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On Nonlinear Dynamics of Fluctuations*)

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A projection-operator method is developed for renormalizing kinetic coefficients by the nonlinear interaction between macroscopic modes in a nonlinear Langevin equation which is often used in studying anomalous transport phenomena near critical points and anomalous transport coefficients in two dimension. It is shown that the renormalization term is given by the time correlation of a new fluctuating force. Other theories are derived from this expression by making approximations. A relation between the Langevin fluctuating force and the transport fluctuating force is derived to clarify the relation between the Fokker-Planck equation approach by Green and Zwanzig and the linear equation-of-motion approach by one of the authors.

§ 1. Introduction

In a previous paper[1] linear transport coefficients were formulated rigorously in terms of the time correlation of fluctuating forces. It is the purpose of the present paper to clarify the relation between this formulation and the Fokker-Planck equation approach which was introduced by Green[2] and formulated by Zwanzig[3] from the statistical-mechanical standpoint. These two approaches led to different expressions for transport coefficients. Recently, however, Zwanzig[4] clearly showed that the difference is important when the nonlinear interaction between macroscopic modes plays a crucial role and then that Green's transport coefficients must be renormalized by the nonlinear interaction. In the present paper this renormalization will be treated in a simple and more complete way by using the projection-operator method.

Let us denote a set of collective variables for describing slowly-varying macroscopic processes in the system by a vector $A \equiv \{A_k\}$, where A_{-k} is included to denote A_k^*. In isotropic Heisenberg ferromagnets they are the long-wavelength Fourier components of the spin and the energy density, and in simple fluids they are the long-wavelength Fourier components of the number, the energy and the momentum density. It is assumed that $\{A_k\}$ denotes the deviations from their invariant parts and are set to be orthonormal so that

$$\langle A_k A_l^* \rangle = \delta_{k,l}, \qquad (1\cdot 1)$$

where the angular brackets denote the equilibrium average. In the present paper

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we study the equilibrium fluctuations of such collective variables with classical statistical mechanics.

Zwanzig started with assuming a nonlinear Langevin equation

$$dA_k/dt = v_k(A) - \sum_l \gamma^0_{kl} A_l + R_k(t), \tag{1·2}$$

where $v_k(A)$ is a streaming velocity. $R_k(t)$ is a Langevin fluctuating force and is assumed to be statistically independent of the collective variables; namely,

$$\langle R_k(t) G(A) \rangle = 0 \tag{1·3}$$

for an arbitrary function $G(A)$ of $\{A_k(0)\}$. Then $v_k(A)$ must be a nonlinear function of $\{A_k\}$. γ^0_{kl} are Green's kinetic coefficients and are related to $R_k(t)$ by

$$\langle R_k(t) R_l{}^*(t') \rangle = 2\gamma^0_{kl} \delta(t-t'). \tag{1·4}$$

Equation (1·2) has been studied by Green,[2] Zwanzig[3] and Kawasaki[5] from different points of view. A derivation of this will be outlined in Appendix A. The most important approximations involved are three-fold. First we have assumed that the time scales of the processes of our interest described by A are distinctly larger than the time scales involved in other dynamical variables coupled with A. Such a clear-cut separation of time scales would not be always possible due to the coupling with intermediate-wavelength components. Secondly we have assumed that the equilibrium distribution of A is Gaussian, and thirdly, that γ^0_{kl} are constants independent of A. Although these assumptions may not be valid when the fluctuations are very large, (1·2) provides us with a useful model in studying anomalous transport phenomena due to the thermal fluctuations of long-wavelength components.[4),5)]

The kinetic coefficients γ^0_{kl} are not the same as the kinetic coefficients observed in linear transport laws, but are "bare" kinetic coefficients in the sense used by Zwanzig.[4] The true kinetic coefficients are obtained from the linear equation of motion:[1]

$$dA_k/dt = i \sum_l \omega_{kl} A_l - \sum_l \int_0^t \varphi_{kl}(s) A_l(t-s) ds + f_k(t), \tag{1·5}$$

where $i\omega_{kl} = \langle \dot{A}_k A_l{}^* \rangle$. $f_k(t)$ is a fluctuating force and satisfies

$$\langle f_k(t) A_l{}^*(0) \rangle = 0, \tag{1·6}$$

$$\langle f_k(t) f_l{}^*(t') \rangle = \varphi_{kl}(t-t'). \tag{1·7}$$

The kinetic coefficients are thus given by the Laplace transform of the memory function $\varphi_{kl}(t)$:

$$\Gamma_{kl}(i\omega) = \int_0^\infty \exp(-i\omega t) \varphi_{kl}(t) dt. \tag{1·8}$$

