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ON NORMAL (f, g, u, v, λ)-STRUCTURES ON SUBMANIFOLDS OF CODIMENSION 2 IN AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

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§ 0. Introduction.

It is well known that a hypersurface of an almost Hermitian manifold admits an almost contact metric structure naturally induced on it.

The study of hypersurfaces of a Euclidean space and of a Kählerian manifold on which the induced almost contact metric structure satisfies certain conditions has been started by one of the present authors [4, 5].

On the other hand Blair [1, 2], Goldberg [3], Ludden [1, 2], Yamaguchi [8] and the present authors [6, 9] started the study of hypersurface of an almost contact manifold and of submanifolds of codimension 2 of an almost complex manifold.

These submanifolds admit, under certain conditions, what we call (f, g, u, v, λ)-structure. An even-dimensional sphere of codimension 2 of an even-dimensional Euclidean space is a typical example of a manifold which admits this kind of structure.

In a previous paper [9], we have studied the (f, g, u, v, λ)-structure and given characterizations of even-dimensional sphere.

In the present paper, we study submanifolds of codimension 2 in an even-dimensional Euclidean space which admit a normal (f, g, u, v, λ)-structure.

In § 1, we consider submanifolds of codimension 2 of an even-dimensional Euclidean space regarded as a flat Kählerian manifold. In the next section, we deal with (f, g, u, v, λ)-structure induced on a submanifold of codimension 2 of an even-dimensional Euclidean space.

In § 3, we find differential equations which f, g, u, v and λ satisfy. § 4 is devoted to the study of relations between the structure equations of the submanifold and the induced (f, g, u, v, λ)-structure.

In § 5 we prove a series of lemmas which are valid for normal (f, g, u, v, λ)-structures and in § 6 we study properties of the mean curvature vector of the submanifold with normal (f, g, u, v, λ)-structure.

In the last § 7, we study hypersurfaces of an odd-dimensional Euclidean space and determine all the hypersurfaces admitting a normal (f, g, u, v, λ)-structure.

Our main theorem appears at the end of § 7.

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§1. Submanifolds of codimension 2 of an even-dimensional Euclidean space.

Let E be a $(2n+2)$-dimensional Euclidean space and denote by X the position vector representing a point of E. Since E is even-dimensional, E can be regarded as a flat Hermitian manifold, and hence there exists a tensor field F of type $(1,1)$ with constant components such that

$$F^2=-I \tag{1.1}$$

and

$$(FX)\cdot(FY)=X\cdot Y \tag{1.2}$$

for any vectors X and Y, where I denotes the identity transformation and a dot the inner product in the Euclidean space E.

We consider an orientable submanifold M of codimension 2 of E covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and throughout the paper the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$.

We put

$$X_i=\partial_i X, \qquad \partial_i=\partial/\partial x^i, \tag{1.3}$$

then X_i are $2n$ linearly independent vector fields tangent to the submanifold M and

$$g_{ji}=X_j\cdot X_i \tag{1.4}$$

give components of the fundamental metric tensor of M regarded as a Riemannian manifold referred to the coordinate system $\{U; x^h\}$. We denote by C and D two mutually orthogonal unit normals to the submanifold M such that X_i, C, D form the positive orientation of E. Then we have

$$\begin{aligned} &X_i\cdot C=0, \qquad X_i\cdot D=0,\\ &C\cdot C=1, \qquad C\cdot D=0, \qquad D\cdot D=1. \end{aligned} \tag{1.5}$$

Now the vectors X_i, C and D being linearly independent, the transforms FX_i of X_i by F can be expressed as

$$FX_i=f_i{}^h X_h+u_iC+v_iD,$$

where $f_i{}^h$ are components of a tensor field of type (1.1) and u_i and v_i are components of 1-forms in M.

As to the transform FC of C by F, we have

$$(FC)\cdot X_i=(F^2C)\cdot(FX_i)=-C\cdot(FX_i)=-u_i,$$

$$(FC)\cdot C=(F^2C)\cdot(FC)=-C\cdot(FC)=0$$

by virtue of (1. 2) and consequently

$$FC=-u^h X_h+\lambda D,$$

where

$$u^h=u_i g^{ih},$$

g^{ih} being contravariant components of the metric tensor and λ a function of M.

As to the transform FD of D by F, we have

$$(FD)\cdot X_i=(F^2D)\cdot(FX_i)=-D\cdot(FX_i)=-v_i,$$

$$(FD)\cdot C=(F^2D)\cdot(FC)=-D\cdot(FC)=-\lambda,$$

$$(FD)\cdot D=(F^2D)\cdot(FD)=-D\cdot(FD)=0,$$

and consequently

$$FD=-v^h X_h-\lambda C,$$

where

$$v^h=v_i g^{ih}.$$

Thus we have

$$\begin{aligned} FX_i&=f_i{}^h X_h+u_i C+v_i D,\\ FC&=-u^h X_h+\lambda D,\\ FD&=-v^h X_h-\lambda C. \end{aligned} \tag{1. 6}$$

We note here that the 1-forms u_i and v_i depend on the choice of unit normals C and D but the function $\lambda=(FC)\cdot D$ does not depend on the choice of C and D. In fact, if we choose another set of mutually orthogonal unit normals C' and D', we have

$$C'=C\cos\theta-D\sin\theta,$$

$$D'=C\sin\theta+D\cos\theta,$$

and consequently

$$\begin{aligned}(FC')\cdot D'&=(FC\cos\theta-FD\sin\theta)(C\sin\theta+D\cos\theta)\\ &=(\lambda D\cos\theta+\lambda C\sin\theta)(C\sin\theta+D\cos\theta)\\ &=\lambda.\end{aligned}$$

§ 2. (f, g, u, v, λ)-structure on a submanifold of codimension 2.

Now applying the operator F to the first equation of (1. 6) and taking account of (1. 6), we find

$$F^2 X_i = f_i{}^t F X_t + u_i F C + v_i F D,$$

$$-X_i = f_i{}^t (f_t{}^h X_h + u_t C + v_t D) + u_i(-u^h X_h + \lambda D) + v_i(-v^h X_h - \lambda C),$$

from which

$$f_i{}^t f_t{}^h = -\delta_i^h + u_i u^h + v_i v^h,$$

$$u_t f_i{}^t = \lambda v_i, \qquad v_t f_i{}^t = -\lambda u_i.$$

Applying the operator F to the second equation of (1. 6), we find

$$F^2 C = -u^i F X_i + \lambda F D,$$

$$-C = -u^i (f_i{}^h X_h + u_i C + v_i D) + \lambda(-v^h X_h - \lambda C),$$

from which

$$f_i{}^h u^i = -\lambda v^h, \qquad u_i u^i = 1 - \lambda^2, \qquad v_i u^i = 0.$$

Applying also the operator F to the last equation of (1. 6), we find

$$F^2 D = -v^i F X_i - \lambda F C,$$

$$-D = -v^i (f_i{}^h X_h + u_i C + v_i D) - \lambda(-u^h X_h + \lambda D),$$

from which

$$f_i{}^h v^i = \lambda u^h, \qquad u_i v^i = 0, \qquad v_i v^i = 1 - \lambda^2.$$

