Tohoku Math. J.
58 (2006), 599–616

# ON NORMALITY OF CONES OVER SYMMETRIC VARIETIES

ROCCO CHIRIVÌ, CORRADO DE CONCINI AND ANDREA MAFFEI

(Received May 25, 2005, revised April 24, 2006)

**Abstract.** Let $G$ be a simply connected semisimple algebraic group and let $K$ be the subgroup of points fixed by an involution of $G$. For certain representations containing a line $r$ preserved by $K$, we study the normality of the closure of the set of vectors which are $G$ conjugate to a vector in $r$. Some applications of our result to the normality of certain classical varieties are given.

**Introduction.** Let $G$ be a simply connected semisimple algebraic group, let $\sigma$ an involution of $G$ and let $H$ be the normalizer of the subgroup $G^\sigma$ of fixed points. Denoting by $X$ the wonderful compactification of $G/H$ and by $\mathcal{L}$ a line bundle generated by global sections, we know that the ring $A(\mathcal{L}) = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^n)$ is generated in degree one by [3]. So it can be identified with the projective coordinate ring of the variety which is the image of $X$ into $\boldsymbol{P}(\Gamma(X, \mathcal{L})^*)$. The variety $X$ is smooth, so the corresponding cone in $\boldsymbol{P}(\Gamma(X, \mathcal{L})^*)$ is normal.

On the other hand, if we consider an irreducible module $V$ with a (necessarily unique) eigenvector $h$ for $G^\sigma$, then one knows that the map from $G/H$ to $\boldsymbol{P}(V)$ defined by $gH \mapsto [g \cdot h]$ extends to a morphism $\pi$ from $X$ to $\boldsymbol{P}(V)$. Set $\mathcal{L} = \pi^*(\mathcal{O}_{\boldsymbol{P}(V)}(1))$ and denote by $B(\mathcal{L})$ the projective coordinate ring of $\pi(X)$. We prove that $A(\mathcal{L})$ is the integral closure of $B(\mathcal{L})$ in its quotient field. In particular, we obtain that the cone over $\pi(X)$ is normal if and only if the highest weight of $V$ is a minuscule weight for the restricted root system of the involution $\sigma$. In the special case of the compactification of the group this has already appeared in [5].

The paper is organized as follows. In Section 1 we give a short description of the construction and the basic properties of complete symmetric varieties. Our result reported above about $A(\lambda)$ and $B(\lambda)$ is proved in Section 2.

In Section 3 we give a slight generalization of our result about normality considering a sort of parabolic induction of a symmetric variety. In the last Section we remark that our result can be applied to proving the normality and non normality of some rather concrete rings. For example, we give a proof of the normality of the subring of the ring of polynomial functions on the symmetric (respectively antisymmetric) matrices of rank less than or equal to a given integer, generated by the minors (respectively pfaffians) of a fixed order. In the last example we apply our result to proving that the closure of a spherical nilpotent orbit of height less or equal to 2 is a normal variety.

We would like to thank Michel Brion for explaning to us his simple proof of Proposition 2.1 and Domingo Luna for suggesting Example 4.4.

2000 *Mathematics Subject Classification.* Primary 14M17; Secondary 14L30.
*Key words and phrases.* Complete symmetric variety, projective normality.

**1. Description of complete symmetric varieties.** In this Section we collect some preliminary results for the sequel, setting up our notation. We review the construction of the wonderful compactification of $G/H$, for details one can see [7, 8].

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field $\boldsymbol{k}$ of characteristic zero, and let $\sigma$ be an order 2 automorphism of $\mathfrak{g}$. Denote by $\mathfrak{h}$ the subalgebra of fixed points of $\sigma$ in $\mathfrak{g}$. If $\mathfrak{t}$ is a $\sigma$-stable toral subalgebra of $\mathfrak{g}$, we can decompose $\mathfrak{t}$ as $\mathfrak{t}_0 \oplus \mathfrak{t}_1$ with $\mathfrak{t}_0$ the $(+1)$-eigenspace of $\sigma$ and $\mathfrak{t}_1$ the $(-1)$-eigenspace. We recall that any $\sigma$-stable toral subalgebra of $\mathfrak{g}$ is contained in a maximal one which is itself $\sigma$-stable. We fix such a $\sigma$-stable maximal toral subalgebra $\mathfrak{t}$ for which $\dim \mathfrak{t}_1$ is maximal and denote this dimension by $l$; we call it the *rank* of $\sigma$.

