$, $y_2(t) > 0, \ldots, y_n(t) > 0$ and we can put $l = n$. At the same time $0 < L_{n-1} \leqq \infty$ which implies that $L_{n-2} = \ldots = L_1 = \infty$.

If $y_{n-1}(t) < 0$ in $[T, \infty)$, then y_n, y_{n-1} are of opposite signs on that interval. Suppose that $y_n(t) > 0$, $y_{n-1}(t) < 0$, $y_{n-2}(t) > 0$, $y_{n-3}(t) < 0, \ldots, y_{l+2}(t) > 0$, $y_{l+1}(t) < 0$ are all consecutive pairs consisting of members with opposite signs in $[T, \infty)$. By Lemma 2, in the sequence $y_n, y_{n-1}, y_{n-2}, \ldots, y_2, y_1$ cannot be two consecutive terms which are negative in $[T, \infty)$, because then y_1 should be negative, too. Hence $y_l(t) > 0$ and as $y_{l-1}(t) < 0$ cannot occur, $y_{l-1}(t) > 0$ and thus, $y_l(t) > 0$, $y_{l-1}(t) > 0, \ldots, y_1(t) > 0$ in $[T, \infty)$. Thus (13) as well as (14) are true whereby $n + l$ is even. As $y_{l+1}(t) < 0$, $y_{l+1}(h_{l+1}(t)) < 0$ in $[T, \infty)$, by the l-th equation of (S) $y_l'(t) \leqq 0$ and hence, $0 \leqq L_l < \infty$. This is also true in the case $l = n$ as was shown above. Hence (15) is valid. By Lemma 2 this implies (16) while the inequality $y_{l-1}'(t) \geqq 0$ yields (17) and, by Lemma 2, (18) is true.

If instead of condition (6) we consider condition (7), then we get a generalization of the second Kiguradze lemma.

Lemma 4. *Suppose that conditions* (4), (5), (7), (8), (9), (12) *are satisfied. Let, further,* $y = (y_1, \ldots, y_n) \in W$ *be a nonoscillatory solution of* (S) *in* $[t_0, \infty)$, $t_0 \geqq a$, *with* $L_i = \lim_{t\to\infty} y_i(t)$, $i = 1, 2, \ldots, n$. *Then there exist an integer* $l \in \{1, 2, \ldots, n\}$ *with* $n + l$ *odd or* $l = n$ *and a* $T \geqq t_0$ *such that*

(13) $$y_i(t)\, y_1(t) > 0 \quad on \quad [T, \infty) \quad for \quad i = 1, \ldots, l$$

and, if $l < n$,

(14) $$(-1)^{l+i}\, y_i(t)\, y_1(t) > 0 \quad on \quad [T, \infty) \quad for \quad i = l + 1, \ldots, n\,.$$

If we denote $\delta = \operatorname{sgn} y_1(t)$ *on* $[T, \infty)$, *then*

for $l = n$

$$(19) \qquad 0 < \delta L_n \leqq \infty\,, \quad \delta L_1 = \ldots = \delta L_{n-1} = \infty\,,$$

while for $l < n$

$$(15) \qquad 0 \leqq \delta L_l < \infty\,,$$

$$(16) \qquad L_{l+1} = \ldots = L_n = 0$$

and if $l > 1$, *then*

$$(17) \qquad 0 < \delta L_{l-1} \leqq \infty\,,$$

whereby for $l > 2$

$$(18) \qquad \delta L_1 = \ldots = \delta L_{l-2} = \infty\,.$$

Proof. The proof is similar to that of Lemma 3. Using the same notation as in that proof, in view of condition (7) $y_1(t) > 0$ in $[T, \infty)$ implies that $y_n'(t) \geqq 0$ and y_n is nondecreasing in that interval. If $y_n(t) > 0$ in $[T, \infty)$, then with help of Lemma 2 we come to (19) and hence $l = n$. In the case $y_n(t) < 0$ in $[T, \infty)$, Lemma 2 implies that $y_{n-1}(t) > 0$ and by analogous considerations as in the proof of Lemma 3 we obtain the assertion that there is an integer $l \in \{1, 2, \ldots, n\}$ with $n + 1$ odd such that (13) and (14) are valid. Again (15) must be fulfilled and this implies (16) while (17), (18) can be proved in the same way as in the proof of Lemma 3.

Remark 3. From inequalities (13), (14) it is clear that the number l in both Lemmas 3 and 4 is uniquely determined. This justifies the following definition.

