On Nonstationary Flow through Porous Media (*) (**).

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Summary. – *Saturated-unsaturated flow of an incompressible fluid through a porous medium is considered in the case of time-dependent water levels. This corresponds to coupling the mass conservation law with a continuous constitutive condition between water content and pressure. An existence result for the corresponding weak formulation is proved. Finally we study the limit as the constitutive relation degenerates into a maximal monotone graph.*

Introduction.

We deal with nonstationary saturated-unsaturated flow of an incompressible fluid through a nonhomogeneous porous medium; this is assumed to form a dam bounded by impervious layers, water reservoirs and air; water levels are assumed to change in time.

By saturated-unsaturated flow one means that the dependence of water content on pressure is given by a continuous function; this corresponds to experiments (see fig. 1). In another model this relationship is represented by a step function; this corresponds to a well-known free boundary problem and can be interpreted as the large scale behaviour of the continuum case (see [2]).

Mathematical work on this problem was started with Baiocchi's fundamental paper [4] treating the stationary free boundary problem via a variational inequality. Another approach introduced by Alt [1] was the approximation of the free boundary flow by the saturated-unsaturated flow in the stationary case.

The nonstationary situation has been studied first by Torelli [11], who gave a time-dependent version of the Baiocchi transformation. Later on working with pressure directly, Gilardi [7] and Visintin [12] treated a more general situation.

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In this paper we apply the general method developed by ALT-LUCKHAUS [3] to nonstationary saturated-unsaturated flow; our results include the one given in [8] by HORNUNG for flow without gravity. Then we study the limit free boundary problem; since in the general case we can just show the weak convergence of the approximated saturation, we prove existence of a solution for a weaker formulation, in which the non-linear relation between water content and conductivity is replaced by a more general condition.

If one can show that the unsaturated region has vanishing measure, which is not true for general inhomogeneous media, then there is equivalence with the physical problem. This is the same situation as in [7, 12].

However in the case of a single space dimension we can prove the strong convergence of the approximating saturation, hence we can take the limit in the non-linear relation between water content and conductivity; this gives a physical meaning to the solution.

1. – The physical situation.

Let $D \subset \boldsymbol{R}^3$ represent the region occupied by the porous medium and Γ_1 be the impervious part of the boundary; let p denote the pressure, θ the relative water content and $\tilde{k}$ the permeability of the porous medium depending on θ and x.

We have the equation of continuity

$$\theta_t + \overline{\nabla}\cdot\bar{q} = 0 \quad \text{in } Q = D \times]0, T[\quad \text{(we bar vectors)}$$

(where $\bar{q}$ is the flux) and Darcy's law

$$\bar{q} = -\tilde{k}(\overline{\nabla}p + \varrho g\bar{e}) \quad \text{in } Q$$

(where ϱ is the density of the fluid, g is the gravity acceleration and $\bar{e}$ is the upward vertical vector); moreover

$$\theta = \varphi(x)\tilde{s}(p) \quad \text{in } Q\,,$$

where $\tilde{s}(p) \in [0, 1]$ denotes saturation and $\varphi(x)$ is the proposity of the medium. The above formulae yield

$$\varphi\tilde{s}(p)_t - \overline{\nabla}\cdot[\tilde{k}(\tilde{s}(p), x)(\overline{\nabla}p + \varrho g\bar{e})] = 0 \quad \text{in } Q\,. \tag{1.1}$$

The following figures represent typical experimental relationships for $\tilde{s}$ and $\tilde{k}$.

On the complement Γ_2 of Γ_1 in ∂D we have the following: where the porous medium is in contact with water the pressure is prescribed and positive, where it is

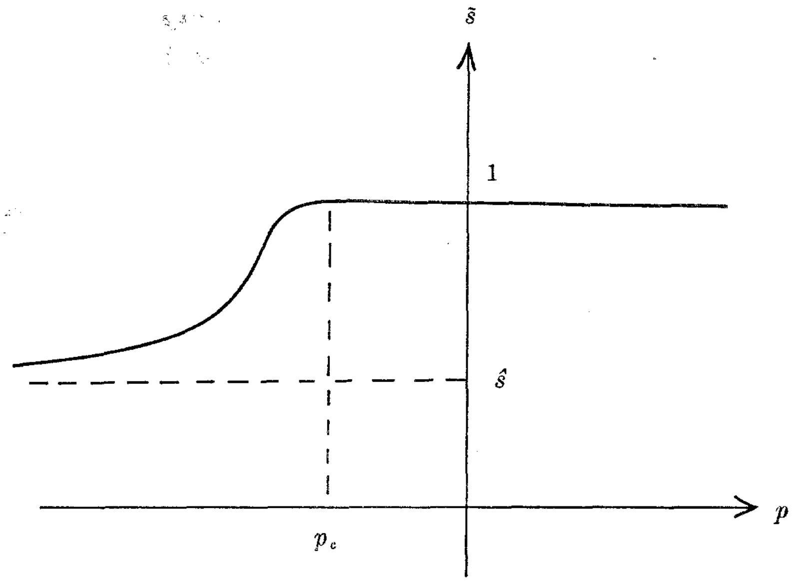

Fig. 1 (cf. [5] fig. 6-13).