Let $\Phi \subset \mathfrak{t}^*$ be the root system of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be the root space decomposition with respect to the action of $\mathfrak{t}$. Observe that $\sigma$ acts also on $\mathfrak{t}^*$ and that it preserves $\Phi$ and the Killing form $(\cdot,\cdot)$ on $\mathfrak{t}$ and $\mathfrak{t}^*$. Let $\Phi_0 = \{\alpha \in \Phi \mid \sigma(\alpha) = \alpha\}$ and $\Phi_1 = \Phi \smallsetminus \Phi_0$. The choice of a $\sigma$-stable toral subalgebra for which $\dim \mathfrak{t}_1$ is maximal is equivalent to the condition $\sigma|_{\mathfrak{g}_\alpha} = \mathrm{id}|_{\mathfrak{g}_\alpha}$ for all $\alpha \in \Phi_0$. Moreover, we can choose the set $\Phi^+$ of positive roots in such a way that $\sigma(\alpha) \in \Phi^-$ for all roots $\alpha \in \Phi^+ \cap \Phi_1$. Let $\Delta$ be the base defined by $\Phi^+$ and put $\Delta_0 = \Delta \cap \Phi_0$, $\Delta_1 = \Delta \cap \Phi_1$. Denote by $\Lambda \subset \mathfrak{t}^*$ the set of integral weights of $\Phi$ and observe that $\sigma$ preserves $\Lambda$. Let $\Lambda^+$ be the set of dominant weights with respect to $\Phi^+$ and let $\omega_\alpha$ be the fundamental weight dual to the simple coroot $\alpha^\vee$ for $\alpha \in \Delta$. For $\lambda \in \Lambda^+$ let also $V_\lambda$ be the irreducible representation of $\mathfrak{g}$ of highest weight $\lambda$.

We say that $\lambda \in \Lambda^+$ is *spherical* if there exists $h \in V_\lambda \smallsetminus \{0\}$ fixed by $\mathfrak{h}$ (i.e., $\mathfrak{h} \cdot h = 0$): in this case the vector $h$ is also unique up to scalar and we denote it by $h_\lambda$. It is called a *spherical vector* and the $G$-module $V_\lambda$ is called a *spherical module*. We denote the set of spherical weights by $\Omega^+$ and the lattice they generate by $\Omega$.

For a root $\alpha$ define $\tilde\alpha \doteq \alpha - \sigma(\alpha)$ and let $\tilde\Phi = \{\tilde\alpha \mid \alpha \in \Phi_1\}$. This is a (not necessarily reduced) root system of rank $l$ with base $\tilde\Delta = \{\tilde\alpha \mid \alpha \in \Delta_1\}$; it is called the *restricted root system*. As a consequence of a result of Helgason (see also [14] or [9] for an algebraic approach), $\Omega \cap \Lambda^+ = \Omega^+$ and $\Omega$ can be identified with the lattice of integral weights of the root system $(\tilde\Phi, \Omega \otimes_{\mathbf{Z}} \boldsymbol{R})$. We denote by $\tilde R$ the root lattice of $\tilde\Phi$ and by $\tilde R^+$ the monoid generated by the base $\tilde\Delta$.

Now we come to the construction of complete symmetric varieties following De Concini and Procesi [7]. Let $G$ be a connected algebraic group over $\boldsymbol{k}$ whose Lie algebra is isomorphic to $\mathfrak{g}$. The action of $\sigma$ on $\mathfrak{g}$ lifts to an automorphism of $G$, still denoted by $\sigma$. Let $H$ be the normalizer in $G$ of the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. As explained in [7], $H$ is the maximal subgroup having $\mathfrak{h}$ as Lie algebra; if $G$ is an adjoint group, $H$ coincides with the fixed point set of $\sigma$ in $G$. The quotient $G/H$ is called a *symmetric variety*. However, since $G/H$ does not depend on the choice of the group $G$ over $\mathfrak{g}$, we prefer to choose $G$ simply connected, so for the rest of the paper $G$ is a simply connected algebraic group with Lie algebra $\mathfrak{g}$ unless otherwise stated. We introduce also the torus $T$ (resp., $T_0$ and $T_1$) whose Lie algebra is $\mathfrak{t}$ (resp., $\mathfrak{t}_0$ and $\mathfrak{t}_1$), and the parabolic subgroup $P$ of $G$ associated to $\Delta_0$.

Consider now a (dominant) spherical weight $\lambda$ with the property $(\lambda, \tilde{\alpha}) \neq 0$ for all $\tilde{\alpha} \in \tilde{\Delta}$, such a weight is called *regular*, and let $x_0 = [h_\lambda] \in \boldsymbol{P}(V_\lambda)$. We define the variety $X = X(\sigma)$ as the closure $\overline{G \cdot x_0} \subset \boldsymbol{P}(V_\lambda)$. One can show that $x_0$ is the unique point fixed by $H$ in $X$ and that the map $g \mapsto gx_0$ induces an embedding $G/H \hookrightarrow X$ which is called the "minimal compactification" of $G/H$. Moreover the variety $X$ constructed in this way is independent of the regular weight $\lambda$ up to isomorphism of $G$-variety.