Since $f_k(t)$ is the fluctuating force which gives correct linear transport coefficients, it will be called the "transport" fluctuating force. It should be also noted that

(1·5) with (1·6) and (1·7) is an exact equation and is not a linearized form of (1·2). The transport fluctuating force $f_k(t)$ is uncorrelated only with the linear function of the collective variables $\{A_k\}$, and thus contains nonlinear terms in the collective variables.[1] When a nonlinear coupling between macroscopic modes is important, therefore, $f_k(t)$ contains macroscopic processes through the nonlinear terms and the memory function $\varphi_{kl}(t)$ has a long-time tail of macroscopic time scale. Thus $f_k(t)$ differs from the Langevin fluctuating force $R_k(t)$, and the difference is important when the nonlinear coupling cannot be neglected. In (1·2) such a nonlinear coupling comes out from the streaming velocity $v_k(A)$.[5] If one expands $v_k(A)$ in powers of A, then one gets

$$v_k(A) = i\sum_l \omega_{kl}A_l + v_k'(A), \tag{1·9}$$

$$v_k'(A) = -i\sum_{pq} \lambda_{kpq}(A_pA_q - \langle A_pA_q\rangle) + \cdots, \tag{1·10}$$

where λ_{kpq} are bilinear coupling constants. Therefore, the bare kinetic coefficients γ^0_{kl} must be renormalized by this nonlinear interaction $v_k'(A)$. This renormalization can be done by transforming (1·2) into a linear form similar to (1·5), thus leading to an approximate expression

$$\varphi_{kl}(t) \doteqdot 2\gamma^0_{kl}\delta(t) + \psi_{kl}(t), \tag{1·11}$$

where $\psi_{kl}(t)$ is a new memory function determined by $v_k'(A)$. This renormalization will be carried out in the next section.

§ 2. Renormalization of γ_{kl}^0 by $v_k'(A)$

Let us denote the representative point of the system in phase space by x and introduce the distribution for the collective variables $A(x) \equiv \{A_k(x)\}$ to have a set of values $a \equiv \{a_k\}$,

$$g_a(x) = \delta(A(x) - a) \equiv \prod_k \delta(A_k(x) - a_k). \tag{2·1}$$

A linear equation of motion for $g_a(t)$ is obtained by using the projection-operator method, as will be outlined in Appendix A. Namely,

$$\partial g_a(t)/\partial t = Dg_a(t) + F_a(t), \tag{2·2}$$

where D is the Fokker-Planck operator:

$$D = -\sum_k \frac{\partial}{\partial a_k}[v_k(a) - \sum_l \gamma^0_{kl}a_l] + \sum_k\sum_l \gamma^0_{kl}\frac{\partial}{\partial a_k}\frac{\partial}{\partial a_l^*}. \tag{2·3}$$

The streaming velocity $v_k(a)$ is given by

$$v_k(a) = \langle \dot{A}_k g_a\rangle / \langle g_a\rangle, \tag{2·4}$$

where $\langle g_a\rangle$ represents the equilibrium distribution of a. $F_a(t)$ is a fluctuating force and is related to the Langevin fluctuating force $R_k(t)$ by

$$R_k(t) = \int a_k F_a(t) da . \tag{2·5}$$

Since $A_k(t)$ is related to $g_a(t)$ by

$$A_k(t) = \int a_k g_a(t) da , \tag{2·6}$$

the Langevin equation (1·2) can be derived from (2·2) directly. Since $F_a(t)$ is orthogonal to $g_a(0)$, we have

$$\langle F_a(t) G(A) \rangle = \langle R_k(t) G(A) \rangle = 0 \tag{2·7}$$

for an arbitrary function $G(A)$ of $\{A_k(0)\}$.