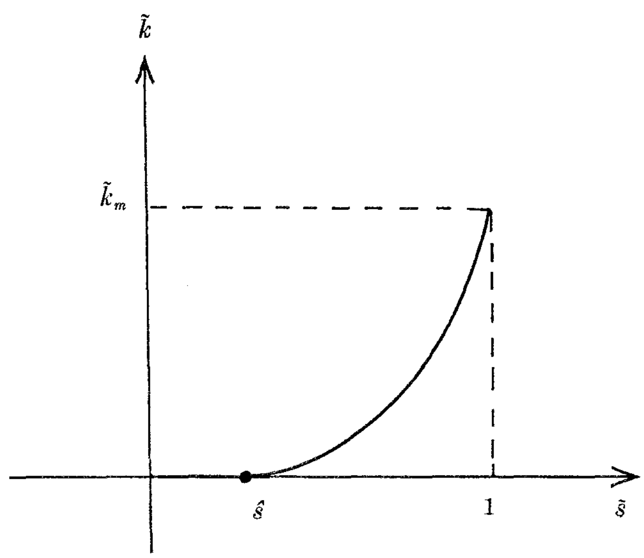

Fig. 2 (cf. [5] fig. 6-16).

is contact with air either the flux is zero and pressure non-positive (non-prescribed) or there is overflow, that is non-negative flux and zero pressure.

So we have (denoting the outer normal by $\bar{\nu}$)

$$\tilde{k}(\overline{\nabla}p + \varrho g\bar{e})\cdot\bar{\nu} = 0 \qquad \text{on } \Sigma_1 = \Gamma_1 \times]0, T[\tag{1.2}$$

(1.3) $$p^+ = p^* \text{ (datum)} \qquad \text{on } \Sigma_2 = \Gamma_2 \times]0, T[$$

(1.4) $$\tilde{k}(\overline{\nabla} p + \varrho g \bar{e}) \cdot \bar{\nu} \leqq 0 \qquad \text{on } \{(\sigma, t) \in \Sigma_2 | p(\sigma, t) = 0\}$$

(1.5) $$\tilde{k}(\overline{\nabla} p + \varrho g \bar{e}) \cdot \bar{\nu} = 0 \qquad \text{on } \{(\sigma, t) \in \Sigma_2 | p(\sigma, t) < 0\}$$

Note that the last two conditions can be formulated as follows

(1.6) $$\tilde{k}(\overline{\nabla} p + \varrho g \bar{e}) \cdot \bar{\nu}(p - v) \leqq 0 \qquad \text{on } \Sigma_2 , \qquad \forall v : \Sigma_2 \to \mathbf{R} \text{ such that } v^+ = p^* .$$

We restrict to the case that

(1.7) $$\tilde{k}(\tilde{s}, x) = k(\tilde{s})\, a(x) \qquad \text{in } Q .$$

We introduce the transformation

(1.8) $$u = \int_0^p k(\tilde{s}(\xi))\, d\xi \qquad \text{in } Q$$

after which (1.1) becomes

(1.9) $$\varphi s(u)_t - \overline{\nabla} \cdot \{a(x)[\overline{\nabla} u + k(s(u))\bar{e}]\} = 0 \qquad \text{in } Q$$

where $s(u) = \tilde{s}(p)$ has the form of fig. 3.

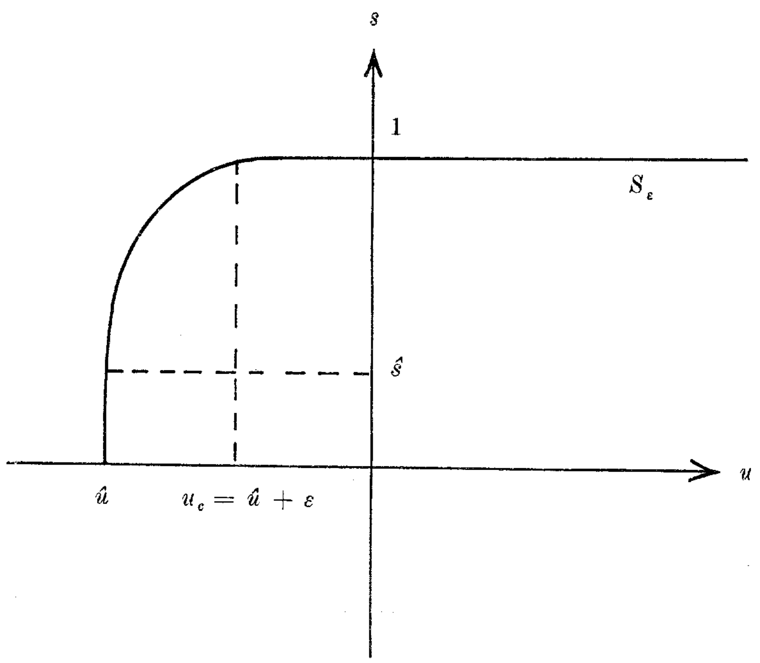

Fig. 3

We shall consider a family of problems with S_ε as in fig. 3 (with $u_c := \hat{u} + \varepsilon$) and we shall study the limit behaviour as $\varepsilon \to 0$.