The following Proposition describes the structure of the compactification.

PROPOSITION 1.1 (Theorem 3.1 in [7]). *Let $X = X(\sigma)$ be the compactification of $G/H$ described above, then the following hold.*

(1) *$X$ is a smooth projective $G$-variety*;

(2) *$X \setminus G \cdot x_0$ is a divisor with normal crossings and smooth irreducible components $S_1, \dots, S_l$*;

(3) *the $G$-orbits of $X$ correspond to the subsets of the indices $1, 2, \dots, l$ so that the orbit closures are the intersections $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}$, with $1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k \leq l$*;

(4) *the unique closed orbit $Y \doteq \bigcap_{i=1}^{l} S_i$ is isomorphic to the partial flag variety $G/P$.*

Fix now a dominant weight $\lambda \in \Lambda^+$ such that there exists a point $r \in \boldsymbol{P}(V_\lambda)$ fixed by $H$. By [8] we know that a point with this property is unique, so we define the variety $X_\lambda$ to be the closure $\overline{G \cdot r}$ in $\boldsymbol{P}(V_\lambda)$ and define $C_\lambda$ as the cone over $X_\lambda$ in $V_\lambda$. By [7] it is known that the map $G/H \ni gH \mapsto g \cdot r \in \boldsymbol{P}(V_\lambda)$ extends to a morphism

$$\pi_\lambda : X \to X_\lambda .$$

In particular, $\pi_\lambda$ is an isomorphism in the case of a regular weight $\lambda$.

We use such a morphism to construct some line bundles on $X$ by setting $\mathcal{L}_\lambda$ corresponding to $\lambda$ in the identification of $\mathrm{Pic}(G/P)$ with a sublattice of the weight lattice $\Lambda$. One can see the following

PROPOSITION 1.2 (Proposition 8.1 in [7]). *The map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ induced by the inclusion is injective.*

So we can identify $\mathrm{Pic}(X)$ with a sublattice of the weight lattice. Furthermore, the line bundles constructed above account for all line bundles, since we have

PROPOSITION 1.3 (Lemma 4.6 in [8]). *Under the described identification $\mathrm{Pic}(X)$ is the lattice generated by the dominant weights $\lambda$ such that $\boldsymbol{P}(V_\lambda)^H$ is not empty.*

Notice that the lattice $\Omega$ generated by the spherical weights is contained in the $\mathrm{Pic}(X)$ by definition. Further this lattice is the Picard group "almost always", i.e., except in few situations. Indeed, one can define an involution $\bar{\sigma}$ on $\Delta_1$ such that for all $\alpha \in \Delta_1$ we have $\sigma(\alpha) = -\bar{\sigma}(\alpha) - \beta$ for a suitable non-negative linear combination $\beta$ of roots in $\Delta_0$. We call a root $\alpha \in \Delta_1$ *exceptional* if $\bar{\sigma}(\alpha) \neq \alpha$ and $(\alpha, \bar{\sigma}(\alpha)) \neq 0$. Then the Picard group $\mathrm{Pic}(X)$ is the lattice generated by $\Omega$ and by the fundamental weights associated to the (simple) exceptional roots.

By construction every line bundle has a natural $G$-linearization. Moreover, if $\lambda$ is not dominant, then $\Gamma(Y, \mathcal{L}_{\lambda|Y}) = \mathrm{Ind}_P^G(\boldsymbol{k}_{-\lambda}) = 0$ and hence $\mathcal{L}_\lambda$ is generated by global sections if and only if $\lambda$ is dominant.

Now we describe the sections of a line bundle $\mathcal{L}$ as a $G$-module. The first useful remark is that any irreducible $G$-module appears in $\Gamma(X, \mathcal{L})$ with multiplicity at most one (see Lemma 8.2 in [7]). We analyze first the case of the divisors $S_i$, $1 \le i \le l$. Let $\tilde{\alpha}_1, \dots, \tilde{\alpha}_l$ be the elements of $\tilde{\Delta}$. Then, up to reindexing the $G$-stable divisors, we have

PROPOSITION 1.4 (Corollary 8.2 in [7]). *There exists a unique up to scalar $G$-invariant section $s_i \in \Gamma(X, \mathcal{L}_{\tilde{\alpha}_i})$ whose divisor is $S_i$.*

For an element $\nu = \sum_{i=1}^{l} n_i \tilde{\alpha}_i \in \tilde{R}^+$, the multiplication by $s^\nu \doteq \prod_i s_i^{n_i}$ gives a linear map

$$\Gamma(X, \mathcal{L}_{\lambda-\nu}) \to \Gamma(X, \mathcal{L}_\lambda)\,.$$

If $\mu \in \mathrm{Pic}(X)$ is dominant, then by construction of $\mathcal{L}_\mu$ we certainly have a submodule of $\Gamma(X, \mathcal{L}_\mu)$ isomorphic to $V_\mu^*$, obtained by the pull back of the homogeneous coordinates of $\boldsymbol{P}(V_\mu)$ to $X$.