Definition 4. Let $y = (y_1, \ldots, y_n) \in W$ be a nonoscillatory solution of (S) in $[t_0, \infty)$. We shall say that the solution y has *the property* P_l with $l \in \{1, 2, \ldots, n\}$ if a) each component y_i of that solution has $\lim_{t\to\infty} y_i(t) = L_i$; b) there exists a $T \geqq t_0$ such that y has the property (13) and if $l < n$ the property (14); c) if we denote $\delta = \operatorname{sgn} y_1(t)$ on $[T, \infty)$, then for L_i, $i = 1, 2, \ldots, n$, the relation (15) and for $l < n$ (16), while for $l > 1$ (17) and for $l > 2$ (18) hold.

Further, we shall say that the solution y has *the property* P_{n+1} if a) each component y_i of that solution has $\lim_{t\to\infty} y_i(t) = L_i$; b) there exists a $T \geqq t_0$ such that y has the property (13) for $i = 1, 2, \ldots, n$; c) if we denote $\delta = \operatorname{sgn} y_1(t)$ on $[T, \infty)$, then for L_i, $i = 1, 2, \ldots, n$, the relations (19) hold.

In terms of properties P_l, Lemmas 3 and 4 can be expressed in the following compact form.

Theorem 1. *Suppose that conditions* (4), (5), (6), (8), (9), (12) (*conditions* (4), (5), (7), (8), (9), (12)) *are satisfied. Then for each nonoscillatory solution* $y \in W$ *of the* *system* (S) *there exists an* $l \in \{1, 2, \ldots, n+1\}$ *such that* $n + l$ *is even* ($n + l$ is odd) *and* y *has the property* P_l.

Remark 4. Theorem 1 in a certain sense generalizes Theorem 1 in [12, p. 121] (Lemma 4 and Lemma 6 from that paper are not extended to the case of (S)).

When all nonoscillatory solutions $y \in W$ of the system (S) have a property P_l, then it is easy to find a sufficient condition for that system to have the property A or the property B.

Lemma 5. i) *If conditions* (4), (5), (6), (8), (9), (12) *are satisfied, and each nonoscillatory solution* $y = (y_1, \ldots, y_n) \in W$ *of* (S) *has the property* P_1 *whereby* $\lim_{t \to \infty} y_1(t) = L_1 = 0$, *then the system* (S) *has the property* A.

ii) *If conditions* (4), (5), (7), (8), (9), (12) *are satisfied and each nonoscillatory solution* $y = (y_1, \ldots, y_n) \in W$ *of* (S) *with* $L_i = \lim_{t \to \infty} y_i(t)$, $i = 1, 2, \ldots, n$, *and* $\delta = \operatorname{sgn} y_1(t)$ *in a sufficiently small neighbourhood of* ∞, *has either the property* P_1 *or* P_{n+1} *whereby for the solution* y *with* P_1 $L_1 = 0$ *is true while for the solution* y *with* P_{n+1} $\delta L_{n+1} = \infty$ *is valid, then the system* (S) *has the property* B.

Proof. Only the statement ii) will be proved. The proof of the first statement would proceed in a similar way. Theorem 1 gives that each nonoscillatory solution y has the property P_l with $n + l$ odd. If n is even, then for P_1 all components y_i of y tend monotonically to 0, while for P_{n+1} $|y_i|$ tend monotonically to ∞ as $t \to \infty$. If n is odd, then each nonoscillatory solution $y \in W$ of (S) has the property P_{n+1} and again $|y_i|$ tend monotonically to ∞ as $t \to \infty$. The proof is complete.

Theorem 2. *Suppose that all conditions of Lemma* 3 *as well as condition* (10) *and condition*

(20) *there is an* $\varepsilon > 0$ *such that for each* $c_1 > 0$

$$\lim_{t \to \infty} \int_a^t f_n(s, c_1, c_1, -\varepsilon, -\varepsilon, \ldots, -\varepsilon, -\varepsilon)\, ds = -\infty$$

and for each $c_1 < 0$

$$\lim_{t \to \infty} \int_a^t f_n(s, c_1, c_1, \varepsilon, \varepsilon, \ldots, \varepsilon, \varepsilon)\, ds = \infty$$

are satisfied. Then the system (S) *has the property* A.

Proof. Let $y = (y_1, \ldots, y_n) \in W$ be a nonoscillatory solution of (S). Without loss of generality we may suppose that $y_1(t) > 0$ in an interval $[T, \infty)$. By Theorem 1, y has a property P_l with $l \in \{1, 2, \ldots, n+1\}$ and $n + l$ is even.