$$\hat{u} := -\int_{-\infty}^{0} k(\tilde{s}(\xi))\, d\xi\,, \qquad u_c := -\int_{v_\varepsilon}^{0} k(\tilde{s}(\xi))\, d\xi$$

S_ε is actually extended into a graph, since for the initial saturation we admit also values under the irreducible wetness $\hat{s}$.

Since ε does not depend on t, in the following we shall take $\varphi = 1$ without any loss of generality.

2. – Existence result for the problem with saturated-unsaturated flow.

Let D be connected and Γ_1, Γ_2 be Lipschitz manifolds. Let

$$p^* \in H^1(Q) \cap C^0([0, T]; H^1(D))\,.$$

Set

$$V = \{v \in H^1(D) | v = 0 \ \text{ on } \ \Gamma_2\}$$

$$K = \{v \in L^2(0, T; H^1(D)) | v^+ = p^* \ \text{ on } \ \Sigma_2\}:$$

$$\forall \varepsilon > 0\,, \quad \forall \xi \in \boldsymbol{R}\,, \quad \text{set } W_\varepsilon(\xi) := \sup_{\hat{u} \leqq \eta < \infty} \left(\xi\eta - \int_0^\eta S_\varepsilon(\lambda)\, d\lambda\right) = \int_0^{S_\varepsilon^{-1}(\xi)} (\xi - S_\varepsilon(\lambda))\, d\lambda\,.$$

Let $s^0 \in L^\infty(D)$, $0 \leqq s^0 \leqq 1$ a.e. in D.

(P_ε) Find $u_\varepsilon \in K$ such that, setting

$$s_\varepsilon = S_\varepsilon(u_\varepsilon) \in L^\infty(Q) \cap H^1(0, T; V')\,, \tag{2.1}$$

$$\begin{cases} -\int_D [W_\varepsilon(s^0(x)) - s^0(x)\, v(x, 0)]\, \alpha(x, 0)\, dx - \iint_Q W_\varepsilon(s_\varepsilon)\, \alpha_t\, dx\, dt \\ \qquad + \iint_Q s_\varepsilon(v\alpha)_t\, dx\, dt + \iint_Q a(\overline{\nabla} u + k(s_\varepsilon)\bar{e}) \cdot \overline{\nabla}[(u_\varepsilon - v)\, \alpha]\, dx\, dt \leqq 0\,, \\ \forall v \in C^0([0, T]; H^1(D)) \cap H^1(Q) \cap K\,, \quad \forall \alpha \in C^2(Q) \quad \text{with } 0 \leqq \alpha \leqq 1\,, \\ \alpha(\cdot, T) = 0 \quad \text{in } D\,. \end{cases} \tag{2.3}$$

INTERPRETATION. – From (2.3), taking $\alpha = 0$ on Σ_2 and integrating by parts we get (for any v as in (2.3))

$$\int_0^T {}_{V'}\langle s_{\varepsilon t} - \overline{\nabla} \cdot [a(\overline{\nabla} u_\varepsilon + k(s_\varepsilon)\bar{e})],\ (u_\varepsilon - v)\, \alpha \rangle_V\, dt \leqq 0 \tag{2.4}$$

whence

$$s_{\varepsilon t} - \overline{\nabla}\cdot[a(\overline{\nabla} u_\varepsilon + k(s_\varepsilon)\bar{e})] = 0 \quad \text{in } V', \text{ a.e. in }]0, T[; \tag{2.5}$$

and then (2.3) yields

$$s_\varepsilon(x, 0) = s^0(x) \quad \text{a.e. in } D, \tag{2.6}$$

indeed by (2.1) s_ε is weakly star continuous from $[0, T]$ to $L^\infty(\Omega)$; we get also (1.2) and

$${}_{(H^{1/2}_{00}(\Gamma_2))'}\left\langle a\left[\frac{\partial u_\varepsilon}{\partial \nu} + k(s_\varepsilon)\bar{e}\right]\cdot\bar{\nu},\ u_\varepsilon - v\right\rangle_{H^{1/2}_{00}(\Gamma_2)} \leq 0 \quad \text{in } H^{-1}(0, T), \tag{2.7}$$

which corresponds to (1.6).

PROPOSITION 1. – $\forall\varepsilon > 0$ problem (P_ε) has at least one solution.

PROOF. – Extend S_ε to all of $\boldsymbol{R}$ by

$$S_\varepsilon(\xi) = \xi - \hat{u} \quad \text{for } \xi < \hat{u}$$

and extend k by zero.