The probability distribution of a is defined by

$$P(a, t) = \langle g_a(x_t) G(A(x)) \rangle , \tag{2·8}$$

where x_t denotes the image of phase x after time t and $G(A)$ is a function of A which satisfies $\langle G(A) \rangle = 1$. The transition probability from a_0 to a in time t, $T(a_0|a;t)$, is obtained from (2·8) by taking the particular initial condition $G(A) = \delta(A-a_0)/w(a_0)$, where $w(a) = \langle g_a \rangle$. From (2·2) and (2·7) we thus obtain the Fokker-Planck equation

$$\partial P(a, t)/\partial t = DP(a, t), \tag{2·9}$$

which has been derived by Green[2] and Zwanzig.[3),4)]

Next let us define an adjoint operator Λ of D by

$$\int [\Lambda f(a)] g(a) da = \int f(a) [Dg(a)] da \tag{2·10}$$

and introduce a new quantity

$$a_k(t) \equiv \exp(t\Lambda) a_k . \tag{2·11}$$

Then from (2·2) and (2·6) we have

$$A_k(t) = \int a_k(t) \delta(a - A(0)) da + \int_0^t ds \int a_k(t-s) F_a(s) da . \tag{2·12}$$

This explicitly states that the time evolution of the collective variables consists of two parts; first, the macroscopic motion $a_k(t)$ governed by the Fokker-Planck operator D, and second, its microscopic fluctuations produced by the fluctuating force $F_a(t)$. We next extract the contribution of the nonlinear interaction between macroscopic modes in this macroscopic motion to the linear transport coefficients. This is done by deriving a linear equation of motion for $a_k(t)$ which corresponds to (1·5) for $A_k(t)$. Namely, we define a Hilbert space of functions of a and introduce the projection operator

$$P_a g(a) \equiv \sum_l \langle g(a) a_l^* \rangle a_l , \tag{2·13}$$

where the angular brackets denote the equilibrium average

$$\langle f(a)\rangle = \int f(a)w(a)da\,. \tag{2·14}$$

Then, as will be shown in Appendix B, we obtain

$$\frac{d}{dt}a_k = \sum_l (i\omega_{kl} - \gamma^0_{kl})a_l - \sum_l \int_0^t \psi_{kl}(s)a_l(t-s)ds + q_k(t), \tag{2·15}$$

where

$$\langle q_k(t)a_l^*(0)\rangle = 0\,, \tag{2·16}$$

$$\psi_{kl}(t) = \langle q_k(t)q_l^*(0)\rangle\,. \tag{2·17}$$

It should be noted that the new memory function $\psi_{kl}(t)$ has the standard form in terms of the new fluctuating force $q_k(t)$. An explicit expression for $q_k(t)$ is given by

$$q_k(t) = \exp[t(1-P_a)\Lambda]v_k'(a), \tag{2·18}$$

where

$$v_k'(a) \equiv (1-P_a)v_k(a) = v_k(a) - \sum_l i\omega_{kl}a_l\,. \tag{2·19}$$

$v_k'(a)$ is the vertical component of $v_k(a)$, and represents the nonlinear interaction between the collective variables $\{a_k\}$, for example, as in (1·10). This nonlinear interaction determines the fluctuating force $q_k(t)$ and the memory function $\psi_{kl}(t)$.

The two equations (1·5) and (2·15) have an intimate relation. First let us consider the relaxation matrix

$$\xi_{km}(t) \equiv \langle A_k(t)A_m^*(0)\rangle = \langle a_k(t)a_m^*(0)\rangle\,, \tag{2·20}$$

where the second equation is obtained from (2·12) and (2·7). Since $\langle f(t)A_m^*\rangle = \langle q_k(t)a_m^*\rangle = 0$, (1·5) and (2·15) thus lead to two equations for $\xi_{km}(t)$. Comparing them we find the relation (1·11), which is inserted into (1·8) to give the renormalized transport coefficients

$$\Gamma_{kl}(i\omega) \doteqdot \gamma^0_{kl} + \psi_{kl}(i\omega)\,. \tag{2·21}$$

As will be shown in Appendix C, we also obtain

$$f_k(t) \doteqdot R_k(t) + \int q_k(t)\delta(a-A(0))da + \int_0^t ds \int q_k(t-s)F_a(s)da\,. \tag{2·22}$$

This means that when (2·2) is approximately valid, the transport fluctuating force $f_k(t)$ is split up into two parts; first, the Langevin fluctuating force $R_k(t)$, and second, the nonlinear fluctuating force $q_k(t)$. The third term represents a coupling between these two parts which, however, does not contribute to the transport coefficients. Equation (1·11) can be derived also directly from (2·22) by using (1·7) and introducing the approximations used in deriving (2·2). Thus (2·22) most clearly shows the relation between the Langevin fluctuating force $R_k(t)$ and the transport fluctuating force $f_k(t)$, and thus the relation between the Fokker-