Let $l \geqq 2$. Then $y_1(t) > 0$, $y_2(t) > 0$ as well as $y_1(h_1(t)) > 0$, $y_2(h_2(t)) > 0$ in an interval $[T_1, \infty)$, $T_1 \geqq T$. In view of the condition (5), the first equation of (S) gives that $y_1'(t) \geqq 0$ in $[T_1, \infty)$ and hence, there is a $c_1 > 0$ such that $y_1(t) \geqq c_1 > 0$ in $[T_1, \infty)$. Similarly, by the property P_l, there exist constants $c_2, \ldots, c_{l-1}, c_l$ such that the components y_i of the solution y satisfy the inequalities

$$
\begin{aligned}
(21) \qquad & 0 < c_1 \leqq y_1(t) < L_1 \leqq \infty\,, \\
& \dots\dots\dots\dots\dots\dots\dots\dots \\
& 0 < c_{l-1} < y_{l-1}(t) < L_{l-1} \leqq \infty\,, \\
& 0 \leqq L_l < y_l(t) < c_l < \infty\,, \\
& -\varepsilon \leqq y_{l+1}(t) < L_{l+1} = 0\,, \\
& 0 = L_{l+2} < y_{l+2}(t) < \varepsilon\,, \\
& \dots\dots\dots\dots\dots\dots\dots\dots \\
& 0 = L_n < y_n(t) < \varepsilon
\end{aligned}
$$

in an interval $[T_2, \infty) \subset [T_1, \infty)$, $L_i = \lim_{t\to\infty} y_i(t)$, $i = 1, 2, \dots, n$, and $\varepsilon > 0$ is involved in condition (20). From the n-th equation of (S) we have, with help of (10) and (21),

$$
\begin{aligned}
(22) \qquad y_n'(t) = f_n(t, y_1(t), y_1(h_1(t)), y_2(t), y_2(h_2(t)), \dots, y_n(t), y_n(h_n(t))) \leqq \\
\leqq f_n(t, c_1, c_1, -\varepsilon, -\varepsilon, \dots, -\varepsilon, -\varepsilon)
\end{aligned}
$$

in an interval $[T_3, \infty)$, $T_3 \geqq T_2$. Integrating the last inequality from T_3 to t, we obtain

$$0 < y_n(t) \leqq y_n(T_3) + \int_{T_3}^{t} f_n(s, c_1, c_1, -\varepsilon, -\varepsilon, \dots, -\varepsilon, -\varepsilon)\, ds$$

which contradicts (20). Hence $l = 1$. If $L_1 > 0$ were true, instead of (22) we should have

$$y_n'(t) \leqq f_n(t, L_1, L_1, -\varepsilon, -\varepsilon, \dots, -\varepsilon, -\varepsilon)$$

and again we come to a contradiction. Now, by Lemma 5, Theorem 2 follows

Remark 5. Theorem 2 represents a certain generalization of Theorem 1 in [5, p. 76].

Theorem 3. *Suppose that all conditions of Lemma* 4 *as well as condition* (11) *and condition*

(23) *there is an* $\varepsilon > 0$ *such that for each* $c_1 > 0$

$$\lim_{t\to\infty} \int_a^t f_n(s, c_1, c_1, -\varepsilon, -\varepsilon, \dots, -\varepsilon, -\varepsilon)\, ds = \infty$$

and for each $c_1 < 0$

$$\lim_{t\to\infty} \int_a^t f_n(s, c_1, c_1, \varepsilon, \varepsilon, \dots, \varepsilon, \varepsilon)\, ds = -\infty$$

are satisfied. Then the system (S) *has the property* B.

Proof. Suppose that $y = (y_1, \dots, y_n) \in W$ is a nonoscillatory solution. By Theorem 1, y has a property P_l with $l \in \{1, 2, \dots, n + 1\}$ and $n + l$ is odd. We shall consider the case $y_1(t) > 0$ in an interval $[T, \infty)$. The opposite case would be handled in a similar way. Assume that $l \geqq 2$. Since in this case the inequalities (21) for $y_1(t), \dots$

..., $y_n(t)$ are true in an interval $[T_2, \infty)$, the condition (11) implies that

$$y_n'(t) = f_n(t, y_1(t), y_1(h_1(t)), \ldots, y_n(t), y_n(h_n(t))) \geqq$$
$$\geqq f_n(t, c_1, c_1, -\varepsilon, -\varepsilon, \ldots, -\varepsilon, -\varepsilon)$$

and hence

$$y_n(t) \geqq y_n(T_3) + \int_{T_3}^{t} f_n(s, c_1, c_1, -\varepsilon, -\varepsilon, \ldots, -\varepsilon, -\varepsilon)\, ds$$

for all $t \geqq T_3$, T_3 sufficiently great. By (23), $\lim_{t \to \infty} y_n(t) = \infty$ which means that y has the property P_{n+1}.

If $l = 1$ and $L_1 > 0$, then we obtain the same result. This means that this case cannot occur. Hence, if $l = 1$, $L_1 = 0$ must be true. Again by Lemma 5 the statement of the theorem follows.