Note that

$$S_\varepsilon(\xi) = S^c_\varepsilon(\xi) + \hat{s}H(\xi - \hat{u}),$$

where H is the Heaviside function, and that $k(S_\varepsilon(\xi)) = k(S^c_\varepsilon(\xi) + \hat{s})$ depends on the continuous part S^c_ε of S_ε only.

To approximate (2.3) we use an implicit time-discretization, i.e. we have to solve

$$\int_{t_1}^{t_2}\int_D \frac{s_h(t, x) - s_h(t-h, x)}{h}(u_h - v)\,dx\,dt + \int_{t_1}^{t_2}\int_D a(x)\left(\overline{\nabla} u_h(t, x) + k\left(S^c_\varepsilon(u_h(t-h, x)) + \hat{s}\right)\bar{e}\right)\cdot\overline{\nabla}(u_h - v)\,dx\,dt \leq 0 \tag{2.3h}$$

for all $0 \leq t_1 \leq t_2 \leq T$, $v \in K_h$; where $u_h \in K_h$, $s_h \in S_\varepsilon(u_h)$, s_h fulfills the initial condition, $s_h(t) = s_0$ for $-h < t \leq 0$. K_h is defined by $K_h = \{v \in L^2(0, T: H^1(D))| v^+ = p^*_h$ on $\Sigma_2\}$, where $p^*_h(t) = p^*([t/h]h)$, [ ] denoting the Gaussbrackets.

The proof of convergence is quite analogous to that in [3, ch. 5]; we outline it here for completeness.

Note that $u_h \in \partial W_\varepsilon(s_h)$, ∂ denoting the subdifferential. So taking $v = p_h^*$ as a test function we arrive at the estimate

$$\int_0^T \int_D |\overline{\nabla} u_h|^2 \leqq \text{const}\,. \tag{2.8}$$

Next letting v range over the set $v = u_h + \hat{v}$, $\hat{v} \in V$ we get

$$\frac{1}{h}\|s_h - s_h(\cdot - h)\|_{V'} = \|\overline{\nabla}\cdot(a(\overline{\nabla} u_h + k\bar{e}))\|_{V'} \leqq \text{const}\,. \tag{2.9}$$

As a consequence

$$\int_{nh}^T \int_D \zeta(s_h - s_h(\cdot - \tau))\cdot(u_h - u_h(\cdot - \tau))\,dx\,dt \leqq C(\|\overline{\nabla}\zeta\|_{L^\infty} + 1)\,\tau \tag{2.10}$$

whenever ζ is Lipschitz and $\zeta \equiv 0$ on Σ_2.

Now since S_ε^c is monotone continuous it fulfills the following estimate, cf. [3, lemma 1.8]

$$|S_\varepsilon^c(x) - S_\varepsilon^c(y)| \leqq \omega((\xi_x - \xi_y)\cdot(x - y)) \qquad \text{for all } y, x, \xi_x, \xi_y$$

such that $\xi_x \in S_\varepsilon(x)$, $\xi_y \in S_\varepsilon(y)$, where ω is a continuous function with $\omega(0) = 0$.

Consequently from the estimates (2.8) and (2.10) we get the compactness of $S_\varepsilon^c(u_h)$ in every L^p with $p < \infty$. Moreover with the help of the same estimates using a compensated compactness result [3, lemma 5.3] we have for the weak limits s, u of s_h, u_h

$$s \in S_\varepsilon(u)\,.$$

Now we show the inequality (2.3) for s, u; we take $(1 - \alpha_h)u_h + \alpha_h v_h$ as test functions, where α, v are as in (2.3) and we have set

$$\alpha_h(t) = \alpha\left(\left[\frac{t}{h}\right]h\right), \qquad v_h(t) = v\left(\left[\frac{t}{h}\right]h\right).$$

Integrating (2.3h) partially with respect to time and going to the limit, using the lower semicontinuity of $\|\overline{\nabla}\cdot\|^2$ we get (2.3).

Finally u is in K because p_h^* converges to p^* strongly and $u \geqq \hat{u}$ follows from the weak maximum principle of the heat equation. ∎

PROPOSITION 2. – If besides the assumptions of proposition 1 we have $\hat{s} \leqq s^0 \leqq 1$, then there is a solution of 2.3 with $s \geqq \hat{s}$.

The proof is the same as that of proposition 1, except that $\hat{s}H(\xi - u)$ has to be replaced by the constant $\hat{s}$. ∎

REMARK. – The continuity condition $v \in C^0(0, T; H^1(D))$ may be weakened to $v\colon]0, T[\to H^1(D)$ piecewise continuous. If we impose on p^* the rather natural condition that p^* is piecewise monotone in time, then as $H^1(Q) \cap K \cap C^0(0, T, H^1(D))$ is dense in $H^1(Q) \cap K$, the above continuity condition on v may be dropped altogether.

3. – Study of the limit free boundary problem.

In this section we assume $s^0 \geq \hat{s}$, which corresponds to the situation usually considered in literature.