Thus summing up, we have

$$\begin{aligned} &f_i{}^t f_t{}^h = -\delta_i^h + u_i u^h + v_i v^h, \\ &u_t t_i{}^t = \lambda v_i, \qquad v_t f_i{}^t = -\lambda u_i, \\ &f_i{}^h u^i = -\lambda v^h, \qquad f_i{}^h v^i = \lambda u^h, \\ &u_i u^i = 1 - \lambda^2, \qquad u_i v^i = 0, \qquad v_i v^i = 1 - \lambda^2. \end{aligned} \tag{2. 1}$$

Now, substituting the first equation of (1. 6) into

$$(F X_j) \cdot (F X_i) = X_j \cdot X_i,$$

we find

$$(f_j{}^t X_t + u_j C + v_j D)(f_i{}^s X_s + u_i C + v_i D) = g_{ji},$$

that is,

$$f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i. \tag{2. 2}$$

If we put

$$f_{it} = f_i{}^s g_{ts}, \tag{2. 3}$$

we find, from the first equation of (2. 1),

$$f_j{}^t f_{ti} = -g_{ji} + u_j u_i + v_j v_i,$$

and from (2. 2)

$$f_j{}^t f_{it} = g_{ji} - u_j u_i - v_j v_i.$$

From these two equations, we find

$$f_j{}^t (f_{ti} + f_{it}) = 0. \tag{2. 4}$$

Transvecting (2. 4) with $f_s{}^j$ and taking account of the first equation of (2. 1), we find

$$(-\delta_s^t + u_s u^t + v_s v^t)(f_{ti} + f_{it}) = 0,$$

from which

$$f_{si} + f_{is} = 0, \tag{2. 5}$$

because of the second and the third equations of (2. 1). Thus the tensor f_{it} defined by (2. 3) is skew-symmetric.

We call an (f, g, u, v, λ)-structure the set of f, g, u, v, and λ satisfying (2. 1) and (2. 2).

§ 3. Differential equations which f, g, u, v and λ satisfy.

We denote by $\{{}_j{}^h{}_i\}$ the Christoffel symbols formed with g_{ji} and by ∇_i the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$. Then the equations of Gauss of M are

$$\nabla_j X_i = \partial_j X_i - \left\{ {h \atop j\ i} \right\} X_h = h_{ji} C + k_{ji} D, \tag{3. 1}$$

where

$$h_{ji} = h_{ij} \quad \text{and} \quad k_{ji} = k_{ij}$$

are the second fundamental tensors of M with respect to the normals C and D respectively.

The equations of Weingarten are

(3. 2) $$\nabla_i C=\partial_i C=-h_i{}^h X_h+l_i D,$$

and

(3. 3) $$\nabla_i D=\partial_i D=-k_i{}^h X_h-l_i C,$$

where

(3. 4) $$h_i{}^h=h_{it}g^{th}, \qquad k_i{}^h=k_{it}g^{th}$$

and l_i are components of the third fundamental tensor with respect to the normals C and D. The l_i define the connection induced in the normal bundle of M.

Now applying the operator ∇_j of covariant differentiation to the first equation of (1. 6) and taking account of $\nabla_j F=0$, we find

$$F\nabla_j X_i=(\nabla_j f_i{}^h)X_h+f_i{}^h\nabla_j X_h+(\nabla_j u_i)C+u_i(\nabla_j C)+(\nabla_j v_i)D+v_i(\nabla_j D),$$

$$F(h_{ji}C+k_{ji}D)=(\nabla_j f_i{}^h)X_h+f_i{}^t(\nabla_j X_t)+(\nabla_j u_i)C+u_i(\nabla_j C)+(\nabla_j v_i)D+v_i(\nabla_j D),$$

or

$$h_{ji}(-u^h X_h+\lambda D)+k_{ji}(-v^h X_h-\lambda C)$$
$$=(\nabla_j f_i{}^h)X_h+f_i{}^t(h_{jt}C+k_{jt}D)+(\nabla_j u_i)C+u_i(-h_j{}^h X_h+l_j D)+(\nabla_j v_i)D+v_i(-k_j{}^h X_h-l_j C),$$

from which

$$\nabla_j f_i{}^h=-h_{ji}u^h+h_j{}^h u_i-k_{ji}v^h+k_j{}^h v_i,$$

$$\nabla_j u_i=-h_{jt}f_i{}^t-\lambda k_{ji}+l_j v_i,$$

$$\nabla_j v_i=-k_{jt}f_i{}^t+\lambda h_{ji}-l_j u_i.$$

Applying the operator ∇_i to the second equation of (1. 6), we find

$$F\nabla_i C=-(\nabla_i u^h)X_h-u^h\nabla_i X_h+(\nabla_i\lambda)D+\lambda(\nabla_i D),$$

$$F(-h_i{}^h X_h+l_i D)=-(\nabla_i u^h)X_h-u^h(\nabla_i X_h)+(\nabla_i\lambda)D+\lambda(\nabla_i D),$$

or

$$-h_i{}^t(f_t{}^h X_h+u_t C+v_t D)+l_i(-v^h X_h-\lambda C)$$
$$=-(\nabla_i u^h)X_h-u^t(h_{it}C+k_{it}D)+(\nabla_i\lambda)D+\lambda(-k_i{}^h X_h-l_i C),$$

from which

$$\nabla_i u^h=h_i{}^t f_t{}^h-\lambda k_i{}^h+l_i v^h,$$

$$\nabla_i\lambda=-h_i{}^t v_t+k_{it}u^t.$$

Applying the operator ∇_i to the last equation of (1. 6), we find

$$F\nabla_i D = -(\nabla_i v^h)X_h - v^t(\nabla_i X_t) - (\nabla_i \lambda)C - \lambda(\nabla_i C),$$

$$F(-k_i{}^t X_t - l_i C) = -(\nabla_i v^h)X_h - v^t(\nabla_i X_t) - (\nabla_i \lambda)C - \lambda(\nabla_i C),$$

or

$$-k_i{}^t(f_t{}^h X_h + u_t C + v_t D) - l_i(-u^h X_h + \lambda D)$$
$$= -(\nabla_i v^h)X_h - v^t(h_{it}C + k_{it}D) - (\nabla_i \lambda)C - \lambda(-h_i{}^h X_h + l_i D),$$

from which

$$\nabla_i v^h = k_i{}^t f_t{}^h + \lambda h_i{}^h - l_i u^h,$$
$$\nabla_i \lambda = -h_{it} v^t + k_i{}^t u_t.$$

Thus, summing up, we have

$$\begin{aligned} &\nabla_j f_i{}^h = -h_{ji} u^h + h_j{}^h u_i - k_{ji} v^h + k_j{}^h v_i, \\ &\nabla_j u_i = -h_{jt} f_i{}^t - \lambda k_{ji} + l_j v_i, \\ &\nabla_j v_i = -k_{jt} f_i{}^t + \lambda h_{ji} - l_j u_i, \\ &\nabla_i \lambda = -h_{it} v^t + k_{it} u^t. \end{aligned} \tag{3. 5}$$

§ 4. Normal (f, g, u, v, λ)-structure.

We now compute

$$S_{ji}{}^h = N_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j)u^h + (\nabla_j v_i - \nabla_i v_j)v^h, \tag{4. 1}$$

where

$$N_{ji}{}^h = f_j{}^t \nabla_t f_i{}^h - f_i{}^t \nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h \tag{4. 2}$$

is the Nijenhuis tensor formed with $f_i{}^h$.