If $\lambda \in \mathrm{Pic}(X)$ is any element such that $\lambda - \mu \in \tilde{R}^+$, we can consider the image of $V_\mu^*$ under the multiplication by $s^{\lambda-\mu}$ from $\Gamma(X, \mathcal{L}_\mu)$ to $\Gamma(X, \mathcal{L}_\lambda)$. We call this image $s^{\lambda-\mu} V_\mu^*$. We order $\mathrm{Pic}(X)$ by setting $\mu \le_\sigma \lambda$ if $\lambda - \mu$ is a non-negative integer combination of $\tilde{\Delta}$ (i.e., if $\lambda - \mu \in \tilde{R}^+$).

PROPOSITION 1.5 (Theorem 5.10 in [7]). *Let $\lambda \in \mathrm{Pic}(X)$. Then*

$$\Gamma(X, \mathcal{L}_\lambda) = \bigoplus_{\mu \le_\sigma \lambda \text{ and } \mu \in \Lambda^+} s^{\lambda-\mu} V_\mu^*\,.$$

**2. The normalization.** Let $X$ be the complete symmetric variety associated to $(G, \sigma)$, and let $\lambda$ be a dominant weight in the Picard group of $X$. Let $A_n(\lambda) = \Gamma(X, \mathcal{L}_{n\lambda})$, and consider the graded ring $A(\lambda) = \bigoplus_{n \in \boldsymbol{N}} A_n(\lambda)$. Let $B(\lambda)$ be the subring of $A(\lambda)$ generated by the module $V_\lambda^* \subset A_1(\lambda)$; also denote by $B_n(\lambda) = B(\lambda) \cap A_n(\lambda)$ the homogeneous components of $B(\lambda)$. In this Section we prove that the ring $A(\lambda)$ is the normalization of the ring $B(\lambda)$.

On the geometric side, let $\tilde{X}_\lambda \subset \boldsymbol{P}(\Gamma(X, \mathcal{L}_\lambda)^*)$ be the image of $X$ under the morphism $\tilde{\pi}_\lambda$ of $X$ defined by the line bundle $\mathcal{L}_\lambda$ generated by its global sections. Since by [3] the ring $A(\lambda)$ is generated in degree 1, it is the coordinate ring of the cone $\tilde{C}_\lambda$ over $\tilde{X}_\lambda$. By construction $B(\lambda)$ is the coordinate ring of the cone $C_\lambda$ over $X_\lambda$, and we have the following commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "\tilde{\pi}_\lambda"'] \arrow[dr, "\pi_\lambda"] & \\ \tilde{X}_\lambda \arrow[rr, "\eta_\lambda"] & & X_\lambda \end{tikzcd}$$

Notice also that, since $X$ is smooth, the ring $A(\lambda)$ is integrally closed (see for example [10] Exercise II.5.14). Hence to prove that $A(\lambda)$ is the normalization of $B(\lambda)$, it is enough to prove that $\eta_\lambda$ is birational (which clearly implies that the two cones are birational) and that $A(\lambda)$ is integral over $B(\lambda)$.

Our original proof of the following proposition was by far too complicated. We thank Michel Brion who explained us the following proof.

PROPOSITION 2.1. *Let $\lambda$ be a dominant weight in* $\mathrm{Pic}(X)$. *Then $A(\lambda)$ is integral over $B(\lambda)$.*

PROOF. Let $\mathcal{S}(\mathcal{L}_\lambda)$ be the symmetric algebra sheaf constructed over $\mathcal{L}_\lambda$, and let $L = \mathrm{Spec}(\mathcal{S}(\mathcal{L}_\lambda))$ be the total space of the dual of $\mathcal{L}_\lambda$. Further, denote by $M$ the total space of the dual of $\mathcal{O}_{\boldsymbol{P}(V_\lambda)}(1)$. By construction we have the following pull back diagram

$$\begin{tikzcd} L \arrow[r, "\bar{\pi}"] \arrow[d] \arrow[dr, phantom, "PB"] & M \arrow[d] \\ X \arrow[r, "\pi_\lambda"] & \boldsymbol{P}(V_\lambda) \end{tikzcd}$$