Now we find another sufficient condition for the property A. Instead of (20) we shall need some other conditions. First we give an estimate for a solution y of the system (S) with a property P_l.

Lemma 6. *Suppose that all conditions of Lemma 3 as well as condition (10) are satisfied. Let there exist a solution $y = (y_1, \ldots, y_n) \in W$ of the system (S) in $[t_0, \infty)$, $t_0 \geqq a$, with a property P_l, $l \in \{1, 2, \ldots, n + 1\}$, and if $l = 1$, let $L_1 = \lim_{t \to \infty} y_1(t) \neq 0$. Then for each $c_l > 0$ ($c_l < 0$) such that the inequalities $0 < c_l < y_l(t)$ ($0 > c_l > y_l(t)$) for $l > 1$, or $0 < L_1 < y(t) < c_1$ ($0 > L_1 > y_1(t) > c_1$) for $l = 1$, are true in a neighbourhood of ∞ and for $\varepsilon > 0$ which is involved in condition (12), there exist a $T \geqq t_0$ and a system of functions*

$$(24) \qquad u_{l,n}, u_{l,n-1}, \ldots, u_{l,2}, u_{l,1}$$

given in $[T, \infty)$ by (28), (31) and (34), respectively (given in $[T, \infty)$ by (28'), (31') and (34'), respectively) such that

$$(25) \qquad \operatorname{sgn} u_{l,i}(t) = \operatorname{sgn} y_i(t) \quad in \quad [T, \infty)\,, \quad i = 1, 2, \ldots, n\,,$$

and

$$(26) \qquad |y_i(t)| > |u_{l,i}(t)| > 0\,, \quad t \in [T, \infty)\,, \quad i = 1, 2, \ldots, n\,.$$

Proof. By Lemma 3, $n + l$ is even and hence we have to consider three cases.

i) $l = n$. Denoting $L_i = \lim_{t \to \infty} y_i(t)$ and assuming that $y_1(t) > 0$ in a neighbourhood of ∞, we obtain the following inequalities for the components of y in an interval $[T, \infty)$:

$$(27) \qquad \begin{aligned} &0 < c_1 < y_1(t) < L_1 \leqq \infty\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &0 < c_{n-1} < y_{n-1}(t) < L_{n-1} \leqq \infty\,, \\ &0 \leqq L_n < y_n(t) < c_n\,, \end{aligned}$$

with some positive constants $c_1, c_2, \ldots, c_n$.

Integrating from T to t in that interval and using the monotonicity properties of f_i given by conditions (9), (10) as well as (27) we get that

$$y_n(t) - L_n = -\int_t^\infty f_n(s, y_1(s), y_1(h_1(s)), \ldots, y_n(s), y_n(h_n(s)))\, ds \geqq$$

$$\geqq -\int_t^\infty f_n(s, c_1, c_1, -\varepsilon, -\varepsilon, \ldots, -\varepsilon, -\varepsilon)\, ds = u_{n,n}(t) > 0$$

in $[T, \infty)$.

Further,

$$y_{n-1}(t) = y_{n-1}(T) + \int_T^t f_{n-1}(s, y_n(s), y_n(h_n(s)))\, ds \geqq$$

$$\geqq \int_T^t f_{n-1}(s, u_{n,n}(s), u_{n,n}(h_n(s)))\, ds = u_{n,n-1}(t) > 0$$

in $[T, \infty)$.

Thus, by finite induction we construct the functions

(28) $$u_{n,n}(t) = -\int_t^\infty f_n(s, c_1, c_1, -\varepsilon, -\varepsilon, \ldots, -\varepsilon, -\varepsilon)\, ds\,,$$

$$u_{n,k}(t) = \int_T^t f_k(s, u_{n,k+1}(s), u_{n,k+1}(h_{k+1}(s)), \ldots, u_{n,n}(s), u_{n,n}(h_n(s)))\, ds\,, \quad t \in [T, \infty)\,,$$

$$k = n - 1, n - 2, \ldots, 1\,.$$

The system (28) satisfies the inequalities

(29) $$y_n(t) > u_{n,n}(t) > 0\,,$$

$$y_k(t) > u_{n,k}(t) > 0\,, \quad k = n - 1, n - 2, \ldots, 1$$

and hence it has both the properties (25) and (26).

In the case that $y_1(t) < 0$ for sufficiently great t, the system (28) should be modified into the form

(28′) $$u_{n,n}(t) = -\int_t^\infty f_n(s, c_1, c_1, \varepsilon, \varepsilon, \ldots, \varepsilon, \varepsilon)\, ds\,,$$

$$u_{n,k}(t) = \int_T^t f_k(s, u_{n,k+1}(s), u_{n,k+1}(h_{k+1}(s)), \ldots, u_{n,n}(s), u_{n,n}(h_n(s)))\, ds\,, \quad t \in [T, \infty)\,,$$

$$k = n - 1, n - 2, \ldots, 1$$

and it would satisfy inequalities converse to (29).

ii) $1 < l < n$. First we construct the system (24) in the case that $y_1(t) > 0$.