Substituting (3. 5) into (4. 1), we find

$$\begin{aligned} S_{ji}{}^h = &f_j{}^t(-h_{ti} u^h + h_t{}^h u_i - k_{ti} v^h + k_t{}^h v_i) \\ &- f_i{}^t(-h_{tj} u^h + h_t{}^h u_j - k_{tj} v^h + k_t{}^h v_j) \\ &- (h_j{}^t u_i - h_i{}^t u_j + k_j{}^t v_i - k_i{}^t v_j) f_t{}^h \\ &- (h_{jt} f_i{}^t - h_{it} f_j{}^t - l_j v_i + l_i v_j) u^h \\ &- (k_{jt} f_i{}^t - k_{it} f_j{}^t + l_j u_i - l_i u_j) v^h, \end{aligned}$$

that is,

$$
\begin{aligned}
S_{ji}{}^h = & (f_j{}^t h_t{}^h - h_j{}^t f_t{}^h) u_i - (f_i{}^t h_t{}^h - h_i{}^t f_t{}^h) u_j \\
& + (f_j{}^t k_t{}^h - k_j{}^t f_t{}^h) v_i - (f_i{}^t k_t{}^h - k_i{}^t f_t{}^h) v_j \\
& + (l_j v_i - l_i v_j) u^h - (l_j u_i - l_i u_j) v^h.
\end{aligned} \tag{4. 3}
$$

When the tensor $S_{ji}{}^h$ vanishes identically, the (f, g, u, v, λ)-structure is said to be *normal.*

Now the equations of Gauss of the submanifold M are

$$
K_{kji}{}^h = h_k{}^h h_{ji} - h_j{}^h h_{ki} + k_k{}^h k_{ji} - k_j{}^h k_{ki}, \tag{4. 4}
$$

where

$$
K_{kji}{}^h = \partial_k \left\{ {h \atop j\ i} \right\} - \partial_j \left\{ {h \atop k\ i} \right\} + \left\{ {h \atop k\ t} \right\} \left\{ {t \atop j\ i} \right\} - \left\{ {h \atop j\ t} \right\} \left\{ {t \atop k\ i} \right\}
$$

are components of the curvature tensor of M, the equations of Codazzi are

$$
\begin{aligned}
& \nabla_k h_{ji} - \nabla_j h_{ki} - l_k k_{ji} + l_j k_{ki} = 0, \\
& \nabla_k k_{ji} - \nabla_j k_{ki} + l_k h_{ji} - l_j h_{ki} = 0,
\end{aligned} \tag{4. 5}
$$

and the equations of Ricci are

$$
\nabla_j l_i - \nabla_i l_j + h_j{}^t k_{ti} - h_i{}^t k_{tj} = 0. \tag{4. 6}
$$

In the sequel, we assume that the connection induced in the normal bundle of M has no curvature, that is, we can choose C and D in such a way that we have $l_i = 0$, and we say in this case that the connection induced in the normal bundle is *trivial.*

In this case, we have, from (4. 5),

$$
\nabla_k h_{ji} - \nabla_j h_{ki} = 0, \qquad \nabla_k k_{ji} - \nabla_j k_{ki} = 0, \tag{4. 7}
$$

which say that the tensors $\nabla_k h_{ji}$ and $\nabla_k k_{ji}$ are both symmetric in all the three indices, and, from (4. 6),

$$
h_j{}^t k_{ti} - h_i{}^t k_{tj} = 0, \tag{4. 8}
$$

or

$$
h_j{}^t k_t{}^i - k_j{}^t h_t{}^i = 0, \tag{4. 9}
$$

which says that $h_i{}^h$ and $k_i{}^h$ are commutative as linear transformations in the tangent space of M.

Now for the normal (f, g, u, v, λ)-structure of M such that the connection induced in the normal bundle is trivial, we have, from (4. 3),

$$
(f_j{}^t h_t{}^h - h_j{}^t f_t{}^h) u_i - (f_i{}^t h_t{}^h - h_i{}^t f_t{}^h) u_j + (f_j{}^t k_t{}^h - k_j{}^t f_t{}^h) v_i - (f_i{}^t k_t{}^h - k_i{}^t f_t{}^h) v_j = 0. \tag{4. 10}
$$

Since 1-forms u_i, v_i and l_i depend on the choice of unit normals C and D, the tensor $S_{ji}{}^h$ also depends on the choice of the normals. However, the left hand side of (4.10) does not depend on the choice of C and D. In fact, if we choose another set of mutually orthogonal unit normals C' and D', we have

$$C'=C\cos\theta-D\sin\theta,\qquad D'=C\sin\theta+D\cos\theta. \tag{4.11}$$

Then the second fundamental tensors h_{ji}' and k_{ji}' with respect to C' and D' are defined by

$$\nabla_j X_i=h_{ji}'C'+k_{ji}'D'. \tag{4.12}$$

Substituting (4.11) into (4.12) and comparing the resulting equation with (3.1), we get

$$h'_{ji}=h_{ji}\cos\theta-k_{ji}\sin\theta,\qquad k'_{ji}=h_{ji}\sin\theta+k_{ji}\cos\theta. \tag{4.13}$$

On the other hand (4.11) and the first equation of (1.16) show that

$$u_i'=u_i\cos\theta-v_i\sin\theta,\qquad v_i'=u_i\sin\theta+v_i\cos\theta. \tag{4.14}$$

Consequently we have

$$(f_j{}^t h'_t{}^h-h'_j{}^t f_t{}^h)u_i'-(f_i{}^t h'_t{}^h-h'_i{}^t f_t{}^h)u'_j+(f_j{}^t k'_t{}^h-k'_j{}^t f_t{}^h)v_i'-(f_i{}^t k'_t{}^h-k'_i{}^t f_t{}^h)v'_j$$
$$=(f_j{}^t h_t{}^h-h_j{}^t f_t{}^h)u_i-(f_i{}^t h_t{}^h-h_i{}^t f_t{}^h)u_j+(f_j{}^t k_t{}^h-k_j{}^t f_t{}^h)v_i-(f_i{}^t k_t{}^h-k_i{}^t f_t{}^h)v_j.$$

This shows that the conditions imposed on M are of intrinsic character.

§ 5. Some lemmas on normal (f, g, u, v, λ)-structure.

As we have seen in § 4, the condition imposed on the submanifold M does not depend on the choice of unit normals C and D.

The main purpose of the following discussions is to determine submanifolds of codimension 2 of E which satisfy (4.10).

Assuming that the function $\lambda(1-\lambda^2)$ does not vanish almost everywhere on M, we prove following series of lemmas.

LEMMA 5.1. *For the normal (f, g, u, v, λ)-structure of M such that the connection induced in the normal bundle is trivial, we have*

$$f_j{}^t h_t{}^h-h_j{}^t f_t{}^h=au_ju^h+b(u_jv^h+v_ju^h)+cv_jv^h \tag{5.1}$$

and

$$f_j{}^t k_t{}^h - k_j{}^t f_t{}^h = b u_j u^h + c(u_j v^h + v_j u^h) + d v_j v^h, \tag{5.2}$$

a, b, c, and d being scalars of M.