Planck equation approach by Green[2] and Zwanzig[3] and the linear equation-of-motion approach.[1]

§ 3. Relation with other theories

From (2·17) and (2·18) the nonlinear memory function is written as

$$\psi_{kl}(t)=\langle v_l'^*(a)\exp[t(1-P_a)\Lambda]v_k'(a)\rangle . \qquad (3\cdot1)$$

Λ is written as $\Lambda=\Lambda_0+\Lambda_1$,

$$\Lambda_0\equiv\sum_k\sum_l\left[(i\omega_{kl}-\gamma_{kl}^0)a_l\frac{\partial}{\partial a_k}+\gamma_{kl}^0\frac{\partial}{\partial a_k}\frac{\partial}{\partial a_l^*}\right], \qquad (3\cdot2)$$

$$\Lambda_1\equiv\sum_k v_k'(a)\frac{\partial}{\partial a_k}. \qquad (3\cdot3)$$

If the nonlinear streaming velocity $v_k'(a)$ is small, then one may replace Λ in (3·1) by its unperturbed part Λ_0, thus obtaining a second-order perturbation expression.

In simple cases one can assume that

$$\xi_{km}(t)=\xi_k(t)\delta_{k,m} \qquad (3\cdot4)$$

and the matrices ω_{kl}, γ_{kl}^0, $\psi_{kl}(t)$ are all diagonal, where the $\boldsymbol{k}$'s denote the wave vectors. One may also assume that the leading nonlinear terms are bilinear in the form

$$v_k'(a)\simeq -i\sum_p{}' V_{kp}a_p a_{k-p}, \qquad (3\cdot5)$$

where $|V_{kp}|^2\sim k^x p^y$ for small values of k and p, x and y being nonnegative numbers, and $\sum_p{}'$ does not include the short-wavelength components. Then since $w(a)$ is Gaussian,

$$(1-P_a)\Lambda_0 a_p a_{k-p}=-(\gamma_p{}^0+\gamma_{k-p}^0)a_p a_{k-p}, \qquad (3\cdot6)$$

where ω_k is assumed to be zero. Thus the second-order perturbation expression leads to

$$\psi_k(t)\simeq 2\sum_p{}'|V_{kp}|^2\exp[-t(\gamma_p{}^0+\gamma_{k-p}^0)]. \qquad (3\cdot7)$$

This is equivalent to the expression obtained by Zwanzig with the use of a second-order perturbation calculation.[4] If one can assume $\gamma_p{}^0\sim p^2$ for small values of p, then the integral over wave vector $\boldsymbol{p}$ in the small wavenumber region leads to

$$\psi_k(t)\sim k^x t^{-(d+y)/2}, \qquad (3\cdot8)$$

where d is the dimensionality of the system. This determines the long-time behavior of the memory function, and leads to $1/t$ in two dimension if $y=0$. Then the renormalized transport coefficients $\Gamma_k(i\omega)$ diverge at low frequency as $\log(1/\omega)$. This kind of anomalous transport phenomena have been investigated by a number of authors.[4),6)]

One way to avoid a perturbation calculation is to find a self-consistent equation. Assuming (1·10) for $v_k'(a)$ and

$$\exp[t(1-P_a)\Lambda]a_p a_q \simeq a_p(t)\,a_q(t) \qquad (3\cdot 9)$$

and replacing the four-body correlations by products of the pair correlations, we obtain

$$\psi_{kl}(t) \simeq 2\sum_{pq}\sum_{fg}\lambda_{kpq}\lambda^*_{lfg}\xi_{pf}(t)\,\xi_{qg}(t). \qquad (3\cdot 10)$$

Multiplying (2·15) by $a_m^*(0)$ and taking the equilibrium average, we then obtain

$$d\xi_{km}(t)/dt - \sum_l (i\omega_{kl}-\gamma^0_{kl})\,\xi_{lm}(t)$$

$$\simeq -2\sum_l\sum_{pq}\sum_{fg}\lambda_{kpq}\lambda^*_{lfg}\int_0^t \xi_{pf}(s)\,\xi_{qg}(s)\,\xi_{lm}(t-s)\,ds. \qquad (3\cdot 11)$$

This is a self-consistent equation for determining the relaxation matrix $\{\xi_{km}(t)\}$, and is equivalent to the equation obtained by Kawasaki with the aid of the diagram technique.[5)] This type of equations are often used in studying anomalous transport phenomena near critical points.[5),7)] If one finds the solution for $\xi_{km}(t)$ and inserts it into (3·10), then (2·21) leads to the renormalized transport coefficients.

