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ON NONLINEAR VARIATIONAL INEQUALITIES

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Abstract. In this note we have given a direct proof of the result which states that if K is a compact convex subset of a linear Hausdorff topological space E over the reals and T is a monotone and hemicontinuous (nonlinear) mapping of K into E^*, then there is a $u_0 \in K$ such that $(T(u_0), v - u_0) > 0$ for all $v \in K$.

Introduction. Browder [1] has proved that if K is a closed convex subset of a reflexive Banach space E such that $0 \in K$ and T is a monotone and hemicontinuous nonlinear mapping of K into E^* satisfying the coercivity condition, then there is a $u_0 \in K$ such that $(T(u_0), v - u_0) \geqslant 0$ for all $v \in K$. Hartman and Stampacchia [3] have independently proved a similar result and made applications to second order nonlinear elliptic equations. This result with $c(u) = 0$ (see Theorem 1.1 of [3]) is a special case of Browder's result [1]. With the closed convex subset K of E as assumed in [3], the coercivity condition on T reduces the problem to proving the existence of u_0 satisfying the above inequality in a closed bounded convex subset of K (see remark following Theorem 1.1 and Lemma 2.2 in [3]). Thus it is of interest to prove the above result in a weakly compact convex subset of an arbitrary Banach space. This would then contain Theorem 1.1 in [3] and the result of [1] as special cases. In fact the main object of this paper is to prove this result in a compact convex subset of a linear topological space over the reals without the coercivity condition on T. The techniques used in [1] and [3] are more or less the same, 'to prove the result in a finite dimensional case and then apply a limiting procedure'. We will give a direct proof of our result by applying a generalized version of a fixed point theorem of Browder [2].

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We first prove a slight generalization of a fixed point theorem of Browder [2, Theorem 1, p. 285] which will suit our purpose.

Theorem 1. *Let K be a nonempty compact convex subset of a Hausdorff linear topological space E. Let T be a multivalued mapping of K into 2^K such that*

(i) *for each $x \in K$, $T(x)$ is a nonempty convex subset of K;*

(ii) *for each $y \in K$, $T^{-1}(y) = \{x \in K: y \in T(x)\}$ contains an open subset*

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O_y of K (O_y may be empty);

(iii) $\bigcup\{O_y : y \in K\} = K$.

Then there exists a point $x_0 \in K$ such that $x_0 \in T(x_0)$.

PROOF. Although the proof is similar to that in [2], we include it for the sake of completeness. Since K is compact, by (iii) there exists a finite family $\{y_1, y_2, \dots, y_n\}$ such that $K = \bigcup_{i=1}^n O_{y_i}$. Let $\{f_1, f_2, \dots, f_n\}$ be a partition of unity corresponding to this finite covering, i.e. each f_i, $i = 1, 2, \dots, n$, is a real valued continuous function defined on K such that f_i vanishes outside O_{y_i}, $0 \leqslant f_i(x) \leqslant 1$, for all $x \in K$ and $\sum_{i=1}^n f_i(x) = 1$ for each $x \in K$.

We define a mapping $p\colon K \to K$ by

$$p(x) = \sum_{i=1}^{n} f_i(x) y_i, \qquad x \in K.$$

Obviously p maps K into K and is continuous. Also for each k with $f_k(x) \neq 0$, $x \in O_{y_k} \subseteq T^{-1}(y_k)$, i.e. $y_k \in T(x)$. As $T(x)$ is convex, this implies that $p(x) \in T(x)$ for each $x \in K$.

Let S be the finite dimensional simplex spanned by $y_1, y_2, \dots, y_n$. Then clearly p maps S into S. Also, since E is Hausdorff linear topological space, the topology on S induced by the topology in E is Euclidean. Hence by the Brouwer fixed point theorem, there is a point $x_0 \in S$ such that $x_0 = p(x_0) \in T(x_0)$.

Let K be a subset of a linear topological space E over the reals and T a single valued (nonlinear) mapping of K into E^*. We say T is *monotone* provided $(T(u) - T(v), u - v) \geqslant 0$ for all $u, v \in K$. Here $(\cdot, \cdot)$ denotes the pairing between E^* and E.

$T\colon K \to E^*$ is said to be hemicontinuous if T is continuous from the line segments in K to the weak topology of E^*.

A point $u_0 \in K$ is said to satisfy the variational inequality if

$$(1) \qquad (T(u_0), v - u_0) \geqslant 0 \quad \text{for all } v \in K \dots.$$

u_0 is also called a solution of (1).

LEMMA. *If K is a convex subset of a linear Hausdorff topological space E, and T is a single valued mapping of K into E^* such that T is monotone and hemicontinuous, then u_0 is a solution of* (1) *if and only if u_0 is a solution of*

$$(2) \qquad (T(v), v - u_0) \geqslant 0 \quad \text{for all } v \in K \dots.$$

PROOF. The proof of this lemma on a Banach space in [1, Lemma 1] or in [3, Lemma 2.3] also holds here. If u_0 satisfies (1), then an application of monotonicity shows that u_0 satisfies (2). Now suppose that u_0 satisfies (2). As in [1] and [3] we employ a device of Minty [4]. Let v be an arbitrary point of K. Then since K is convex, $v_t = (1 - t)u_0 + tv \in K$ for $0 < t \leqslant 1$. By (2) we have

$$0 \leqslant (T(v_t), t(v - u_0)) = t(T(v_t), v - u_0).$$

Since $t > 0$, $(T(v_t), v - u_0) \geqslant 0$.