Proof. We put

$$P_j{}^h = f_j{}^t h_t{}^h - h_j{}^t f_t{}^h, \qquad Q_j{}^h = f_j{}^t k_t{}^h - k_j{}^t f_t{}^h \tag{5.3}$$

and note that

$$P_{ji} = f_j{}^t h_{ti} + f_i{}^t h_{tj}, \qquad Q_{ji} = f_j{}^t k_{ti} + f_i{}^t k_{tj} \tag{5.4}$$

are both symmetric with respect to j and i.

Then equation (4.10) can be written as

$$P_j{}^h u_i - P_i{}^h u_j + Q_j{}^h v_i - Q_i{}^h v_j = 0, \tag{5.5}$$

from which, by transvection with u^i,

$$P_j{}^h(1-\lambda^2) - (P_i{}^h u^i) u_j - (Q_i{}^h u^i) v_j = 0$$

by virtue of (2.1), that is, $P_j{}^h$ is of the form

$$P_j{}^h = u_j P^h + v_j Q^h, \tag{5.6}$$

and consequently P_{ji} is of the form

$$P_{ji} = u_j P_i + v_j Q_i. \tag{5.7}$$

Since P_{ji} is symmetric, we have, from (5.7),

$$u_j P_i - u_i P_j + v_j Q_i - v_i Q_j = 0, \tag{5.8}$$

from which we see that P_i must be of the form

$$P_i = a u_i + b v_i \tag{5.9}$$

and Q_i of the form

$$Q_i = d u_i + c v_i. \tag{5.10}$$

Substituting (5.9) and (5.10) into (5.8), we find

$$u_j(a u_i + b v_i) - u_i(a u_j + b v_j) + v_j(d u_i + c v_i) - v_i(d u_j + c v_j) = 0,$$

or

$$(b-d)(u_j v_i - u_i v_j) = 0.$$

from which, u_i and v_i being orthogonal to each other, we have

$$b=d,$$

and consequently we have

$$P_i=au_i+bv_i, \qquad Q_i=bu_i+cv_i,$$

or

$$P^h=au^h+bv^h, \qquad Q^h=bu^h+cv^h.$$

Substituting these into (5. 6), we obtain

$$P_j{}^h=au_ju^h+b(u_jv^h+v_ju^h)+cv_jv^h. \tag{5. 11}$$

Similarly, we have

$$Q_j{}^h=\bar{a}u_ju^h+\bar{b}(u_jv^h+v_ju^h)+\bar{d}v_jv^h. \tag{5. 12}$$

Substituting these into (5. 5), we find

$$(u_jv_i-u_iv_j)\{(b-\bar{a})u^h+(c-\bar{b})v^h\}=0,$$

from which

$$\bar{a}=b, \qquad \bar{b}=c. \tag{5. 13}$$

Equations (5. 11), (5. 12) and (5. 13) prove the lemma.

LEMMA 5. 2. *For the normal (f, g, u, v, λ)-structure of M such that the connection induced in the normal bundle is trivial, we have*

$$\begin{aligned} h_i{}^hu^i&=\alpha u^h+\beta v^h, & h_i{}^hv^i&=\beta u^h+\gamma v^h, \\ k_i{}^hu^i&=\bar{\alpha}u^h+\bar{\beta}v^h, & k_i{}^hv^i&=\bar{\beta}u^h+\bar{\gamma}v^h. \end{aligned} \tag{5. 14}$$

Proof. Transvecting (5. 1) with $f_k{}^j$, we find

$$(-\delta_k^t+u_ku^t+v_kv^t)h_t{}^h-f_k{}^jh_j{}^tf_t{}^h=a\lambda v_ku^h+b\lambda(v_kv^h-u_ku^h)-c\lambda u_kv^h$$

by virtue of (2. 1), or

$$-h_{kh}+u_k(u^th_{th})+v_k(v^th_{th})+h_{ts}f_k{}^tf_h{}^s=a\lambda v_ku_h+b\lambda(v_kv_h-u_ku_h)-c\lambda u_kv_h,$$

from which, taking the skew-symmetric part,

$$u_k(u^th_{th})-u_h(u^th_{tk})+v_k(v^th_{th})-v_h(v^th_{tk})=-\lambda(a+c)(u_kv_h-u_hv_k).$$

This equation shows that u^th_{th} and v^th_{th} should be respectively of the form

$$u^th_{th}=\alpha u_h+\beta v_h, \qquad v^th_{th}=\beta u_h+\gamma v_h,$$

that is,

$$h_i{}^h u^i = \alpha u^h + \beta v^h, \qquad h_i{}^h v^i = \beta u^h + \gamma v^h.$$

The other two equations will be proved in a similar way.
We note here that α, β, γ and $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are given by

$$(1-\lambda^2)\alpha = h_{ji} u^j u^i, \qquad (1-\lambda^2)\beta = h_{ji} u^j v^i, \qquad (1-\lambda^2)\gamma = h_{ji} v^j v^i,$$

$$(1-\lambda^2)\bar{\alpha} = k_{ji} u^j u^i, \qquad (1-\lambda^2)\bar{\beta} = k_{ji} u^j v^i, \qquad (1-\lambda^2)\bar{\gamma} = k_{ji} v^j v^i.$$

LEMMA 5. 3. *In Lemma* 5. 2, *we have*

(5. 15) $$2\beta = \bar{\alpha} - \bar{\gamma}, \qquad 2\bar{\beta} = \gamma - \alpha.$$

Proof. In (4. 10), we contract with respect ot h and i, then we obtain

(5. 16) $$f_j{}^t (h_t{}^i u_i) - h_j{}^t (f_t{}^i u_i) + f_j{}^t (k_t{}^i v_i) - k_j{}^t (f_t{}^i v_i) = 0$$

because of

$$f_i{}^t h_t{}^i = f^{it} h_{ti} = 0, \qquad f_i{}^t k_t{}^i = f^{it} k_{ti} = 0.$$

Substituting (2. 1) and (5. 14) into (5. 16), we find

$$f_j{}^t (\alpha u_t + \beta v_t) - \lambda h_j{}^t v_t + f_j{}^t (\bar{\beta} u_t + \bar{\gamma} v_t) + \lambda k_j{}^t u_t = 0,$$

and consequently

$$\alpha\lambda v_j - \beta\lambda u_j - \lambda(\beta u_j + \gamma v_j) + \bar{\beta}\lambda v_j - \bar{\gamma}\lambda u_j + \lambda(\bar{\alpha} u_j + \bar{\beta} v_j) = 0,$$

or

$$-\lambda(2\beta + \bar{\gamma} - \bar{\alpha}) u_j + \lambda(2\bar{\beta} + \alpha - \gamma) v_j = 0,$$

from which

$$2\beta = \bar{\alpha} - \bar{\gamma}, \qquad 2\bar{\beta} = \gamma - \alpha.$$

LEMMA 5. 4. *In Lemma* 5. 2, *we have*

(5. 17) $$\beta = 0, \qquad \bar{\beta} = 0$$

and consequently

$$\alpha = \gamma, \qquad \bar{\alpha} = \bar{\gamma}.$$

Proof. Transvecting the first equation

$$h_i{}^h u^i = \alpha u^h + \beta v^h$$

of (5. 14) with $k_h{}^t$ and using (5. 14), we obtain

$$k_h{}^t h_i{}^h u^i = \alpha(\bar{\alpha}u^t + \bar{\beta}v^t) + \beta(\bar{\beta}u^t + \bar{\gamma}v^t),$$
$$= (\alpha\bar{\alpha} + \beta\bar{\beta})u^t + (\alpha\bar{\beta} + \beta\bar{\gamma})v^t.$$