By definition $A(\lambda) = \Gamma(L, \mathcal{O}_L) = \Gamma(M, \bar{\pi}_*\mathcal{O}_L)$ and the image of the natural morphism $\Gamma(M, \mathcal{O}_M) \to \Gamma(M, \bar{\pi}_*\mathcal{O}_L)$ is the subring $B(\lambda)$. Now observe that $\bar{\pi}$ is projective, so $\bar{\pi}_*\mathcal{O}_L$ is a coherent sheaf on $M$. Hence $A(\lambda)$ is finite as a $\Gamma(M, \mathcal{O}_M)$-module, or equivalently, as a $B(\lambda)$-module. □

We now prove that $A(\lambda)$ and $B(\lambda)$ have the same quotient field by proving that the map $\eta_\lambda : \tilde{X}_\lambda \to X_\lambda$ is birational. We need the following simple lemma on the dominant order; only for its statement $\Phi$ is an arbitrary irreducible root system.

LEMMA 2.2. *Let $\Phi$ be an irreducible root system and $\Delta \subset \Phi$ a set of simple roots. If $\lambda \in \Lambda \setminus \{0\}$ is a dominant weight, then there exists a positive integer $m$ and a dominant weight $\mu$ such that $\langle \mu, \alpha^\vee \rangle > 0$ for all simple roots $\alpha$ and $m\lambda \geq \mu$ with respect to the dominant order.*

PROOF. Set $\mathrm{supp}(\lambda) \doteq \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \neq 0\}$. If $\mathrm{supp}(\lambda) = \Delta$, there is nothing to prove. So suppose the contrary. Let $\gamma \in \Delta$ be such that there exists $\alpha \in \mathrm{supp}(\lambda)$ with $\langle \gamma, \alpha^\vee \rangle < 0$. Taking $m$ to be any integer greater than 2, we have $\mathrm{supp}(m\lambda - \alpha) \supset \mathrm{supp}(\lambda) \cup \{\gamma\}$. Using this remark and the irreducibility of $\Phi$, the claim follows by an easy induction. □

We recall that an involution $\sigma$ is said to be *simple* if $\mathfrak{g}$ has no $\sigma$-stable proper ideal. It is known (Lemma 15.5.6 in [14]) that for a simple involution $\tilde{\Phi}$ is irreducible and either $\mathfrak{g}$ is simple or $\mathfrak{g} \simeq \mathfrak{g}_1 \times \mathfrak{g}_1$ with $\mathfrak{g}_1$ simple and $\sigma(x, y) = (y, x)$.

LEMMA 2.3. *Let $\sigma$ be a simple involution. Let $\lambda$ be a dominant weight in* $\mathrm{Pic}(X)$ *which is not a multiple of the fundamental weight corresponding to an exceptional root, and let $x_\lambda \in \boldsymbol{P}(V_\lambda)$ be the point fixed by $H$. Then the stabilizer in $G$ of $x_\lambda$ is $H$.*

PROOF. By hypothesis we have $H \subset \mathrm{Stab}_G(x_\lambda)$ and by [7, §1] we know that $H$ is a maximal subgroup having $\mathfrak{h}$ as Lie algebra. So it suffices to show that $\dim \mathrm{Stab}_G(x_\lambda) = \dim H$, or, equivalently, $\dim X_\lambda = \dim X$.

By Proposition 2.1 we also have that $\dim X_\lambda = \dim \operatorname{Proj} A(\lambda)$. Hence we need to verify that the dimension of $\Gamma(X, \mathcal{L}_{n\lambda})$, when $n$ goes to infinity, grows with the same order of infinity of the polynomial $n \mapsto n^{\dim X+1}$. More precisely, since we already know that $\dim X_\lambda \leq \dim X$, it is enough to show that the dimension of $\Gamma(X, \mathcal{L}_{n\lambda})$ grows at least with this order of infinity. Observe that to verify this property we can substitute $\lambda$ by any multiple $m\lambda$, with $m$ a positive integer.

Now, if $\lambda$ is spherical, using the previous lemma 2.2 applied to $\tilde{\Phi}$, we can choose $m$ such that there exists a regular weight $\mu$ with the properties $\mu \leq_\sigma m\lambda$. Then, by Proposition 1.5 we have

$$\dim \Gamma(X, \mathcal{L}_{nm\lambda}) \geq \dim \Gamma(X, \mathcal{L}_{n\mu})\,.$$

Since $\mu$ is regular, $\mathcal{L}_\mu$ is an ample line bundle on $X$. Hence the dimension of the right-hand side has the desired order of growth, by a standard property of the Hilbert polynomial (see Theorem 7.5 in [10]).

Let us now assume that we are in the exceptional case and that $\lambda$ is not a multiple of one of the two exceptional fundamental weights. Write $\lambda = \lambda_0 + n\omega$ with $\lambda_0$ spherical (different from 0 by hypothesis), $n$ positive and $\omega$ an exceptional fundamental weight. In particular, by what we have proved for spherical weights, we can assume the lemma proved for $\lambda_0$.