Instead of (27) we now have the system

$$(30) \qquad \begin{aligned} &0 < c_1 < y_1(t) < L_1 \leqq \infty\,, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &0 < c_{l-1} < y_{l-1}(t) < L_{l-1} \leqq \infty\,, \\ &0 \leqq L_l < y_l(t) < c_l\,, \\ &-\varepsilon < y_{l+1}(t) < L_{l+1} = 0\,, \\ &L_{l+2} = 0 < y_{l+2}(t) < \varepsilon\,, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &L_n = 0 < y_n(t) < \varepsilon \end{aligned}$$

with some positive constants $c_1, \ldots, c_{l-1}, c_l$ and ε given in (12). (30) is valid in an interval $[T, \infty)$. Using integration from T to $t > T$ as well as the inequalities (30), conditions (9) and (10) we get that the system

$$(31) \qquad \begin{aligned} u_{l,n}(t) &= -\int_t^\infty f_n(s, c_1, c_1, -\varepsilon, -\varepsilon, \ldots, -\varepsilon, -\varepsilon)\,\mathrm{d}s\,, \\ u_{l,n-1}(t) &= -\int_t^\infty f_{n-1}(s, u_{l,n}(s), u_{l,n}(h_n(s)))\,\mathrm{d}s\,, \\ u_{l,n-2}(t) &= -\int_t^\infty f_{n-2}(s, u_{l,n-1}(s), u_{l,n-1}(h_{n-1}(s)), \varepsilon, \varepsilon)\,\mathrm{d}s\,, \\ u_{l,n-3}(t) &= -\int_t^\infty f_{n-3}(s, u_{l,n-2}(s), u_{l,n-2}(h_{n-2}(s)), \\ &\qquad -\varepsilon, -\varepsilon, u_{l,n}(s), u_{l,n}(h_n(s)))\,\mathrm{d}s\,, \\ u_{l,n-4}(t) &= -\int_t^\infty f_{n-4}(s, u_{l,n-3}(s), u_{l,n-3}(h_{n-3}(s)), \\ &\qquad \varepsilon, \varepsilon, u_{l,n-1}(s), u_{l,n-1}(h_{n-1}(s)), \varepsilon, \varepsilon)\,\mathrm{d}s\,, \end{aligned}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\begin{aligned} u_{l,l}(t) &= -\int_t^\infty f_l(s, u_{l,l+1}(s), u_{l,l+1}(h_{l+1}(s)), \varepsilon, \varepsilon, u_{l,l+3}(s), u_{l,l+3}(h_{l+3}(s)), \\ &\qquad \varepsilon, \varepsilon, \ldots, \varepsilon, \varepsilon)\,\mathrm{d}s\,, \\ u_{l,l-1}(t) &= \int_T^t f_{l-1}(s, u_{l,l}(s), u_{l,l}(h_l(s)), -\varepsilon, -\varepsilon, u_{l,l+2}(s), u_{l,l+2}(h_{l+2}(s)), \\ &\qquad -\varepsilon, -\varepsilon, \ldots, u_{l,n}(s), u_{l,n}(h_n(s)))\,\mathrm{d}s\,, \\ u_{l,l-2}(t) &= \int_T^t f_{l-2}(s, u_{l,l-1}(s), u_{l,l-1}(h_{l-1}(s)), u_{l,l}(s), \\ &\qquad u_{l,l}(h_l(s)), -\varepsilon, -\varepsilon, u_{l,l+2}(s), u_{l,l+2}(h_{l+2}(s)) \ldots, u_{l,n}(s), u_{l,n}(h_n(s)))\,\mathrm{d}s\,, \end{aligned}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\begin{aligned} u_{l,1}(t) &= \int_T^t f_1(s, u_{l,2}(s), u_{l,2}(h_2(s)), u_{l,3}(s), u_{l,3}(h_3(s)) \ldots, u_{l,l}(s), u_{l,l}(h_l(s)), \\ &\qquad -\varepsilon, -\varepsilon, u_{l,l+2}(s), u_{l,l+2}(h_{l+2}(s)), \ldots, u_{l,n}(s), u_{l,n}(h_n(s)))\,\mathrm{d}s \end{aligned}$$

satisfies the inequalities

(32)
$$\begin{aligned} &y_n(t) > u_{l,n}(t) > 0\,,\\ &y_{n-1}(t) < u_{l,n-1}(t) < 0\,,\\ &\dots\dots\dots\dots\dots\\ &y_l(t) > u_{l,l}(t) > 0\,,\\ &y_{l-1}(t) > u_{l,l-1}(t) > 0\,,\\ &\dots\dots\dots\dots\dots\\ &y_1(t) > u_{l,1}(t) > 0 \end{aligned}$$

and hence (25) and (26) are valid in $[T, \infty)$.