Also transvecting the third equation

$$k_i{}^h u^i = \bar{\alpha}u^h + \bar{\beta}v^h$$

of (5. 14) with $h_h{}^t$ and using (5. 14), we obtain

$$h_h{}^t k_i{}^h u^i = \bar{\alpha}(\alpha u^t + \beta v^t) + \bar{\beta}(\beta u^t + \gamma v^t)$$
$$= (\alpha\bar{\alpha} + \beta\bar{\beta})u^t + (\bar{\alpha}\beta + \bar{\beta}\gamma)v^t.$$

Thus, $h_i{}^h$ and $k_i{}^h$ being commutative,

$$\alpha\bar{\beta} + \beta\bar{\gamma} = \bar{\alpha}\beta + \bar{\beta}\gamma,$$

or

$$(\gamma - \alpha)\bar{\beta} + (\bar{\alpha} - \bar{\gamma})\beta = 0,$$

or using (5. 15),

$$2\bar{\beta}^2 + 2\beta^2 = 0,$$

from which the lemma follows.

Combining Lemma 5. 2 and Lemma 5. 4, we have

LEMMA 5. 5. *For the normal (f, g, u, v, λ)-structure of M such that the connection induced in the normal bundle is trivial, we have*

$$h_i{}^h u^i = \alpha u^h, \qquad h_i{}^h v^i = \alpha v^h,$$
$$k_i{}^h u^i = \bar{\alpha}u^h, \qquad k_i{}^h v^i = \bar{\alpha}v^h. \tag{5. 18}$$

We also have

LEMMA 5. 6. *For the normal (f, g, u, v, λ)-structure of M such that the connection induced in the normal bundle is trivial, $h_i{}^h$ and $k_i{}^h$ commute with $f_i{}^h$.*

Proof. Transvecting (4. 10) with u^i and using (2. 1) and (5. 18), we obtain

$$(f_j{}^t h_t{}^h - h_j{}^t f_t{}^h)(1-\lambda^2) + (\lambda h_t{}^h v^t + \alpha f_t{}^h u^t)u_j + (\lambda k_t{}^h v^t + \bar{\alpha} f_t{}^h u^t)v_j = 0,$$

or

$$(f_j{}^t h_t{}^h - h_j{}^t f_t{}^h)(1-\lambda^2) = 0,$$

that is,

$$f_j{}^t h_t{}^h = h_j{}^t f_t{}^h.$$

Similarly we can prove

$$f_j{}^t k_t{}^h = k_j{}^t f_t{}^h.$$

LEMMA 5.7. *For the normal (f, g, u, v, λ)-structure of M such that the connection induced in the normal bundle is trivial, we have*

$$h_i{}^t h_t{}^h = \alpha h_i{}^h, \qquad k_i{}^t k_t{}^h = \bar{\alpha} k_i{}^h. \tag{5.19}$$

Proof. Differentiating

$$h_{ji} u^i = \alpha u_j$$

covariantly, we obtain

$$(\nabla_k h_{ji}) u^i + h_{ji}(\nabla_k u^i) = (\nabla_k \alpha) u_j + \alpha(\nabla_k u_j),$$

or using (3.5)

$$(\nabla_k h_{ji}) u^i + h_{ji}(h_k{}^t f_t{}^i - \lambda k_k{}^i) = (\nabla_k \alpha) u_j + \alpha(-h_{kt} f_j{}^t - \lambda k_{kj})$$

and consequently taking skew-symmetric part

$$h_{ji} h_k{}^t f_t{}^i - h_{ki} h_j{}^t f_t{}^i = (\nabla_k \alpha) u_j - (\nabla_j \alpha) u_k - \alpha h_{kt} f_j{}^t + \alpha h_{jt} f_k{}^t$$

because of

$$\nabla_k h_{ji} - \nabla_j h_{ki} = 0, \qquad h_{ji} k_k{}^i = h_{ki} k_j{}^i.$$

But $h_k{}^t$ and $f_t{}^i$ commute and consequently

$$2 h_{ji} h_k{}^t f_t{}^i = (\nabla_k \alpha) u_j - (\nabla_j \alpha) u_k - 2\alpha h_{kt} f_j{}^t, \tag{5.20}$$

from which, transvecting with u^k,

$$2 h_{ji} \alpha u^t f_t{}^i = (u^k \nabla_k \alpha) u_j - (\nabla_j \alpha)(1-\lambda^2) - 2\alpha^2 u_t f_j{}^t,$$

or

$$-2\alpha\lambda h_{ji} v^i = (u^k \nabla_k \alpha) u_j - (\nabla_j \alpha)(1-\lambda^2) - 2\alpha^2 \lambda v_j,$$

that is,

$$(\nabla_j \alpha)(1-\lambda^2) = (u^k \nabla_k \alpha) u_j. \tag{5.21}$$

Thus, $\nabla_j \alpha$ being proportional to u_j, we find from (5.20)

$$h_{ji} h_k{}^t f_t{}^i = -\alpha h_{kt} f_j{}^t,$$

or

$$h_{ji} f_k{}^t h_t{}^i = \alpha h_{jt} f_k{}^t,$$

since $h_k{}^t$ and $f_t{}^i$ commute.

Transvecting this equation with $f_h{}^k$, we find

$$(h_{ji}h_t{}^i)(-\delta_h^t+u_h u^t+v_h v^t)=\alpha h_{jt}(-\delta_h^t+u_h u^t+v_h v^t),$$

or using (5. 18)

$$h_{ji}h_h{}^i=\alpha h_{jh},$$

or

$$h_i{}^t h_t{}^h=\alpha h_i{}^h.$$

Similarly, we can prove

$$k_i{}^t k_t{}^h=\bar{\alpha} k_i{}^h.$$

LEMMA 5. 8. *In Lemma* 5. 5 *and Lemma* 5. 7, α *and* $\bar{\alpha}$ *are both constants.*

Proof. Differentiating the second equation of (5. 18) covariantly and taking account of (3. 5), we find

$$(\nabla_j h_i{}^h)v_h-h_i{}^h(k_{jt}f_h{}^t-\lambda h_{jh})=(\nabla_j\alpha)v_i-\alpha(k_{jt}f_i{}^t-\lambda h_{ji}),$$

from which, taking the skew-symmetric part

$$h_j{}^h k_{it}f_h{}^t-h_i{}^h k_{jt}f_h{}^t=(\nabla_j\alpha)v_i-(\nabla_i\alpha)v_j+\alpha(k_{it}f_j{}^t-k_{jt}f_i{}^t),$$

because of the equation of Codazzi (4. 7).

Transvecting the above equation with v^j and making use of (2. 1) and (5. 18), we obtain

$$(1-\lambda^2)\nabla_j\alpha=(v^k\nabla_k\alpha)v_j. \tag{5. 22}$$

Thus, $\nabla_j\alpha$ is proportional to v_j, but (5. 21) shows that $\nabla_j\alpha$ is proportional to u_j. u_j and v_j being orthogonal to each other, we have $\nabla_j\alpha=0$ and hence $\alpha=$const.

§ 6. The mean curvature vector.