§ 4. Short summary

It was shown that the Fokker-Planck equation approach by Green and Zwanzig is most completely formulated from the equation of motion (2·2), although they developed their theories within the framework of the Fokker-Planck equation (2·9).[2),3),4)] Namely, the transport fluctuating force $f_k(t)$ whose time correlation gives the correct linear transport coefficients was shown to be written approximately as (2·22) in terms of the fluctuating forces $F_a(t)$ and $q_k(t)$, leading to the approximate expression (2·21) for the transport coefficients $\Gamma_{kl}(i\omega)$. This equation for $f_k(t)$ bridges the Fokker-Planck equation approach and the linear equation-of-motion approach.[1)]

It would be worth noting that the basic equations (1·5), (2·2) and (2·15) were all derived from the equation of motion (A·4) by taking as $\{A_\mu\}$ the collective variables $\{A_k\}$, their distribution $\prod_k\delta(A_k-a_k)$ and the vector $\{a_k\}$ in a space, respectively. Taking $\prod_k\delta(A_k-a_k)$ means eliminating the other degrees-of-freedom than those of the collective variables $\{A_k\}$. Taking $\{a_k\}$ means eliminating the nonlinear interaction between the collective variables, which was shown to amount to producing the new memory function $\psi_{kl}(t)$ and the new fluctuating force $q_k(t)$. The renormalization of transport coefficients by the nonlinear interaction was thus given by the time correlation of $q_k(t)$ which is determined by the macroscopic processes.

The transport coefficients were thus separated into two parts, the rapidly-relaxing part of microscopic time scale and the long-time tail of macroscopic time scale. It should be noted that this separation has become possible under the assumption that the distribution of time scales is distinctly separated into two parts, as discussed in § 1. When the fluctuations of A are large, (2·3), (2·17) and (2·18) would not be valid. Even then, however, (2·2), (2·15), (2·21) and (2·22) are still valid if one uses (A·19) for D, (B·2) for Λ, (B·5) for $q_k(t)$, and (B·8) for $\psi_{kl}(t)$. Thus the present theory would be useful not only for a deeper understanding of irreversible processes, but also in studying nonlinear dynamics of large fluctuations of collective variables.

Appendix A

——*Derivation of (1·2) and (2·2)*——

Let us first consider an orthonormal set of variables $\{A_\mu\}$ whose time evolution is given by

$$dA_\mu(t)/dt = i\mathcal{L}A_\mu(t), \tag{A·1}$$

where $i\mathcal{L}$ is a time-independent linear operator. As shown previously,[1),8)] an exact equation of motion for $\{A_\mu(t)\}$ can be derived by introducing the projection operator

$$\mathcal{P}G(t) \equiv \sum_\nu (G(t), A_\nu(0))A_\nu(0)\ , \tag{A·2}$$

where (F, G) denotes the inner product and is defined by $\langle FG^* \rangle$ in the classical case. Namely, introducing the frequency matrix

$$i\Omega_{\mu\nu} = (i\mathcal{L}A_\mu, A_\nu), \tag{A·3}$$

we obtain

$$\frac{d}{dt}A_\mu = \sum_\nu i\Omega_{\mu\nu}A_\nu - \sum_\nu \int_0^t \Phi_{\mu\nu}(s)A_\nu(t-s)ds + F_\mu(t), \tag{A·4}$$

where

$$\Phi_{\mu\nu}(t) = -(i\mathcal{L}F_\mu(t), A_\nu(0)), \tag{A·5}$$

$$F_\mu(t) = \exp[t(1-\mathcal{P})i\mathcal{L}](1-\mathcal{P})i\mathcal{L}A_\mu(0). \tag{A·6}$$