If $y_1(t) < 0$ for all sufficiently great t, then for determining $u_{l,n}(t), u_{l,n-1}(t), \dots$ $\dots, u_{l,1}(t)$, instead of (31) we should have the relations (31′) which differ from (31) by opposite signs at ε. In this case all inequalities in (32) turn into opposite ones.

iii) $l = 1$. In this case we suppose that $L_1 \neq 0$. Then for $y_1(t) > 0$ in a neighbourhood of ∞ we obtain the system of inequalities

(33)
$$\begin{aligned} &0 < L_1 < y_1(t) < c_1\,,\\ &-\varepsilon < y_2(t) < L_2 = 0\,,\\ &0 = L_3 < y_3(t) < \varepsilon\,,\\ &\dots\dots\dots\dots\dots\\ &0 = L_n < y_n(t) < \varepsilon \end{aligned}$$

with a positive constant c_1 and ε from (12). The system (33) is valid in an interval $[T, \infty)$. Using the same argument as above we come to the system of functions

(34)
$$\begin{aligned} u_{1,n}(t) &= -\int_t^\infty f_n(s, L_1, L_1, -\varepsilon, -\varepsilon, \dots, -\varepsilon, -\varepsilon)\,ds\,,\\ u_{1,n-1}(t) &= -\int_t^\infty f_{n-1}(s, u_{1,n}(s), u_{1,n}(h_n(s)))\,ds\,,\\ u_{1,n-2}(t) &= -\int_t^\infty f_{n-2}(s, u_{1,n-1}(s), u_{1,n-1}(h_{n-1}(s)), \varepsilon, \varepsilon)\,ds\,,\\ u_{1,n-3}(t) &= -\int_t^\infty f_{n-3}(s, u_{1,n-2}(s), u_{1,n-2}(h_{n-2}(s))\,,\\ &\qquad -\varepsilon, -\varepsilon, u_{1,n}(s), u_{1,n}(h_n(s)))\,ds\,,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ u_{1,1}(t) &= -\int_t^\infty f_1(s, u_{1,2}(s), u_{1,2}(h_2(s)), \varepsilon, \varepsilon,\\ &\qquad u_{1,4}(s), u_{1,4}(h_4(s)), \varepsilon, \varepsilon, \dots, \varepsilon, \varepsilon)\,ds \end{aligned}$$

which satisfies in $[T, \infty)$ the inequalities

(35)
$$\begin{aligned} &y_n(t) > u_{1,n}(t) > 0\,,\\ &y_{n-1}(t) < u_{1,n-1}(t) < 0\,,\\ &\dots\dots\dots\dots\dots\\ &y_1(t) > u_{1,1}(t) > 0\,. \end{aligned}$$

Hence (25) and (26) are satisfied. If $y_1(t) < 0$ for all sufficiently great t, then $L_1 < 0$ and we define the functions $u_{1,n}(t), u_{1,n-1}(t), \ldots, u_{1,1}(t)$ by the system (34′) which differs from (34) by opposite signs at ε. This system satisfies (35) with opposite signs. Lemma 6 is proved.

With help of this lemma we can prove the following theorem.

Theorem 4. *Suppose that all conditions of Lemma* 3, *the condition* (10) *as well as the conditions*

(36) *For each* $c_1 > 0$ *and for* $\varepsilon > 0$ *from the condition* (12) *the systems* (24) *given by* (28), (31) *and* (34), *respectively, and for each* $c_1 < 0$ *the systems* (24) *given by* (28′), (31′) *and* (34′), *respectively, for* $l \in \{1, 2, \ldots, n+1\}$, $n + l$ *even, are well defined.*