The mean curvature vector of the submanifold M is defined to be

$$\frac{1}{2n}g^{ji}\nabla_j X_i=\frac{1}{2n}h_i{}^i C+\frac{1}{2n}k_i{}^i D, \tag{6. 1}$$

and the mean curvature H of the submanifold M is defined to be the length of the mean curvature vector, that is,

$$H^2=\frac{1}{4n^2}[(h_i{}^i)^2+(k_i{}^i)^2]. \tag{6.2}$$

If the mean curvature vector vanishes identically on M, then M is said to be *minimal.*

A necessary and sufficient condition for M to be minimal is that

$$h_i{}^i=0, \qquad k_i{}^i=0. \tag{6.3}$$

We have

LEMMA 6.1. *Suppose that the submanifold M is such that the connection induced in the normal bundle is trivial and the (f, g, u, v, λ)-structure induced on M is normal. Then the mean curvature of M is constant.*

Proof. Let α' be an eigenvalue of $h_i{}^h$ at a point of M and p^i the eigenvector corresponding to α' at the point. Then we have

$$h_i{}^h p^i=\alpha' p^h.$$

Applying this $h_h{}^j$ and taking account of (5.19), we find

$$\alpha\alpha' p^j=\alpha'^2 p^j,$$

from which

$$\alpha'=\alpha \qquad \text{or} \qquad \alpha'=0.$$

Thus the only eigenvalue of $h_i{}^h$ is α or 0. Moreover, by Lemma 5.8, α being constant, the eigenvalues of $h_i{}^h$ are constant.

Similarly we can show that $k_i{}^h$ has only two constant eigenvalues $\bar{\alpha}$ and 0.

Now, let r and s be multiplicities of the eigenvalues α of $h_i{}^h$ and of $\bar{\alpha}$ of $k_i{}^h$ respectively. Then, α and $\bar{\alpha}$ being constant, r and s are also constant. So we have

$$h_i{}^i=r\alpha, \qquad k_i{}^i=s\bar{\alpha}.$$

Substituting this into (6.2), we obtain

$$H^2=\frac{1}{4n^2}(r^2\alpha^2+s^2\bar{\alpha}^2)=\text{const.} \tag{6.4}$$

This lemma shows that, in the sequel, we have to consider only two cases. One of these is the case where the submanifold is minimal and the another is the case where the mean curvature vector does not vanish everywhere on M.

Suppose first that the submanifold M is minimal. Then from Lemma 5.7 we find

$$h_{ji}h^{ji}=0, \qquad k_{ji}k^{ji}=0,$$

from which

(6. 5) $$h_{ji}=0, \qquad k_{ji}=0.$$

Thus equations of Weingarten give

$$\nabla_j C=0, \qquad \nabla_j D=0,$$

and consequently, the unit normals C and D being constant vectors, M is a $2n$-dimensional plane. Thus we have

THEOREM 6. 1. *Let M of codimension 2 of E be such that the connection induced on the normal bundle of M is trivial and the (f, g, u, v, λ)-structure on M is normal. If M is minimal, then M is a plane of codimension 2.*

Suppose next that the mean curvature vector does not vanish everywhere on M, and choose the first unit normal C along the direction of the mean curvature vector and choose the second unit normal D in such a way that X_i, C, D form the positive orientation of E.

Then the 1-forms u_i and v_i are completely determined when M is given. We say that such an (f, g, u, v, λ)-structure is *intrinsic*.

Since the first unit normal C is chosen in the direction of the mean curvature vector, we see, from (6. 1), that

(6. 6) $$k_i{}^i=0.$$

Thus if M is such that the connection induced in the normal bundle is trivial and the (f, g, u, v, λ)-structure induced on M is normal, then we have, from (5. 19),

$$k_{ji}k^{ji}=0,$$

from which

(6. 7) $$k_{ji}=0.$$

Thus, equations of Gauss and Weingarten become respectively

$$\nabla_j X_i=h_{ji}C, \qquad \nabla_j C=-h_j{}^h X_h \qquad \nabla_j D=0,$$

from which D is a constant vector and consequently

$$\nabla_j(X\cdot D)=0,$$

that is,

$$X\cdot D=\text{const},$$

which shows that M lies in a $(2n+1)$-dimensional plane. Thus we have

THEOREM 6. 2. *Let M of codimension 2 be such that the connection induced in the normal bundle of M is trivial and the mean curvature vector does not vanish everywhere. If the (f, g, u, v, λ)-structure induced on M is normal, then there exists a $(2n+1)$-dimensional plane E^{2n+1} such that M is a hypersurface of it.*

§ 7. Hypersurfaces of an odd-dimensional Euclidean space.

By theorem 6. 2, there exists a $(2n+1)$-dimensional plane E' such that the submanifold M under consideration is a hypersurface of it. So, in this section, we regard M as a hypersurface of a $(2n+1)$-dimensional Euclidean space E', which is of course in a $(2n+2)$-dimensional Euclidean space E.

We consider a linear coordinate system in E' consisting of $2n+1$ linearly independent vectors E_λ, E_λ and D forming a linear coordinate system of E, where here and in the sequel the indices $\kappa, \lambda, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$. We put

$$g_{\mu\lambda}=E_\mu \cdot E_\lambda, \tag{7. 1}$$

all the $g_{\mu\lambda}$ being constant. We also have

$$E_\lambda \cdot D=0. \tag{7. 2}$$

Now, F being a complex structure of the $(2n+2)$-dimensional Euclidean space E, we can put

$$\begin{aligned} FE_\lambda &= \varphi_\lambda{}^\mu E_\mu+\eta_\lambda D, \\ FD &= -\eta^\mu E_\mu, \end{aligned} \tag{7. 3}$$

where

$$\eta^\kappa=\eta_\lambda g^{\lambda\kappa}, \tag{7. 4}$$

$g^{\lambda\kappa}$ being contravariant components of $g_{\mu\lambda}$.

From the first equation of (7. 3), we find

$$F^2 E_\lambda=\varphi_\lambda{}^\mu(\varphi_\mu{}^\kappa E_\kappa+\eta_\mu D)-\eta_\lambda \eta^\kappa E_\kappa,$$

from which, F^2 being equal to $-I$,

$$\varphi_\lambda{}^\mu \varphi_\mu{}^\kappa=-\delta_\lambda^\kappa+\eta_\lambda \eta^\kappa,$$

$$\varphi_\lambda{}^\mu \eta_\mu=0.$$

From the second equation of (7. 3), we find

$$F^2D=-\eta^\lambda(\varphi_\lambda{}^\kappa E_\kappa+\eta_\lambda D),$$

from which

$$\varphi_\lambda{}^\kappa\eta^\lambda=0, \qquad \eta_\lambda\eta^\lambda=1.$$

Moreover, from the first equation of (7. 3), we have

$$(FE_\mu)\cdot(FE_\lambda)=(\varphi_\mu{}^\nu E_\nu+\eta_\mu D)\cdot(\varphi_\lambda{}^\kappa E_\kappa+\eta_\lambda D),$$

from which, $(FE_\mu)\cdot(FE_\lambda)$ being equal to $E_\mu\cdot E_\lambda=g_{\mu\lambda}$,

$$g_{\mu\lambda}=\varphi_\mu{}^\nu\varphi_\lambda{}^\kappa g_{\nu\kappa}+\eta_\mu\eta_\lambda.$$

Summing up, we have

$$\varphi_\lambda{}^\mu\varphi_\mu{}^\kappa=-\delta_\lambda^\kappa+\eta_\lambda\eta^\kappa,$$

(7. 5) $$\varphi_\lambda{}^\mu\eta_\mu=0, \qquad \varphi_\lambda{}^\kappa\eta^\lambda=0, \qquad \eta_\lambda\eta^\lambda=1,$$

$$\varphi_\mu{}^\nu\varphi_\lambda{}^\kappa g_{\nu\kappa}=g_{\mu\lambda}-\eta_\mu\eta_\lambda,$$

that is, $(\varphi_\lambda{}^\kappa, \eta_\lambda, g_{\mu\lambda})$ defines an almost contact metric structure of the $(2n+1)$ dimensional Euclidean space E'.