The most important feature of (A·4) is the fact that its systematic part is linear and $F_\mu(t)$ represents a fluctuating force, satisfying

$$(F_\mu(t), A_\nu(0)) = 0\ . \tag{A·7}$$

Equation (2·2) can be derived from (A·4) by taking $g_a(x)$ as $A_\mu(x)$. Then the projection (A·2) takes the form

$$P_Z G(x_t) = \int dx' \rho(x') G(x_t') \delta(A(x') - A(x))/w(A(x)), \tag{A·8}$$

where

$$w(a) \equiv \langle g_a \rangle = \int \delta(A(x)-a)\rho(x)dx \tag{A·9}$$

and $\rho(x)$ is the equilibrium phase-space distribution function. This is identical to the projection operator introduced by Zwanzig.[3] The frequency matrix (A·3) becomes

$$i\Omega_{ab} = -\sum_k \frac{\partial}{\partial a_k}[v_k(a)\delta(a-b)], \tag{A·10}$$

where $v_k(a)$ is the streaming velocity defined by (2·4). The fluctuating force (A·6) is written as

$$F_a(t) = -\exp[t(1-P_Z)iL]\sum_k R_k(0)\frac{\partial}{\partial a_k}\delta(A(0)-a), \tag{A·11}$$

where L is the Liouville evolution operator and we have introduced

$$R_k(t) \equiv \exp[t(1-P_Z)iL](1-P_Z)iLA_k(0). \tag{A·12}$$

Equation (2·5) can be easily obtained from (A·11). The memory kernel (A·5) takes the form

$$\Phi_{ab}(t) = \langle F_a(t)F_b{}^*(0)\rangle / w(b). \tag{A·13}$$

Thus we obtain

$$\frac{\partial}{\partial t}g_a(t) + \sum_k \frac{\partial}{\partial a_k}[v_k(a)g_a(t)]$$
$$= -\int_0^t ds \int db \Phi_{ab}(s) g_b(t-s) + F_a(t). \tag{A·14}$$

If one neglects $F_a(t)$, then this agrees with Zwanzig's generalized Fokker-Planck equation.[3] Using (2·6) we also obtain

$$\frac{d}{dt}A_k = v_k(A) + \sum_l \int_0^t \zeta_{kl}(A(t-s),s)ds + R_k(t), \tag{A·15}$$

where

$$\zeta_{kl}(a,s) = \frac{1}{w(a)}\frac{\partial}{\partial a_l{}^*}\langle \delta(A-a)R_k(s)R_l{}^*(0)\rangle. \tag{A·16}$$

It should be noted that (A·14) is equivalent to Kawasaki's kinetic equation for an infinite set of products of the macroscopic variables, and (A·15) corresponds to his nonlinear kinetic equation for the macroscopic variables.[5]

Now we introduce approximations. In $\Phi_{ab}(s)$ and (A·15), we assume that

$$F_a(s) \simeq -\sum_k R_k(s)\frac{\partial}{\partial a_k}\delta(A(0)-a), \tag{A·17}$$

$$\zeta_{kl}(A(t-s), s) \simeq \zeta_{kl}(A(t), s), \tag{A·18}$$

which means that the time scales for an appreciable change of $A(t)$ are distinctly larger than the typical time scales involved in the Langevin fluctuating force $R_k(t)$. Then (A·14) leads to (2·2) with the generalized Fokker-Planck operator

$$Dg_a = -\sum_k \frac{\partial}{\partial a_k}[v_k(a) g_a] + \sum_k \sum_l \frac{\partial}{\partial a_k}\left[w(a)\gamma_{kl}(a)\frac{\partial}{\partial a_l^*}\left(\frac{1}{w(a)}g_a\right)\right], \tag{A·19}$$

where

$$\gamma_{kl}(a) = \int_0^\infty ds \langle R_k(s) R_l^*(0) g_a\rangle / \langle g_a\rangle . \tag{A·20}$$

Similarly (A·15) leads to

$$\frac{dA_k}{dt} = v_k(A) + \sum_l \frac{1}{w(A)} \frac{\partial}{\partial A_l^*}[w(A)\gamma_{kl}(A)] + R_k(t). \tag{A·21}$$

Secondly we assume that the equilibrium distribution of a is Gaussian,

$$w(a) = C \exp\left(-\frac{1}{2}\sum_k a_k a_k^*\right). \tag{A·22}$$

Thirdly we assume that $\gamma_{kl}(a)$ in (A·19) and (A·21) can be replaced by their average

$$\gamma_{kl}^0 \equiv \langle \gamma_{kl}(a)\rangle = \int_0^\infty \langle R_k(s) R_l^*(0)\rangle ds . \tag{A·23}$$

Recently Kawabata[9] showed that the Landau-Lifshitz friction term in the equation of motion for a spin interacting with its surroundings can be derived from the a dependence of $\gamma_{kl}(a)$. In our problem, however, the above assumptions would be valid when the fluctuations of A are not too large. Under these assumptions (A·21) and (A·19) reduce to (1·2) and (2·3), respectively.