(37) *For the systems* (24) *given by* (28), (31), (34) *for an arbitrary* $c_1 > 0$ (*given by* (28′), (31′), (34′) *for an arbitrary* $c_1 < 0$) *the relations*

$$\lim_{t\to\infty} \int_T^t f_n(s, u_{n,1}(s), u_{n,1}(h_1(s)), u_{n,2}(s), u_{n,2}(h_2(s)), \ldots$$
$$\ldots, u_{n,n}(s), u_{n,n}(h_n(s)))\, \mathrm{d}s = -\delta\infty$$

and for $1 < l < n$, $n + l$ *even,*

$$\lim_{t\to\infty} \int_T^t f_n(s, u_{l,1}(s), u_{l,1}(h_1(s)), \ldots, u_{l,l}(s),$$
$$u_{l,l}(h_l(s)), -\delta\varepsilon, -\delta\varepsilon, u_{l,l+2}(s), u_{l,l+2}(h_{l+2}(s)),$$
$$-\delta\varepsilon, -\delta\varepsilon, \ldots, u_{l,n}(s), u_{l,n}(h_n(s)))\, \mathrm{d}s = -\delta\infty$$

and if n *is odd,*

$$\lim_{t\to\infty} \int_T^t f_n(s, u_{1,1}(s), u_{1,1}(h_1(s)), -\delta\varepsilon, -\delta\varepsilon, u_{1,3}(s),$$
$$u_{1,3}(h_3(s)), -\delta\varepsilon, -\delta\varepsilon, \ldots, u_{1,n}(s), u_{1,n}(h_n(s))\, \mathrm{d}s = -\delta\infty,$$

where $\delta = \operatorname{sgn} c_1$, *hold.*

Then the system (S) *has the property* A.

Proof. By Lemma 5, all assumptions of which are satisfied, it suffices to show that there is no solution $y = (y_1, \ldots, y_n) \in W$ of (S) with the property P_l, $l > 1$, $n + l$ even, and with P_1 whereby $L_1 = \lim_{t\to\infty} y_1(t) \neq 0$. Suppose that such a solution exists in an interval $[t_0, \infty)$. We shall consider only the case $1 < l < n$ with $y_1(t) > 0$ in $[t_0, \infty)$. In the other cases we should proceed similarly.

Let $c_1 > 0, \ldots, c_{l-1} > 0$, $\varepsilon > 0$ be such that (30) is satisfied. Then, by condition (36), the system $u_{l,n}, u_{l,n-1}, \ldots, u_{l,1}$ determined by (31) in an interval $[T, \infty)$ satisfies the inequalities (32) in that interval. The n-th equation of the system (S), by virtue of (32) and (6), gives that

$$(38)\qquad y_n(t) = y_n(T) + \int_T^t f_n(s, y_1(s), y_1(h_1(s)), \ldots, y_n(s), y_n(h_n(s)))\,ds \leqq$$

$$\leqq y_n(T) + \int_T^t f_n(s, u_{l,1}(s), u_{l,1}(h_1(s)), \ldots, u_{l,l}(s),$$

$$u_{l,l}(h_l(s)), -\varepsilon, -\varepsilon, u_{l,l+2}(s), u_{l,l+2}(h_{l+2}(s)), \ldots$$

$$\ldots, -\varepsilon, -\varepsilon, u_{l,n}(s), u_{l,n}(h_n(s)))\,ds\,.$$

The condition (37) now implies that $\lim_{t\to\infty} y_n(t) = -\infty$ which contradicts (30). The proof is complete.

Theorems 2, 3 and 4 will be applied to the second order linear differential equation

$$(39)\qquad y''(t) = p(t)\,y(t)$$

where $p \in C([a, \infty), R)$,

(40) $p(t) \leqq 0$ in $[a, \infty)$ and $p(t) \equiv 0$ does not hold on any subinterval $[t_1, \infty)$ of the interval $[a, \infty)$.

If we put (39) into the form of the system

$$(41)\qquad \begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= p(t)\,y_1(t) \end{aligned}$$

we see that the conditions (4), (5), (6), (8), (9), (10) and (12) are satisfied. The condition (20) as well as the condition (23) is equivalent to

$$(42)\qquad \int_a^\infty |p(s)|\,ds = \infty\,.$$

If

$$(43)\qquad \int_a^\infty |p(s)|\,ds < \infty\,,$$

then the condition (36) is fulfilled, since the system $u_{2,2}, u_{2,1}$, determined by (28),

$$u_{2,2}(t) = (-c_1)\int_t^\infty p(s)\,ds$$

$$u_{2,1}(t) = (-c_1)\left[\int_T^t (u - T)\,p(u)\,du + (t - T)\int_t^\infty p(u)\,du\right]$$

is well defined in $[a, \infty)$. At the same time the condition (37) means that

$$\lim_{t\to\infty} (-c_1)\left[\int_T^t p(s)\left\{\int_T^s (u - T)\,p(u)\,du + (s - T)\int_s^\infty p(u)\,du\right\} ds\right] = \operatorname{sgn}(-c_1)\,\infty$$

and hence, (37) is equivalent to

$$(44)\qquad \lim_{t\to\infty} \int_T^t p(s)\left\{\int_T^s (u - T)\,p(u)\,du + (s - T)\int_s^\infty p(u)\,du\right\} ds = \infty\,.$$

As

$$\int_T^t p(s)\left\{\int_T^s (u - T)\, p(u)\, \mathrm{d}u + (s - T)\int_s^\infty p(u)\, \mathrm{d}u\right\} \mathrm{d}s \leqq \int_T^t p(s)\,(s - T)\int_T^\infty p(u)\, \mathrm{d}u\, \mathrm{d}s,$$

the condition (44) implies that

$$\lim_{t\to\infty} \int_T^t |p(s)|\,(s - T)\, \mathrm{d}s = \infty\,. \tag{45}$$

Thus (44) is stronger than (45). Theorems 2 and 4 imply the following corollary.