As we shall see, difficulties are encountered in proving that the approximated saturation s_ε converges strongly. Therefore here we introduce a weak formulation of the limit free boundary problem in which the relationship between saturation and permeability is expressed in terms of the closed convex hull L of the graph of $k|_{[\hat{s},1]}$ (see fig. 4). The limit saturation graph is obtained taking $u_c \to \hat{u}$ (i.e. $\varepsilon \to 0$) in fig. 3.

$$S(\xi) := \begin{cases} \{\hat{s}\} & \text{if } \xi < \hat{u}\,, \\ [0, 1] & \text{if } \xi = \hat{u}\,, \\ \{1\} & \text{if } \xi > \hat{u}\,. \end{cases}$$

<u>(P)</u> Find$(u, s, \varkappa)$ such that

$$u \in K\,, \quad s \in L^\infty(Q) \cap H^1(0, T; V')\,, \tag{3.1}$$

$$s \in S(u) \qquad \text{a.e. in } Q \tag{3.2}$$

$$(s, \varkappa) \in L \qquad \text{a.e. in } Q \tag{3.3}$$

$$\begin{cases} \displaystyle\int_D s^0(x)\, v(x, 0)\, \alpha(x, 0)\, dx + \iint_Q s(v\alpha)_t\, dx\, dt + \iint_Q a(\overline{\nabla} u + \varkappa \bar{e}) \cdot \overline{\nabla}[(u - v)\, \alpha]\, dx\, dt \leqq 0 \\ \forall v \in C^0([0, T]; H^1(D)) \cap H^1(Q) \cap K\,, \quad \forall \alpha \in C^2(Q) \quad \text{with } 0 \leqq \alpha \leqq 1\,, \\ \alpha(\cdot, T) = 0 \quad \text{in } D\,. \end{cases} \tag{3.4}$$

REMARK. – (3.2) and (3.3) entail

$$\varkappa = k_m \quad \text{where } u > \hat{u} \quad \text{a.e. in } Q\,. \tag{3.5}$$

The hitherto known existence results of [7, 12] correspond to a linear relationships for k which of course is preserved at the limit:

$$\varkappa = \frac{k_m}{1 - \hat{s}}(s - \hat{s}) \qquad \text{a.e. in } Q\,. \tag{3.6}$$

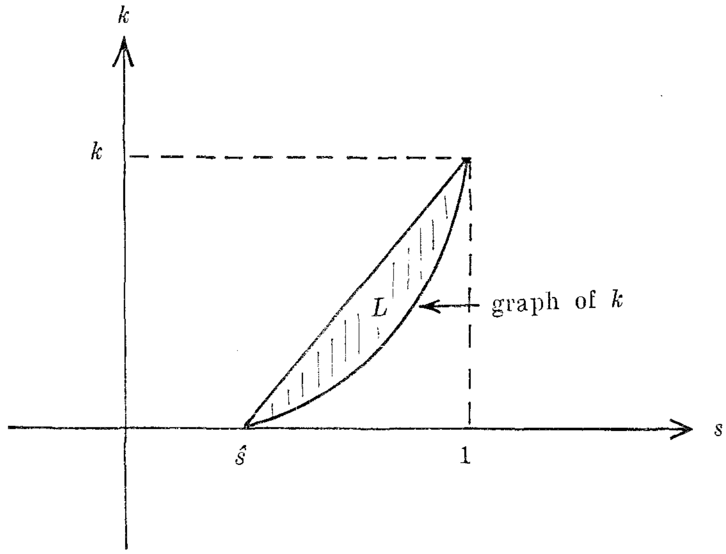

Fig. 4

Of course if the unsaturated region $\{(x, t) \in Q \,|\, \hat{s} < s(x, t) < 1\}$ has vanishing measure, then (3.3), (3.5) and (3.6) are equivalent.

THEOREM 1. – There exists a sequence of solutions $\{u_\varepsilon\}$ of (P_ε) (where $\varepsilon \to 0$) and $u, s, \varkappa$ such that

(3.7) $u_\varepsilon \to u$ weakly in $L^2(0, T, H^1(D))$

(3.8) $S_\varepsilon(u_\varepsilon) \to s$ weakly star in $L^\infty(Q)$, weakly in $H^1(0, T, V')$

(3.9) $k(S_\varepsilon(u_\varepsilon)) \to \varkappa$ weakly star in $L^\infty(Q)$.

Moreover this entails that $(u, s, \varkappa)$ is a solution of problem (P).

PROOF. – Since the W_ε's are uniformly bounded, (2.3) yields

$$\|u_\varepsilon\|_{L^2(0,T;H^1(D))} \leqq \text{constant (independent of } \varepsilon);$$

then by (2.5)

$$\|s_{\varepsilon t}\|_{L^2(0,T;V')} \leqq \text{constant;}$$

moreover of course

$$\|s_\varepsilon\|_{L^\infty(Q)} \leqq \text{constant}, \qquad \|k(s_\varepsilon)\|_{L^\infty(Q)} \leqq \text{constant}.$$

Therefore there exists a sequence $\{u_\varepsilon\}(\varepsilon \to 0)$ and u, s, $\varkappa$ such that (3.7), (3.8), (3.9) hold.

