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On normality of a noetherian ring

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In the present article, we understand by a ring a commutative ring with identity; any local or semi-local rings are assumed to be noetherian. A ring is called *quasi-normal*[1] if it is integrally closed in its total quotient ring. If such a ring has no nilpotent elements (except zero), then it is the direct sum of some normal domains under certain finiteness condition, and several results are known in such a case.

Our main result is a characterization of a quasi-normal noetherian ring, which asserts as follows:[2]

A noetherian ring R is quasi-normal if and only if the following two conditions are satisfied:

(1) If P is a prime ideal of height one in R, and if P contains a non-zero-divisor, then R_P is a discrete valuation ring.

(2) If a non-unit a of R is not a zero-divisor, then aR has no imbedded prime divisor.

As for normal rings (under any one of the definitions in foot-note 1), the normality is carried over to its rings of quotients. But quasi-normality may not be carried over to rings of quotients. The present article deals also with some topics related to this fact.

For a ring R, $Q(R)$ denotes the total quotient ring of R.

1. Rings of quotients with respect to prime ideals.

It is well known that if R is a local quasi-normal ring, then either $R=Q(R)$ or R has no nilpotent elements (except zero). More generally we have:

Proposition 1.1. *Let P be a prime ideal of a quasi-normal noetherian ring R. If P contains a non-zero-divisor p, then R_P has no nilpotent elements.*[3]

Proof. Let ϕ: $R \rightarrow R_P$ be the natural homomorphism. If a/r (a, $r \in R$;

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1) As far as the writers know, there are two definitions of normality. In one definition, a normal ring means a normal domain (and this is rather common). In the other definition, a normal ring is the direct sum of a finite number of normal domains.

2) This result was announced in [6] as an exercise (without proof).

3) This result was obtained while the writers made discussion with Professor Satoshi Suzuki.

$r\notin P$) is a nilpotent element of R_P, then ϕa is nilpotent and $sa^m=0$ with a natural number m and $s\notin P$ ($s\in R$). Then sa is nilpotent and sa/p^n ($\in Q(R)$) is integral over R for any natural number n. By the quasi-normality, we have $sa\in p^nR$. Thus $\phi a\in\bigcap(\phi p)^n R_P=0$. Q.E.D.

It is obvious that the direct sum of rings $R_1, \cdots, R_n$ is quasi-normal if and only if every R_i is quasi-normal. This fact and Proposition 1.1 give us:

Corollary 1.2. *Any quasi-normal noetherian ring R is the direct sum of a finite number of quasi-normal noetherian rings R_i so that each R_i has no proper idempotent elements. Each R_i is either an integral domain or a ring in which every maximal prime divisor of zero is a maximal ideal.*[4]

Proof. The first assertion is obvious, and we assume that $R=R_1$ has no proper idempotent elements and that R is not an integral domain. Let P be a maximal prime divisor of zero and let M be a maximal ideal containing P. If $M\neq P$, then M contains a non-zero-divisor and R_M has no nilpotent elements. Thus P must be a primary component of the zero ideal. Let I be the intersection of other primary components of zero. Since P is a primary component of zero, it follows that $I\nsubseteq P$. Since P is a maximal prime divisor of zero, there are elements p, q of P, I, respectively, such that $p+q$ is not a zero-divisor. Then $e=p/(p+q)$ is a proper idempotent element of $Q(R)$; e is obviously integral over R, a contradiction. Q.E.D.

Corollary 1.3. *Let R be a ring in which every maximal ideal contains some non-zero-divisors. If the integral closure of R in $Q(R)$ is noetherian, then R has no nilpotent elements (except zero).*

Corollary 1.4. *Let R be a Macaulay ring (in the sense of [5]) of Krull dimension ≥ 1. Then, either R has no nilpotent elements or the integral closure of R in $Q(R)$ is not noetherian.*

2. Key lemmata.

Lemma 2.1. *Let P be a prime ideal of height one in a quasi-normal ring R such that the set of zero-divisors is a finite union of prime ideals. If P contains a non-zero-divisor p, then R_P is also quasi-normal. Therefore, if furthermore R is noetherian, then R_P is a discrete valuation ring.*

Remark. For our main result (Theorem 3.1), we need noetherian case only, which follows from (12.5) in [5].[5]

Proof. Considering R_S with the set S of non-zero-divisors not contained in P, we may assume that P is the unique prime ideal containing non-zero-

4) This result was announced in [5] as an exercise, but there was an error so that a kind of converse was claimed (which is false as is easily seen by our example in §4 below).

5) This was announced in [6] as an exercise, where the condition that ht $P=1$ is missing by an error.

divisors. Then $Q(R)$ coincides with $R[p^{-1}]$. Let N be the kernel of the natural homomorphism $R \to R_P$ and set $R^* = R/N$. Let x^* be an element of $Q(R^*)$ $(= Q(R_P))$ integral over R_P. Then there are a natural number m and an element c of $R - P$ such that cx^* is integral over R^* and $p^m cx^*$ is in R^*. Let b be a representative of $p^m cx^*$ in R. We want to show that b/p^m is in R. Since cx^* is integral over R^*, there is an element c' of $R - P$ such that bc'/p^m is integral over R. This implies that bc'/p^m is in R. Suppose that b/p^m is not in R. Then $c' \in p^m R : b \subseteq P$, which is a contradiction. Thus b/p^m is contained in R, and therefore cx^* is in R^*. Since c is not in P, it follows that x^* is in R_P. *Q.E.D.*

The proof above can be adapted to the following case:

P is a prime ideal of a quasi-normal ring R and p a non-zero-divisor. S is the set of non-zero-divisors not contained in P. ϕ is the natural homomorphism $R_S \to R_P$. $R^* = \phi(R_S)$. We extend ϕ to a homomorphism of $Q(R_P)$ into $Q(R_P)$.