Now we consider a $2n$-dimensional submanifold M in E' in E and represent it by the position vector

$$X=X(x)=X^\kappa(x)E_\kappa,$$

the origin of the coordinate system being on E'.

The vectors X_i tangent to M and the unit normal vector C to M can be expressed as

(7. 6) $$X_i=B_i{}^\kappa E_\kappa, \qquad C=C^\kappa E_\kappa,$$

respectively, where

$$B_i{}^\kappa=\partial_i X^\kappa.$$

Applying the operator F to the both sides of the first equation of (7. 6), we find

$$FX_i=B_i{}^\kappa FE_\kappa,$$

$$f_i{}^h X_h+u_i C+v_i D=B_i{}^\lambda(\varphi_\lambda{}^\kappa E_\kappa+\eta_\lambda D),$$

or

$$f_i{}^h B_h{}^\kappa E_\kappa+u_i C^\kappa E_\kappa+v_i D=B_i{}^\lambda(\varphi_\lambda{}^\kappa E_\kappa+\eta_\lambda D)$$

by virtue of (1. 6), (7. 3) and (7. 6), from which

$$\varphi_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa,$$

$$\eta_\lambda B_i{}^\lambda = v_i.$$

Applying the operator F to the both sides of the second equation of (7. 6), we find

$$FC = C^\lambda F E_\lambda,$$

$$-u^i X_i + \lambda D = C^\lambda(\varphi_\lambda{}^\kappa E_\kappa + \eta_\lambda D),$$

or

$$-u^i(B_i{}^\kappa E_\kappa) + \lambda D = C^\lambda(\varphi_\lambda{}^\kappa E_\kappa + \eta_\lambda D),$$

by virtue of (1. 6), (7. 3) and (7. 6), from which

$$\varphi_\lambda{}^\kappa C^\lambda = -u^i B_i{}^\kappa,$$

$$\eta_\lambda C^\lambda = \lambda.$$

Summing up, we have

$$\varphi_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa,$$

$$\varphi_\lambda{}^\kappa C^\lambda = -u^i B_i{}^\kappa \tag{7. 7}$$

$$\eta_\lambda B_i{}^\lambda = v_i, \qquad \eta_\lambda C^\lambda = \lambda.$$

It will be easily verified that, $\varphi_\lambda{}^\kappa$, η_λ, $g_{\mu\lambda}$ defining an almost contact metric structure, $f_i{}^h$, g_{ji}, u_i, v_i, λ define an (f, g, u, v, λ)-structure.

Now the equations of Gauss and Weingarten of M in E' are respectively

$$\nabla_j X_i = h_{ji} C, \tag{7. 8}$$

and

$$\nabla_j C = -h_j{}^i X_i, \tag{7. 9}$$

or

$$\nabla_j B_i{}^\kappa = h_{ji} C^\kappa \tag{7. 10}$$

and

$$\nabla_j C^\kappa = -h_j{}^i B_i{}^\kappa.$$

Differening the first equation of (7. 7) covariantly, we find

$$\varphi_\lambda{}^\kappa h_{ji}C^\lambda=(\nabla_j f_i{}^h)B_h{}^\kappa+f_i{}^t h_{jt}C^\kappa+(\nabla_j u_i)C^\kappa-u_i h_j{}^t B_t{}^\kappa,$$

from which, taking account of the second equation of (7. 7),

$$\nabla_j f_i{}^h=-h_{ji}u^h+h_j{}^h u_i,$$

$$\nabla_j u_i=-h_{jt}f_i{}^t.$$

Differentiating the second equation of (7. 7) covariantly and taking account of (7. 7), we find

$$\varphi_\lambda{}^\kappa(-h_j{}^t B_t{}^\lambda)=-(\nabla_j u^i)B_i{}^\kappa-u^i h_{ji}C^\kappa,$$

or

$$-h_j{}^t(f_t{}^h B_h{}^\kappa+u_t C^\kappa)=-(\nabla_j u^i)B_i{}^\kappa-u^i h_{ji}C^\kappa,$$

from which

$$\nabla_j u^i=h_j{}^t f_t{}^i.$$

Differentiating the third equation of (7. 7) covariantly, we find

$$\eta_\lambda h_{ji}C^\lambda=\nabla_j v_i,$$

from which

$$\nabla_j v_i=\lambda h_{ji}.$$

Finally differentiating the last equation of (7. 7) covariantly, we find

$$\eta_\lambda(-h_j{}^h B_h{}^\lambda)=\nabla_j\lambda,$$

from which

$$\nabla_j\lambda=-h_{jt}v^t.$$

Summing up, we have

$$\begin{aligned}&\nabla_j f_i{}^h=-h_{ji}u^h+h_j{}^h u_i,\\ &\nabla_j u_i=-h_{jt}f_i{}^t,\\ &\nabla_j v_i=\lambda h_{ji},\\ &\nabla_j\lambda=-h_{jt}v^t.\end{aligned}\tag{7.12}$$

We assumed that (f, g, u, v, λ)-structure on M is normal, that is,

(7. 13) $$S_{ji}{}^h=f_j{}^t\nabla_t f_i{}^h-f_i{}^t\nabla_t f_j{}^h-(\nabla_j f_i{}^t-\nabla_i f_j{}^t)f_t{}^h+(\nabla_j u_i-\nabla_i u_j)u^h+(\nabla_j v_i-\nabla_i v_j)v^h=0.$$

As we have seen in § 6, the only eigenvalue of the tensor $h_i{}^h$ is α or 0. We denote the eigenspaces corresponding to the eigenvalues α and 0 by V_α and V_0 respectively. Since the multiplicity r of α is constant, $V_\alpha(x)$ at x and $V_0(x)$ at x, $x\in M$, define respectively r- and $(2n-r)$-dimensional distributions D_α and D_0 over M. They are complementary in the sense that they are mutually orthogonal and their Whiteney sum is $T(M)$.

LEMMA 7. 1. *The distributions D_α and D_0 are both integrable.*

Proof. Let p^h and q^h be two arbitrary eigenvectors of $h_i{}^h$ with constant eigenvalue $\alpha\neq 0$, then we have

$$h_i{}^h p^i=\alpha p^h, \qquad h_i{}^h q^i=\alpha q^h,$$

from which

$$(\nabla_j h_i{}^h)p^i+h_i{}^h(\nabla_j p^i)=\alpha(\nabla_j p^h),$$

$$(\nabla_j h_i{}^h)q^i+h_i{}^h(\nabla_j q^i)=\alpha(\nabla_j q^h).$$

Thus

$$h_i{}^h(p^j\nabla_j q^i-q^j\nabla_j p^i)=\alpha(p^j\nabla_j q^h-q^j\nabla_j p^h)$$

by virtue of the equations of Codazzi, that is, if p^h and q^h belong to D_α, then $[p, q]^h$ also belongs to D_α. Consequently the distribution D_α spanned by eigenvectors of $h_i{}^h$ with eigenvalue $\alpha\neq 0$ is integrable.