Consider the projective morphism $\tilde{\pi} = \tilde{\pi}_{\lambda_0} \times \tilde{\pi}_{n\omega} : X \to \tilde{X}_{\lambda_0} \times \tilde{X}_{n\omega}$. It is now clear $\tilde{\pi}$ factors through $\tilde{X}_\lambda$ and this proves our claim. □

In the case $\lambda = m\omega_\beta$ with $\beta$ an exceptional root, we observe that $X$ is not birational to $X_\lambda$. Nevertheless, $\Gamma(X, \mathcal{L}_\lambda) = V_\lambda^*$ in this case, so $\tilde{X}_\lambda$ is equal to $X_\lambda$ and in particular $\eta_\lambda$ is birational.

THEOREM 2.4. *If $\lambda$ is a dominant weight in* $\operatorname{Pic}(X)$*, then $A(\lambda)$ is the normalization of $B(\lambda)$.*

PROOF. We have already observed that $A(\lambda)$ is integrally closed and in Proposition 2.1 have proved that $A(\lambda)$ is integral over $B(\lambda)$. Finally, Lemma 2.3 and the remark above imply that $\eta_\lambda$ is birational, and we see that $A(\lambda)$ and $B(\lambda)$ have the same quotient field. □

**3. Parabolic induction.** In this Section we introduce a family of varieties related to the complete symmetric varieties; one can think of these varieties as a parabolic induction. They share most properties with the complete symmetric varieties; also the proofs of such properties are almost similar. More details can be found in [5] for the case of the group compactification; here we simply generalize the framework of [5], reporting the main results.

As in the previous sections, $G$ will denote a semisimple simply connected algebraic group, $T \subset G$ a maximal torus and $B \subset G$ a Borel subgroup containing $T$. Also, $\sigma : G \to G$ is an involution with the fixed point set $H$. Furthermore, $\bar{G}$ is the adjoint quotient of $G$ and $\bar{H}$ is the fixed point set of the induced involution $\sigma : \bar{G} \to \bar{G}$.

Now we take another semisimple simply connected group $\mathcal{G}$, a maximal torus $\mathcal{T}$ in $\mathcal{G}$ and a Borel subgroup $\mathcal{B} \supset \mathcal{T}$ in $\mathcal{G}$. We shall denote the character group of $\mathcal{T}$ by $\Lambda_{\mathcal{G}}$ and the monoid of dominant weights with respect to $\mathcal{B}$ by $\Lambda^+_{\mathcal{G}}$.

We shall assume that $\mathcal{G}$ contains a parabolic subgroup $\mathcal{P} \subset \mathcal{B}$ having the following property. If $\mathcal{S} \subset \mathcal{P}$ denotes the solvable radical of $\mathcal{P}$, we have a surjective homomorphism $\pi : \mathcal{P}/\mathcal{S} \to \bar{G}$ with finite kernel. Composing $\pi$ with the quotient homomorphism $\mathcal{P} \to \mathcal{P}/\mathcal{S}$, we get a surjection $\pi' : \mathcal{P} \to \bar{G}$. Equivalently, we can assume that we have an isogeny from $G$ to a semisimple Levi factor of $\mathcal{P}$. In particular, we have an action of $G$ on $\mathcal{S}$ and a surjective homomorphism $\gamma : \mathcal{S} \rtimes G \to \mathcal{P}$ with finite kernel. Using $\gamma$, we can consider any $\mathcal{P}$-module, and hence any $\mathcal{G}$-module, as a $\mathcal{S} \rtimes G$-module.

Furthermore, we can clearly assume that the homomorphism $\pi' : \mathcal{P} \to \bar{G}$ takes the Borel subgroup $\mathcal{B}$ to the Borel subgroup $\bar{B}$, which is the image in $\bar{G}$ of $B$ and the maximal torus $\mathcal{T}$ to the maximal torus $\bar{T}$, which is the image in $\bar{G}$ of $T$. We can also assume that under the homomorphism $\gamma : \mathcal{S} \rtimes G \to \mathcal{P}$, $T$ is mapped to $\mathcal{T}$. So we can identify the chosen base $\Delta$ of the root system of $G$ with a subset of $\Delta_{\mathcal{G}}$.

We set $\mathcal{H}$ equal to the preimage under $\pi'$ of the subgroup $\bar{H}$ of $\bar{G}$.

