On nonstrict means

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Abstract

We investigate a class of aggregation operators : the well-known mean values. Kolmogoroff [6] and Nagumo [8] established a fundamental result about mean values. In their definition a mean value M is a sequence of functions:

$$M^{(1)}(x_1) = x_1, M^{(2)}(x_1, x_2), \ldots, M^{(m)}(x_1, \ldots, x_m), \ldots,$$

each function of this sequence having to satisfy the following conditions: $M^{(m)}(x_1, \ldots, x_m)$ must be a function $[a,b]^m \to [a,b]$ which is continuous, symmetric, strictly increasing on each argument, and idempotent, that is $M^{(m)}(x, \ldots, x) = x$. The elements of this sequence are linked by a *pseudo-associativity* called the *decomposability* property by several authors (see [3, Chapter 5]):

$$M^{(m)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) = M^{(m)}(M_k, \ldots, M_k, x_{k+1}, \ldots, x_m)$$

for all $k \in \{1, \ldots, m\}$, with $M_k = M^{(m)}(x_1, \ldots, x_k)$.

The result of Kolmogoroff-Nagumo is the following one :

Theorem 1 *An aggregation operator $M : \bigcup_{m=1}^{\infty} [a,b]^m \to [a,b]$ is continuous, symmetric, strictly increasing, idempotent and decomposable iff for all $m \in N_0$,*

$$M^{(m)}(x_1, \ldots, x_m) = f^{-1}\left[\frac{1}{m}\sum_i f(x_i)\right]$$

(generalized mean) where f is any continuous strictly monotonic function on $[a,b]$.

On the other hand, Aczél [1] (see also [2]) proved that a function $M(x,y) : [a,b]^2 \to [a,b]$ of two variables is continuous, symmetric, strictly increasing on each argument, idempotent and fulfils the *bisymmetry equation*

$$M[M(x_{11}, x_{12}), M(x_{21}, x_{22})] = M[M(x_{11}, x_{21}), M(x_{12}, x_{22})]$$

if and only if

$$M(x,y) = f^{-1}\left[\frac{f(x)+f(y)}{2}\right]$$

(generalized mean) where f is any continuous strictly monotonic function on $[a,b]$.

As we can see, the decomposability and bisymmetry properties play similar roles and as Horváth [5] showed, the result of Aczél is a trivial consequence of the one of Kolmogoroff-Nagumo.

Our purpose is to describe the family of continuous, symmetric, increasing, idempotent and decomposable aggregation operators thus relaxing strict increasing monotonicity of M into weak increasing monotonicity.

For example, letting $\mathcal{D}_{a,b,\theta}$ be the set of aggregation operators $M : \bigcup_{m=1}^{\infty}[a,b]^m \to [a,b]$ which are continuous, symmetric, increasing, idempotent, decomposable and such that $M(a,b) = \theta$, θ being a given number in $[a,b]$, we have the following result:

Theorem 2 *An aggregation operator $M : \bigcup_{m=1}^{\infty}[a,b]^m \to [a,b]$ is continuous, symmetric, increasing, idempotent and decomposable iff there exists two numbers α and β fulfilling $a \leq \alpha \leq \beta \leq b$, such that, for all $m \in N_0$,*

$$\begin{array}{lll} i) & M(x_1,\ldots,x_m) = M_{a,\alpha,\alpha}(x_1,\ldots,x_m) & \text{if } \max_i x_i \in [a,\alpha]; \\ ii) & M(x_1,\ldots,x_m) = M_{\beta,b,\beta}(x_1,\ldots,x_m) & \text{if } \min_i x_i \in [\beta,b]; \\ iii) & M(x_1,\ldots,x_m) = f^{-1}\left[\frac{1}{m}\sum_i f[median(\alpha,x_i,\beta)]\right] & \text{otherwise,} \end{array}$$

where $M_{a,\alpha,\alpha} \in \mathcal{D}_{a,\alpha,\alpha}$, $M_{\beta,b,\beta} \in \mathcal{D}_{\beta,b,\beta}$ and where f is any continuous strictly monotonic function on $[\alpha,\beta]$.

The sets $\mathcal{D}_{a,\alpha,\alpha}$ and $\mathcal{D}_{\beta,b,\beta}$ can also be described.

We also show that the linkage between decomposability and bisymmetry noted above still holds if we relax strict increasing monotonicity.

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