Lemma 2.2. *Under the circumstances, if $x \in Q(R)$ and if ϕx is integral over R^*, then $x \in R_S$ and $\phi x \in R^*$.*

In particular, we have:

Lemma 2.3. *With R, p, S, R^* as above, if y^* is an element of $Q(R^*)$ integral over R^* and if there is a natural number n so that $p^n y^* \in R^*$ then, $y^* \in R^*$.*

3. The main theorem.

Theorem 3.1. *A noetherian ring R is quasi-normal if and only if the following two conditions are satisfied:*

(1) *If P is a prime ideal of height one and if P contains a non-zero-divisor, then R_P is a discrete valuation ring.*

(2) *If a non-unit a of R is not a zero-divisor, then aR has no imbedded prime divisor.*

Proof. Assume that R is quasi-normal. Lemma 2.1 shows the validity of (1). As for (2), we assume that Q is an imbedded prime divisor of aR. Then QR_Q is an imbedded prime divisor of aR_Q. It follows that there is a proper integral extension of R_Q having QR_Q as a conductor (see [5]); a contradiction by Lemma 2.3. Thus (2) holds good. Conversely, assume that (1) and (2) hold. Assume that a/b ($a, b \in R$; b non-zero-divisor) is integral over R. Consider the shortest representation of bR as an intersection of primary ideals: $bR = Q_1 \cap \cdots \cap Q_s$. Condition (2) implies that $P_i = \sqrt{Q_i}$ are of height one, and hence R_{P_i} are discrete valuation rings (by (1)). Consider the natural homomorphism $\phi_i : R \to R_{P_i}$. $\phi_i(a/b)$ is integral over $\phi_i R \subseteq R_{P_i}$ and $\phi_i(a/b) \in R_{P_i}$. Thus $\phi_i a \in bR_{P_i}$ and $a \in \phi_i^{-1}(\phi_i R \cap Q_i R_{P_i}) = Q_i$ for every i. Therefore $a \in \cap Q_i = bR$ and $a/b \in R$. *Q.E.D.*

4. Additional remarks.

We first consider the polynomial ring F in variables X, Y, Z over an arbitrary field K. Set $M=XF+YF+ZF$ and $I=XYZM$. Consider the ring $R=F/I$. Let ψ be the natural homomorphism of F onto R and set $x=\psi X$, $y=\psi Y$, $z=\psi Z$. We want to show:

Proposition 4.1. *R is a quasi-normal noetherian ring, in which $P^*=xR+yR+(1-z)R$ is a prime ideal of height two and contains a non-zero-divisor. But R_{P^*} is not an integral domain, and consequently R_{P^*} is not quasi-normal.*

Proof. By our construction, prime divisors of zero in R are xR, yR, zR and M/I $(=xR+yR+zR)$. We show first the validity of (1) and (2) in Theorem 3.1. Assume that P is a prime ideal of height one containing a non-zero-divisor. $\bar{P}=\psi^{-1}(P)$ is a prime ideal of F containing $XYZM$, and therefore $\bar{P}$ contains one of X, Y, Z. Because of symmetry, we may assume that $\bar{P}$ contains X. Since P contains a non-zero-divisor, $\bar{P}$ contains a polynomial $f(X, Y, Z)$ such that $f(0, 0, 0)\neq 0$. Set $J=xF+fF$. Since ht $\bar{P}=2$, $\bar{P}$ must be a minimal prime divisor of J. Since J contains XF and since F/XF is a UFD, we see that $P=XF+gF$ with a polynomial g such that $g(0,0,0)\neq 0$. Now, $R_P\cong F_{\bar{P}}/XF_{\bar{P}}$ (because Y, Z are not in $\bar{P}$ and therefore $IF_{\bar{P}}=XF_{\bar{P}}$) and R_P is a discrete valuation ring. Thus (1) is proved. Let a be a non-zero-divisor as in (2). Take $h\in F$ such that $\psi h=a$. Then $h(0, 0, 0)\neq 0$. Consider $J'=hF+XYZM$. J' is not contained in M and therefore $J'=hF+XYZF$. $J_X'=hF+XF$, $J_Y'=hF+YF$, $J_Z'=hF+ZF$ are intersections of primary ideals of height 2 containing X, Y, Z, respectively. $J'=J_X'\cap J_Y'\cap J_Z'$ and therefore J' has no imbedded prime divisor. Thus (2) holds good and R is quasi-normal by Theorem 3.1. Now we consider P^*. Set $\bar{P}^*=\psi^{-1}(P^*)$. Then $\bar{P}^*=XF+YF+(1-Z)F$ and we see that P^* is a prime ideal of height 2. $R_{P^*}\cong F_{\bar{P}^*}/IF_{\bar{P}^*}=F_{\bar{P}^*}/XYF_{\bar{P}^*}$ and therefore R_{P^*} is not an integral domain.

Q.E.D.

In closing this article, we add the following remarks.

Proposition 4.2. *Let R be a quasi-normal noetherian ring. If zero has no imbedded prime divisor, then R is the direct sum of a finite number of quasi-normal noetherian rings R_i so that each R_i is either a normal domain or an Artin local ring.*

Proof is easy because idempotent elements of $Q(R)$ are integral over R.

Note also that if R is a quasi-normal ring (not necessary to be noetherian) in which zero has no imbedded prime divisor, then any ring of quotients of R is quasi-normal. In the noetherian case, we have furthermore:

Corollary 4.3. *For a noetherian ring R such that zero has no imbedded prime divisor, R is quasi-normal if and only if R_M is quasi-normal for every maximal ideal M.*

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