Let us consider now the wonderful compactification $X$ of $G/H$ and define

$$\mathcal{X} \doteq \mathcal{G} \times_{\mathcal{P}} X\,.$$

We want to make a study of some of the properties of $\mathcal{X}$. This study is in fact essentially identical to that of $X$. First of all, notice that, since we have an obvious $\mathcal{G}$-equivariant fibration

$$p : \mathcal{X} \to \mathcal{G}/\mathcal{P}$$

with fiber $X$, we immediately deduce that all $\mathcal{G}$-orbits in $\mathcal{X}$ are of the form $\mathcal{G} \times_{\mathcal{P}} Z$, $Z$ being a $G$-orbit in $X$. This gives a codimension preserving bijection between $\mathcal{G}$-orbits in $\mathcal{X}$ and $G$-orbits in $X$ with the property that, since $\mathcal{G}/\mathcal{P}$ is projective, if $Z$ is a $G$-orbit in $X$, then $\overline{\mathcal{G} \times_{\mathcal{P}} Z} = \mathcal{G} \times_{\mathcal{P}} \bar{Z}$. In particular, each orbit closure in $\mathcal{X}$ is smooth. The complement of the open orbit, which is isomorphic to $\mathcal{G}/\mathcal{H}$, is a divisor $\mathcal{D}$ with normal crossings and smooth irreducible components $\mathcal{D}_i$, $i = 1, \dots, l$, each of which is the closure of a $\mathcal{G}$-orbit.

Furthermore, each orbit closure in $\mathcal{X}$ is the transversal intersection of those among the $\mathcal{D}_i$'s which contain it. Finally, $\mathcal{X}$ contains a unique closed orbit $\bigcap_{i=1}^{l} \mathcal{D}_i$ which is isomorphic to $\mathcal{G} \times_{\mathcal{P}} G/P \simeq \mathcal{G}/\mathcal{Q}$, where $\mathcal{Q}$ is the minimal parabolic subgroup of $\mathcal{G}$ containing $\mathcal{B}$ such that $\mathcal{Q} \cap G = P$.

We are now going to determine the Picard group of $\mathcal{X}$. Recall that, by Proposition 1.2, the homomorphism $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(G/P)$ induced by the inclusion $i : G/P \hookrightarrow X$ as the closed orbit, is injective. Consider now the inclusion $j : \mathcal{G}/\mathcal{Q} \hookrightarrow \mathcal{X}$ as the closed orbit and the inclusion $h : G/P \hookrightarrow \mathcal{G}/\mathcal{Q}$ as the fiber on $\mathcal{P}$ of the fibration $\mathcal{G}/\mathcal{Q} \to \mathcal{G}/\mathcal{P}$. We have

PROPOSITION 3.1. *The homomorphism* $j^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{G}/\mathcal{Q})$ *is injective and has the lattice* $(h^*)^{-1}\big(i^*(\mathrm{Pic}(X))\big)$ *as image.*

Now, having computed the Picard group of $\mathcal{X}$, we can pass to analyze the space of sections of a line bundle on $\mathcal{X}$. Notice that we have an injective map $k : \Lambda \to \Lambda_{\mathcal{G}}$ corresponding to the inclusion $\Delta \subset \Delta_{\mathcal{G}}$ and a surjective map $r : \Lambda_{\mathcal{G}} \to \Lambda$ defined by the restriction of the characters of $\mathcal{T}$ to $T$. Using this last map, we can express the Picard group as $\mathrm{Pic}(\mathcal{X}) = r^{-1}(\mathrm{Pic}(X))$, identifying $\mathrm{Pic}(X)$ with a subset of $\Lambda$. We can further define the elements $\tilde{\beta}_i = k(\tilde{\alpha}_i)$ for $i = 1, \ldots, l$.

PROPOSITION 3.2. *Let $\lambda \in \mathrm{Pic}(\mathcal{X})$ and $L_\lambda$ the corresponding line bundle on $\mathcal{X}$. Then the restriction map*

$$\Gamma(\mathcal{X}, L_\lambda) \to \Gamma(\mathcal{G}/\mathcal{Q}, L_\lambda)$$

*is surjective.*

Once the above result has been established, the following properties of $\mathcal{X}$ are proven exactly as in [7]. The first one is

PROPOSITION 3.3. *We can order the divisors $\mathcal{D}_i$, $i = 1, \ldots, l$ in such a way that, under the above identification, the class in $\mathrm{Pic}(\mathcal{X})$ of $\mathcal{O}(\mathcal{D}_i)$ is $\tilde{\beta}_i$.*

Let us now choose for each $i = 1, \ldots, l$, a non zero section $t_i \in \Gamma(\mathcal{X}, L_{\tilde{\beta}_i})$ whose set of zeros is $\mathcal{D}_i$. Consider the ring

$$\mathcal{A} = \bigoplus_{\lambda \in \mathrm{Pic}(\mathcal{X})} \Gamma(\mathcal{X}, L_\lambda)\,.$$