Corollary 1. *Suppose that the condition* (40) *is satisfied. Then either the condition* (42) *or* (43), (44) *are sufficient for all solutions of the equation* (39) *to be oscillatory.*

Remark 6. By [9, pp. 343, 351], the condition (42) has been derived by W. B. Fite. Conditions (43), (44) are closed to that given by M. Zlámal which reads $\int_T^\infty s^{1-\varepsilon}|p(s)|\,\mathrm{d}s = \infty$.

Similarly when we replace (40) by the condition

(46) $p(t) \geqq 0$ in $[a, \infty)$ and $p(t) \equiv 0$ does not hold on any subinterval $[t_1, \infty)$ of the interval $[a, \infty)$,

we see that the conditions (4), (5), (7), (8), (9), (11), (12) are satisfied. Further, by (46) we obtain for each solution y of (39) that $(y'(t)\, y(t))' = y'^2(t) + p(t)\, y^2(t)$ in $[a, \infty)$ and hence the function $y'(t)\, y(t)$ is nondecreasing and not constant on any interval $[t_1, \infty)$ with $t_1 > a$. Thus no solution of (39) is oscillatory. Theorem 3 gives a known result (see Corollary 3.1 in [2, p. 29]).

Corollary 2. *Suppose that the conditions* (46), (42) *are satisfied. Then each solution y of* (39) *either tends monotonically to* 0 *together with its first derivative or* $|y|$, $|y'|$ *tend monotonically to* ∞.

Literature

[1] *Чантурия, Т. А.* О монотонных и колеблющихся решениях обыкновенных дифференциальных уравнений высших порядков, Ann. Polon. Math. *37* (1978), 195—210.

[2] *Чантурия, Т. А.* О колеблемости решений линейных дифференциальных уравнений высших порядков, Семинар Института прикладной математики имени И. Н. Векуа, Доклады Тбил. Унив. *16* (1982), 1—74.

[3] *Elias, U.:* Generalizations of an Inequality of Kiguradze, J. Math. Anal. Appl. *97* (1983), 277—290.

[4] *Kitamura, Y., Kusano, T.:* Nonlinear Oscillation of Higher Order Functional Differential Equations with Deviating Arguments, J. Math. Anal. Appl. *77* (1980), 100—119.

[5] *Marušiak, P.:* On the Oscillation of Nonlinear Differential Systems with Retaded Arguments, Math. Slovaca *34* (1984), 73—88.

[6] *Philos, Ch., G.:* Oscillatory and Asymptotic Behavior of the Bounded Solutions of Differential Equations with Deviating Arguments, Hiroshima Math. J. *8* (1978), 31—48.

[7] *Philos, Ch. G.:* Oscillatory and Asymptotic Behavior of all Solutions of Differential Equations with Deviating Arguments, Proc. Roy. Soc. Edinburgh *81* A (1978), 195—210.

[8] *Philos, Ch. G.* and *Staikos, V. A.*: Asymptotic Properties of Nonoscillatory Solutions of Differential Equations with Deviating Argument, Pacific J. Math. *70* (1977), 221—242.
[9] *Ráb, M.*: Kriterien für die Oszillation der Lösungen der Differentialgleichung $[p(x)\, y']' + q(x)\, y = 0$, Čas. Pěst. Mat. *84* (1959), 335—370.
[10] *Staikos, V. A., Philos, Ch. G.*: On the Asymptotic Behavior of Nonoscillatory Solutions of Differential Equations With Deviating Arguments, Hiroshima Math. J. *7* (1977), 9—31.
[11] *Šeda, V.*: Nonoscillatory Solutions of Differential Equations with Deviating Argument, Czech. Math. J. *36 (111)* (1986), 93—107.
[12] *Švec, M.*: Behavior of Nonoscillatory Solutions of Some Nonlinear Differential Equations, Acta Math. Univ. Comen. *39* (1980), 115—130.

Author's address: 842 15 Bratislava, Mlynská dolina, Czechoslovakia (Matematicko-fyzikálna fakulta Univerzity Komenského).