Now letting $t \to 0$ and using hemicontinuity of T, $T(v_t) \to T(u_0)$ weakly in E^*. Hence $(T(u_0), v - u_0) \geqslant 0$.

REMARK. We note that in the proof of the first part the convexity of K is not needed. In fact, if T: $K \to E^*$ is a monotone mapping of any set $K \subseteq E$ into E^*, then given $u \in K$, the set $\{v: (T(u), v - u) \geqslant 0\} \subseteq \{v: (T(v), v - u) \geqslant 0\}$. This follows from the definition of monotonicity, i.e., $(T(v), v - u) \geqslant (T(u), v - u)$.

THEOREM 2. *Let K be compact convex subset of a linear Hausdorff topological space E. Let T be a* (*single valued*) *monotone* (*nonlinear*) *mapping of K into E^*. Suppose further that*

(*) *for each $v \in K$ there exists $u \in K$ such that* $(T(u), u - v) < 0$.

Then there is a solution u_0 of (1), *i.e. there is $u_0 \in K$ such that* $(T(u)_0, v - u_0) \geqslant 0$ *for all $v \in K$.*

PROOF. We assume that there is no solution of (1). Then for each $u \in K$, the set $\{v \in K: (T(u), v - u) < 0\}$ is nonempty. We define a multivalued mapping F: $K \to 2^K$ by

$$F(u) = \{v \in K: (T(u), v - u) < 0\}.$$

$F(u)$ is nonempty and clearly convex for each $u \in K$. We now consider

$$F^{-1}(u) = \{v \in K: u \in F(v)\} = \{v \in K: (T(v), u - v) < 0\}.$$

For each $u \in K$, $[F^{-1}(u)]^c$ = the complement of $F^{-1}(u)$ in

$$K = \{v: (T(v), u - v) \geqslant 0\} \subseteq \{v: (T(u), u - v) \geqslant 0\}$$

by monotonicity of $T = B(u)$, say. Obviously $B(u)$ is a closed and convex subset of K. Thus the complement of $B(u) = [B(u)]^c$ is open in K. Since $[F^{-1}(u)]^c \subseteq B(u)$, it follows that $[B(u)]^c \subseteq F^{-1}(u)$. Thus for each $u \in K$, $F^{-1}(u)$ contains an open set $[B(u)]^c$ of K. Now from the hypothesis that for each $v \in K$, there exists $u \in K$ such that $(T(u), u - v) < 0$, it follows that $\bigcup\{[B(u)]^c, u \in K\} = K$. Thus F satisfies all the conditions of our Theorem 1. Hence there exists a point $w \in K$ such that $w \in F(w)$, i.e. $0 > (T(w), w - w) = 0$, which is impossible.

COROLLARY. *Let K be a compact convex subset of a linear Hausdorff topological space E. Let T be a monotone and hemicontinuous* (*nonlinear*) *mapping of K into E^*. Then there is a solution u_0 of* (1), *i.e., there is $u_0 \in K$ such that* $(T(u_0), v - u_0) \geqslant 0$ *for all $v \in K$.*

PROOF. If (*) of Theorem 2 holds, then we have a solution u_0 of (1) by Theorem 2. If (*) does not hold, then it means precisely that there is $u_0 \in K$ such that $(T(u), u - u_0) \geqslant 0$ for all $u \in K$. Since T is hemicontinuous, the lemma implies that $(T(u_0), u - u_0) \geqslant 0$ for all $u \in K$, i.e. u_0 is a solution of the variational inequality.

REMARK. It has already been pointed out in the introduction that our corollary contains the result of [1] and Theorem 1.1 of [3] as a special case. It is also worth noting that

(i) it follows from the proof of our theorem that we can replace the monotonicity condition by a weaker condition that for each $u \in K$, $\{v: (T(v), u - v) \geqslant 0\} \subseteq \{v: (T(u), u - v) \geqslant 0\}$;

(ii) in case of a locally convex Hausdorff topological space E, it does not matter whether we assume K to be compact or weakly compact. The corollary still remains true as T remains hemicontinuous in either case.

REFERENCES

1. F. E. Browder, *Nonlinear monotone operators and convex sets in Banach spaces*, Bull. Amer. Math. Soc. **71** (1965), 780–785.

2. ______, *The fixed point theory of multivalued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.

3. P. Hartman and G. Stampacchia, *On some non-linear elliptic differential-functional equations*, Acta Math. **115** (1966), 271–310 .

4. G. J. Minty, *Monotone* (*non-linear*) *operators in Hilbert spaces*, Duke Math. J. **29** (1962), 341–346.

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