Appendix B

——Derivation of (2·15) and (2·17)——

Let us consider the time evolution of

$$a_k(t) \equiv \exp(t\Lambda) a_k \tag{B·1}$$

with the generalized expression for Λ

$$\Lambda = \sum_k \left\{ v_k(a) + \sum_l \frac{1}{w(a)} \frac{\partial}{\partial a_l^*}[w(a)\gamma_{kl}(a)]\right\}\frac{\partial}{\partial a_k} + \sum_k \sum_l \gamma_{kl}(a) \frac{\partial}{\partial a_l^*}\frac{\partial}{\partial a_k} . \tag{B·2}$$

We apply (A·4) to (B·1) by taking Λ as $i\mathcal{L}$ and using the projection operator (2·13). Since

$$\Lambda a_k = v_k(a) + \sum_l \frac{1}{w(a)} \frac{\partial}{\partial a_l^*}[w(a)\gamma_{kl}(a)], \tag{B·3}$$

the frequency matrix (A·3) becomes

$$i\Omega_{kl} = i\omega_{kl} - \gamma_{kl}^0, \tag{B·4}$$

where (A·23) has been used. The fluctuating force (A·6) becomes

$$q_k(t) = \exp[t(1-P_a)\Lambda] q_k(a), \tag{B·5}$$

where

$$q_k(a) \equiv v_k'(a) + \sum_l \left\{ \frac{1}{w(a)} \frac{\partial}{\partial a_l^*}[w(a)\gamma_{kl}(a)] + \gamma_{kl}^0 a_l \right\}. \tag{B·6}$$

Thus we obtain

$$da_k/dt = \sum_l (i\omega_{kl} - \gamma_{kl}^0) a_l - \sum_l \int_0^t \psi_{kl}(s) a_l(t-s) ds + q_k(t), \tag{B·7}$$

where

$$\psi_{kl}(t) = -\langle \Lambda q_k(t) a_l^* \rangle = \langle q_k(t) \tilde{q}_l^* \rangle. \tag{B·8}$$

The quantity $\tilde{q}_l$ is defined by

$$\begin{aligned} \tilde{q}_l &\equiv -(1-P_a) w(a)^{-1} D[a_l w(a)] \\ &= v_l'(a) - \sum_m \left\{ \frac{1}{w(a)} \frac{\partial}{\partial a_m^*}[w(a)\gamma_{ml}^*(a)] + \gamma_{ml}^{0*} a_m \right\}, \end{aligned} \tag{B·9}$$

where use has been made of $Dw(a)=0$. Equation (B·7) gives us (2·15). If one can neglect the a dependence of $\gamma_{kl}(a)$ and assume the Gaussian distribution for $w(a)$, then (B·8) leads to (2·17).

Appendix C

——Derivation of (2·22)——

Let us introduce a linear operator Q by

$$QA_k(t) \equiv \frac{d}{dt} A_k(t) - \sum_l i\omega_{kl} A_l(t) + \sum_l \int_0^t ds \varphi_{kl}(s) A_l(t-s). \tag{C·1}$$

Then (1·5) and (2·15) lead to

$$f_k(t) = QA_k(t), \tag{C·2}$$

$$q_k(t) = Qa_k(t), \tag{C·3}$$

where (1·11) has been used. Substituting (2·12) into (C·2), we obtain

$$f_k(t)=Q\int\left\{a_k(t)\delta(a-A(0))+\int_0^t a_k(t-s)F_a(s)ds\right\}da\,. \qquad (\text{C}\cdot 4)$$

The first term of (C·4) can be written as

$$\int q_k(t)\delta(a-A)da\,.$$

The second term gives us

$$Q\int_0^t a_k(t-s)F_a(s)ds=a_k(0)F_a(t)+\int_0^t [Qa_k(t-s)]F_a(s)ds\,.$$

Thus (C·4) leads to

$$f_k(t)=\int q_k(t)\delta(a-A(0))da+\int a_k F_a(t)da$$
$$+\int_0^t ds\int q_k(t-s)F_a(s)da\,. \qquad (\text{C}\cdot 5)$$

Substituting (2·5) into the second term of (C·5) we obtain (2·22).

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