As $W_\varepsilon \to 0$, taking the inferior limit as $\varepsilon \to 0$ in (2.3) we get (3.4).

By a compactness result due to Aubin (se [9], p. 57, e.g.) (3.8) yields for instance

$$s_\varepsilon \to s \quad \text{strongly in } L^2(0, T; V');$$

therefore for any $\eta \in C_0^\infty(Q)$ with $\eta \geqq 0$ we have

$$\iint_Q s_\varepsilon u_\varepsilon \eta\, dx\, dt \geqslant \int_0^T {}_{V'}\langle s_\varepsilon, u_\varepsilon \eta\rangle_V\, dt \to \int_0^T {}_{V'}\langle s, u\eta\rangle_V\, dt = \iint_Q su\eta\, dx\, dt;$$

now let Ψ_ε, Ψ: $\boldsymbol{R} \to \boldsymbol{R}$ be convex functions such that $\partial\Psi_\varepsilon = S_\varepsilon$, $\partial\Psi = S$, $\Psi_\varepsilon \to \Psi$ pointwise; by (2.1) we have

$$\iint_Q s_\varepsilon(u_\varepsilon - v)\,\eta\, dx\, dt \geqq \iint_Q \big(\Psi_\varepsilon(u_\varepsilon) - \Psi_\varepsilon(v)\big)\,\eta\, dx\, dt$$

whence taking the inferior limit as $\varepsilon \to 0$

$$\iint_Q s(u - v)\,\eta\, dx\, dt \geqq \iint_Q \big(\Psi(u) - \Psi(v)\big)\,\eta\, dx\, dt$$

which yields (3.2), by the genericity of η.

Finally (3.3) is obtained applying the following result. ∎

LEMMA 1. – Let Ω be a measurable set, C a subset of $\boldsymbol{R}^M$ $(M \geqq 1)$ and $v, v_n\colon \Omega \to C$ $(n \in \boldsymbol{N})$ measurable functions such that $v_n \to v$ weakly in $(L^p(\Omega))^M$ $(1 \leqq p < +\infty)$ or weakly star in $(L^\infty(\Omega))^M$.

Then a.e. in Ω $v(x) \in \overline{\operatorname{conv}(C)}$ (closed convex hull of C). ∎

A natural question arises: is it possible to prove

$$\varkappa = k(s) \quad \text{a.e. in } Q \tag{3.10}$$

for $\varkappa$, s given by (3.8), (3.9)? In the next section we shall show that in the case of a single space dimension this holds; in the general situation the question remains open.

About this point we remark that lemma 1 has a converse:

LEMMA 2. – Let Ω be a measurable set, C a subset of $\boldsymbol{R}^M$ $(M \geqq 1)$

$$v\colon \Omega \to \operatorname{conv}(C) \quad \text{and} \quad v \in \big(L^p(\Omega)\big)^M \quad (1 \leqq p \leqq \infty)\,.$$

Then there exists a sequence of measurable functions $\{v_n: \Omega \to C\}_{n\in N}$ such that $v_n \to v$ weakly in $(L^p(\Omega))^M$ if $1 \leqq p < \infty$, weakly star in $(L^\infty(\Omega))^M$ if $p = \infty$.

For the proof, see [10]. ■

From this result we get that (3.3) cannot be improved using only (3.9) and the fact that $s_\varepsilon \to s$ weakly star in $L^\infty(Q)$.

We have also the following result:

LEMMA 3. – Let C be a closed, convex subset of $\boldsymbol{R}^M$, let $\varphi: C \to \boldsymbol{R}$ be strictly convex and lower semi-continuous; let Ω be a measurable set with finite measure; let v, $v_n: \Omega \to C$ $(n \in \boldsymbol{N})$ be measurable functions such that

$$\begin{aligned} &v_n \to v && \text{weakly in } (L^p(\Omega))^M \qquad (1 < p < \infty)\\ &\varphi(v_n) \to \varphi(v) && \text{weakly in } L^p(\Omega)\,. \end{aligned}$$

Then $\forall q \in [1, p[$

$$\begin{aligned} &v_n \to v && \text{strongly in } (L^q(\Omega))^M\\ &\varphi(v_n) \to \varphi(v) && \text{stronlgy in } L^q(\Omega)\,. \end{aligned}$$

For the proof see [13]. ■

Hence, if (3.10) held then (3.8) and (3.9) would yield for any $q \in [1, \infty[$

$$\begin{aligned} &s_\varepsilon \to s && \text{strongly in } L^q(\Omega)\\ &k(s_\varepsilon) \to \varkappa && \text{strongly in } L^q(\Omega)\,. \end{aligned}$$

4. – Strong convergence of the saturation in one space dimension.