Similarly we can also prove that the distribution D_0 spanned by eigenvectors of $h_i{}^h$ with eigenvalue 0 is integrable.

LEMMA 7. 2. *Each integral manifold of D_α is totally geodesic in M and so is each integral manifold of D_0.*

Proof. Let p^h and q^h be two arbitrary vectors belonging to the distribution D_α. Then we have

(7. 14) $$h_i{}^h p_h=\alpha p_i, \qquad h_i{}^h q_h=\alpha q_i.$$

Differentiating the first equation of (7. 14) covariantly, we obtain

$$(\nabla_j h_i{}^h)p_h+h_i{}^h(\nabla_j p_h)=\alpha\nabla_j p_i,$$

from which

$$h_i{}^h(\nabla_j p_h)-h_j{}^h(\nabla_i p_h)=\alpha(\nabla_j p_i-\nabla_i p_j)$$

by virtue of the equations of Codazzi. Transvecting this equation with q^j and taking account of (7. 14), we have

$$h_i^h(q^j\nabla_j p_h)-\alpha q^h(\nabla_i p_h)=\alpha q^j(\nabla_j p_i-\nabla_i p_j),$$

from which

$$h_i^h(q^j\nabla_j p_h)=\alpha(q^j\nabla_j p_i),$$

or

$$h_i^h(q^j\nabla_j p^i)=\alpha(q^j\nabla_j p^h),$$

which shows that if p^h and q^h are two arbitrary vectors belonging to the distribution D_α, then $q^j\nabla_j p^h$ also belongs to the distribution D_α. Thus each integral manifold of D_α is totally geodesic in M.

Similarly we can prove that each integral manifold of D_0 is totally geodesic in M.

Moreover, if p^i and w^i belong respectively to D_0 and D_α, we have

$$(w^j\nabla_j h_i^h)p^i=w^j\nabla_j(h_i^h p^i)-h_i^h w^j\nabla_j p^i=-h_i^h w^j\nabla_j p^i$$

and

$$(p^j\nabla_j h_i^h)w^i=p^j\nabla_j(h_i^h w^i)-h_i^h p^j\nabla_j w^i=\alpha p^j\nabla_j w^h-h_i^h p^j\nabla_j w^i,$$

that is,

$$(w^j\nabla_j h_i^h)p^i=-\alpha(w^j\nabla_j p^i)_\alpha, \tag{7. 15}$$

and

$$\begin{aligned}(p^j\nabla_j h_i^h)w^i&=\alpha(p^j\nabla_j w^h)-\alpha(p^j\nabla_j w^h)_\alpha\\&=\alpha(p^j\nabla_j w^h)_0,\end{aligned} \tag{7. 16}$$

vector of the form q^h being written as $(q^h)_\alpha+(q^h)_0$, where $(q^h)_\alpha$ and $(q^h)_0$ respectively denote the D_α and D_0 components of q^h. Hence we get

$$-(w^j\nabla_j p^i)_\alpha=(p^j\nabla_j w^h)_0,$$

because of the equation of Codazzi.

Consequently we have

$$(w^j\nabla_j p^i)_\alpha=0, \qquad \text{that is,} \qquad w^j\nabla_j p^i\in D_0, \tag{7. 17}$$

and

$$(p^j \nabla_j w^h)_0 = 0, \qquad \text{that is,} \qquad p^j \nabla_j w^i \in D_\alpha. \tag{7.18}$$

Thus we see that the distributions D_0 and D_α are parallel. So, using de Rham's decomposition theorem [7], we have

LEMMA 7.3. *If the submanifold M is complete, then M is a product of M_α and M_0, M_α corresponding to the integral manifold of D_α and M_0 to that of D_0.*

LEMMA 7.4. *The M_α is totally umbilical in E' and M_0 is totally geodesic in E'.*

Proof. We represent M_α by

$$x^h = x^h(u^a), \tag{7.19}$$

where u^a are local coordinates on M_α. Thus we have

$$X^\kappa = X^\kappa(x(u)), \tag{7.20}$$

from which

$$B_b{}^\kappa = B_b{}^h B_h{}^\kappa, \tag{7.21}$$

where

$$B_b{}^\kappa = \partial_b X^\kappa, \qquad B_b{}^h = \partial_b x^h \qquad (\partial_b = \partial / \partial u^b).$$

From (7.21), we find by covariant differentiation

$$\nabla_c B_b{}^\kappa = B_c{}^j B_b{}^i \nabla_j B_i{}^\kappa$$

because of $\nabla_c B_b{}^i = 0$, from which

$$\nabla_c B_b{}^\kappa = B_c{}^j B_b{}^i h_{ji} C^\kappa$$

or

$$\nabla_c B_b{}^\kappa = \alpha g_{cb} C^\kappa, \tag{7.22}$$

because $B_b{}^h$ are eigenvectors of $h_j{}^h$ with eigenvalue α. Equation (7.22) shows that M_α is totally umbilical in E'.

We can similarly prove that M_0 is totally geodesic in E'.

LEMMA 7.5. *The M_α is a sphere and M_0 is a plane.*

Proof. The M_0 being totally geodesic in E', it is a plane. Thus M_α is a hypersurface of a Euclidean space.

For the covariant derivative of C^κ along M_α, we have

$$\begin{aligned} \nabla_c C^\kappa &= B_c{}^j(\nabla_j C^\kappa) \\ &= -B_c{}^j h_j{}^i B_i{}^\kappa \\ &= -\alpha B_c{}^j B_j{}^\kappa, \end{aligned}$$

$B_c{}^j$ being an eigenvector of $h_j{}^i$ with eigenvalue α, from which

$$\nabla_c C^\kappa + \alpha B_c{}^\kappa = 0$$

and consequently

$$C^\kappa + \alpha X^\kappa = A^\kappa,$$

A^κ being a constant vector. This equation shows that M_α lies on a sphere. Thus, M_α being the intersection of a plane and a sphere, M_α is itself a sphere.

From these lemmas, we have

THEOREM 7. 1. *Let M be a $2n$-dimensional complete differentiable hypersurface in a $(2n+1)$-dimensional Euclidean space E'. If the (f, g, u, v, λ)-structure induced on M is normal, then M is a product of a sphere and a plane.*

Combining Theorems 6. 1, 6. 2 and Theorem 7. 1, we obtain

THEOREM 7. 2. *Let a complete differentiable submanifold M of codimension 2 of an even-dimensional Euclidean space be such that the connection induced in the normal bundle of M is trivial. If the (f, g, u, v, λ)-structure induced on M is normal, then M is a sphere, a plane, or a product of a sphere and a plane.*

As a special case of Theorem 7. 2, we have from (4. 3)

THEOREM 7. 3. *Let a complete differentiable submanifold M of codimension 2 of an even-dimensional Euclidean space be such that the connection induced in the normal bundle of M is trivial. If the linear transformations $h_j{}^i$ and $k_j{}^i$ which are defined by the second fundamental tensors of M commute with $f_j{}^i$, then M is a sphere, a plane, or a product of a sphere and a plane.*

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