As we saw at the end of the last paragraph, strong convergence of the saturation is equivalent to fulfilling the nonlinear equation

$$s_t = \nabla\cdot\big[a\big(\overline{\nabla} u + k(s)\bar{e}\big)\big] \quad \text{in } \mathcal{D}'(Q)\,.$$

So at least for one space dimension we are going to prove strong convergence of the s_ε, for nonconstant a, i.e. also in situations where unsaturated regions will appear in the limit. Let us point out that our method for getting L^1 estimates on $s_{\varepsilon t}$ is the one used in the theory of L^1-contractions, see [6] e.g.

THEOREM 2. – Let u_ε be the solutions of (2.3), and $s_\varepsilon = S_\varepsilon(u_\varepsilon)$ be the corresponding saturations in $D =]x_1, x_2[\subset \boldsymbol{R}^1$. Suppose p independent of time, $s_0 \geqq \hat{s}$,

$s_0 = S_\varepsilon(u_0^\varepsilon)$ with $(u_0^\varepsilon)^+ = p^*$ on Γ_2 and that a and $a(u_{0x}^\varepsilon + k(s_0))$ are bounded in $BV(D)$. We assume that k is strictly monotone on $]\hat{s}, 1[$ and that for each $\varepsilon > 0$, $k\circ S_\varepsilon \in C^\beta_{\text{loc}}(]\hat{u}_\varepsilon, \infty[)$ with a Hölder-exponent $\beta > \frac{1}{2}$.

Then a subsequence of s_ε converges almost everywhere.

PROOF. – To prove the theorem we show that $s_{\varepsilon t}$ is bounded in $L^1(Q)$ and $k(s_\varepsilon)_x$ is bounded in $L^1_{\text{loc}}(Q)$. Since k is strictly monotone we conclude that

$$\iint_Q |s_\varepsilon - s_\varepsilon(\cdot - \tau)|\, dx\, dt \underset{\tau\to 0}{\longrightarrow} 0 \qquad \text{uniformly in } \varepsilon .$$

So s_ε are compact in $L^1(Q)$, which finishes the proof, once the a priori estimates are obtained.

First let us prove the estimate on $s_{\varepsilon t}$: To be absolutely correct one should have to prove the estimate on the solutions s_ε^h of the approximating inequalities (2.3h). But for simplicity of notation, we work with s_ε directly. Take ψ_δ as an approximation of the sign.

$$\psi_\delta(\xi) = \begin{cases} -1 & \text{for } \quad \xi \leqq -\delta \\ \xi/\delta & \text{for } -\delta \leqq \xi \leqq \delta \\ 1 & \text{for } \quad \delta \leqq \xi . \end{cases}$$

Since p^* is constant in time,

$$u_\varepsilon(t) - \delta\psi_\delta\big(u_\varepsilon(t) - u_\varepsilon(t-\tilde{h})\big) \qquad \text{and} \qquad u_\varepsilon(t-\tilde{h}) + \delta\psi_\delta\big(u_\varepsilon(t) - u_\varepsilon(t-\tilde{h})\big)$$

are admissible functions for the variational inequality (2.3) at time t and $t-\tilde{h}$ respectively. We get

$$\int_{\tilde{h}}^{T}\int_{x_1}^{x_2}\big(s_{\varepsilon t}(t) - s_{\varepsilon t}(t-\tilde{h})\big)\,\psi_\delta\big(u_\varepsilon(t) - u_\varepsilon(t-\tilde{h})\big)\,dx\,dt + \int_{\tilde{h}}^{T}\int_{x_1}^{x_2} a\big(u_{\varepsilon x}(t) - u_{\varepsilon x}(t-\tilde{h})\big)^2 \cdot$$
$$\cdot\,\psi_\delta'\big(u_\varepsilon(t) - u_\varepsilon(t-\tilde{h})\big)\,dx\,dt +$$
$$+ \int_{\tilde{h}}^{T}\int_{x_1}^{x_2} a\Big(K\big(s_\varepsilon(t)\big) - K\big(s_\varepsilon(t-\tilde{h})\big)\Big)\big(u_{\varepsilon x}(t) - u_{\varepsilon x}(t-\tilde{h})\big)\,\psi_\delta'\big(u_\varepsilon(t) - u_\varepsilon(t-\tilde{h})\big)\,dx\,dt \leqq 0 .$$

Using the Cauchy inequality we derive the estimate

$$\int_{\tilde{h}}^{T}\int_{x_1}^{x_2}\big(s_{\varepsilon t}(t) - s_{\varepsilon t}(t-\tilde{h})\big)\,\psi_\delta\big(u_\varepsilon(t) - u_\varepsilon(t-\tilde{h})\big)\,dx\,dt - \int_{\tilde{h}}^{T}\int_{x_1}^{x_2}\frac{a}{2}\big[k\big(s_\varepsilon(t)\big) - k\big(s_\varepsilon(t-\tilde{h})\big)\big]^2 \cdot$$
$$\psi_\delta'\big(u_\varepsilon(t) - u_\varepsilon(t-\tilde{h})\big)\,dx\,dt \leqq 0 .$$

Now assuming for the moment that $k\circ S_\varepsilon$ is in $C^\beta([\hat{u}, \infty[)$, the last integral converges to zero with δ. If instead $k\circ S_\varepsilon \in C^\beta_{\mathrm{loc}}(]\hat{u}, \infty[)$ only, one has to modify the test function taking $\psi_\delta\big(\max(u_\varepsilon(t), \hat{u}+\alpha) - \max(u_\varepsilon(t-\tilde{h}), \hat{u}+\alpha)\big)$ with $\alpha > 0$. In any case we get letting δ tend to zero

$$\int_{\tilde{h}}^{T}\int_{x_1}^{x_2} |s_\varepsilon(t) - s_\varepsilon(t-\tilde{h})|_t(1 - \chi_{\{s_\varepsilon(t)<S_\varepsilon(\hat{u}+\alpha)\}}\,\chi_{\{s_\varepsilon(t-\tilde{h})<S_\varepsilon(\hat{u}+\alpha)\}})\,dx\,dt \leqq 0 .$$

So we have the estimate

$$\sup_{(\tilde{h},T)}\int_{x_1}^{x_2}|s_\varepsilon(t) - s_\varepsilon(t-\tilde{h})|\,dx \leqq \int_{x_1}^{x_2}|s_\varepsilon(\tilde{h}) - s_0|\,dx .$$

Now we have to estimate the initial time difference. Multiplying the inequality by $\psi_\delta(u_\varepsilon(t) - u^0_\varepsilon)$ and integrating from zero to h, we obtain

$$\int_0^{\tilde{h}}\int_{x_1}^{x_2} s_{\varepsilon t}\,\psi_\delta(\ldots)\,dx\,dt + \int_0^{\tilde{h}}\int_{x_1}^{x_2} a\big(u_{\varepsilon x} + K(s_\varepsilon)\big)\,\psi_\delta(\ldots)_x\,dx\,dt \leqq 0$$

and it follows that

$$\int_0^{\tilde{h}}\int_{x_1}^{x_2}(s_\varepsilon - s_0)_t\,\psi_\delta(\ldots)\,dx\,dt + \int_0^{\tilde{h}}\int_{x_1}^{x_2} a(u_{\varepsilon x} - u^\varepsilon_{0x})\,\psi_\delta(\ldots)_x\,dx\,dt +$$
$$+\int_0^{\tilde{h}}\int_{x_1}^{x_2} a\big(k(s_\varepsilon) - k(s_0)\big)\,\psi_\delta(\ldots)_x\,dx\,dt \leqq -\int_0^{\tilde{h}}\int_{x_1}^{x_2} a\big(u^\varepsilon_{0x} + k(s_0)\big)\,\psi_\delta(\ldots)_x\,dx\,dt$$

so finally taking $\delta \to 0$ as before

$$\frac{1}{\tilde{h}}\int_0^{\tilde{h}}\int_{x_1}^{x_2}|s_\varepsilon(t) - s_0|\,dx\,dt \leqq \tilde{h}\,\|a\big(u^\varepsilon_{0x} + K(s_0)\big)\|_{BV(D)} .$$

which gives the desired estimate.

After the estimate on $s_{\varepsilon t}$ we proceed to the proof of the estimate on $k(s_\varepsilon)_x$. In (2.3h) we can use $u^\varepsilon_h \pm \psi_\delta\big((\max(u^\varepsilon_h, \hat{u}+\alpha))_x\big)\eta$ as a test function, where η is a cut off and $\alpha > 0$; because by the assumption on $k\circ S_\varepsilon$, the second derivative $(au^\varepsilon_{hx})_x$ is in $L^p(\{u^\varepsilon_h > \hat{u}+\alpha\})$ with some $p > 1$.

Letting first δ then α then h tend to zero we get the estimate

$$\int_0^{T}\int_{x_1}^{x_2} s_{\varepsilon t}\,\mathrm{sign}(u_{\varepsilon x})\,\eta\,dx\,dt - \int_0^{T}\int_{x_1}^{x_2}|au_{\varepsilon x}|_x\,\eta\,dx\,dt -$$
$$-\int_0^{T}\int_{x_1}^{x_2} a\big(k(s_\varepsilon)\big)_x\,\mathrm{sign}(u_{\varepsilon x})\,\eta\,dx\,dt \geqq -\ \|a\|_{BV(D)}\,T\cdot\sup(k) .$$

So we get in the end

$$\int_0^T\int_{x_1}^{x_2} a|k(s_\varepsilon)_x|\eta\,dx\,dt \leqq \int_0^T\int_{x_1}^{x_2}|s_{\varepsilon t}|\,dx\,dt + \sup|\eta_x|\int_0^T\int_{x_1}^{x_2} a\big(|u_x|^2+1\big)\,dx\,dt + \sup(k)\,T\|a\|_{BV(D)}. \quad ■$$

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