Given sequences $\underline{h} = (h_1, \ldots, h_l)$ and $\underline{k} = (k_1, \ldots, k_l)$ of non negative integers, we shall say that $\underline{k} \geq \underline{h}$ if $k_i \geq h_i$ for each $i = 1, \ldots, l$ and set $|\underline{h}| = h_1 + \cdots + h_l$. If we now fix such a sequence $\underline{h}$, we set $\mathcal{A}_{\underline{h}}(\lambda)$ equal to the image of the map

$$\Gamma(\mathcal{X}, L_{\lambda - \sum h_i \tilde{\beta}_i}) \to \Gamma(\mathcal{X}, L_\lambda)$$

given by multiplication by $t_1^{h_1} \cdots t_l^{h_l}$. Clearly, $\mathcal{A}_{\underline{k}}(\lambda) \subset \mathcal{A}_{\underline{h}}(\lambda)$ if and only if $\underline{k} \geq \underline{h}$ and $\bigoplus_{\lambda \in \mathrm{Pic}(\mathcal{X})} \mathcal{A}_{\underline{h}}(\lambda)$ is the ideal generated by $t_1^{h_1} \cdots t_l^{h_l}$.

THEOREM 3.4. (1) *For each $\lambda \in \mathrm{Pic}(\mathcal{X})$,*

$$\mathcal{A}_{\underline{h}}(\lambda) / \sum_{\underline{k} > \underline{h}} \mathcal{A}_{\underline{k}}(\lambda)$$

*is isomorphic, as a $\mathcal{G}$-module, to $\Gamma(\mathcal{G}/\mathcal{B}, L_{\lambda - \sum_i h_i \tilde{\beta}_i})$. In particular, as a $\mathcal{G}$-module, we have an isomorphism*

$$\Gamma(\mathcal{X}, L_\lambda) \simeq \bigoplus_{(h_1, \ldots, h_l)} \Gamma(\mathcal{G}/\mathcal{B}, L_{\lambda - \sum_i h_i \tilde{\beta}_i})\,.$$

(2) *If we set*

$$\mathcal{C} = \bigoplus_{\lambda \in \mathrm{Pic}(\mathcal{X})} \Gamma(\mathcal{G}/\mathcal{B}, L_\lambda)$$

*and*

$$\mathcal{A}_i = \bigoplus_{|\underline{h}| = i, \lambda \in \mathrm{Pic}(\mathcal{X})} \mathcal{A}_{\underline{h}}(\lambda) = \sum_{|\underline{h}| = i} t_1^{h_1} \cdots t_l^{h_l} \mathcal{A}\,,$$

*then the associated graded ring*

$$\operatorname{Gr}\mathcal{A} = \bigoplus_{i\geq 0} \mathcal{A}_i/\mathcal{A}_{i+1}$$

*is isomorphic to the polynomial ring* $\mathcal{C}[x_1, \ldots, x_l]$, *where for* $j = 1, \ldots, l$, $x_j$ *is the image of the* $t_j$ *in* $\mathcal{A}_1/\mathcal{A}_2$.

(3) *Let* $\lambda \in \operatorname{Pic}(\mathcal{X})$ *be a dominant weight. Then the ring*

$$\mathcal{A}_\lambda \doteq \bigoplus_{n\geq 0} \Gamma(\mathcal{X}, L_{n\lambda})$$

*is normal with rational singularities.*

As a consequence of the surjectivity of the multiplication maps of $X$ we have

PROPOSITION 3.5. *Let* $\lambda, \mu \in \operatorname{Pic}(\mathcal{X}) \cap \Lambda^+_{\mathcal{G}}$. *Then the multiplication map*

$$\Gamma(\mathcal{X}, L_\lambda) \otimes \Gamma(\mathcal{X}, L_\mu) \to \Gamma(\mathcal{X}, L_{\lambda+\mu})$$

*is surjective.*

We are now going to use the properties of $\mathcal{X}$ to study certain orbit closures. Let us take a representation $M$ of $\mathcal{G}$ and a non zero vector $v \in M$ which, as we can suppose without loss of generality, spans $M$ as a $\mathcal{G}$-module. The assumptions we are going to make on $v$ are

ASSUMPTIONS 3.6. (1) There is a character $\chi : \mathcal{H} \to k^*$ such that $hv = \chi(h)v$ for all $h \in \mathcal{H}$.

(2) Let $W \subset M$ be the $G$-module spanned by $v$. Then $W$ has a highest weight vector.

Let us now make some considerations. By assumption (1) the subgroup $H$ in $G$ fixes $[v]$, so that the orbit map $g \to g \cdot [v]$ factors through the map $G \to G/H$. Moreover, since in an irreducible representation the line fixed by $H$ is unique, the decomposition of $W$ in irreducible modules is multiplicity free. Thus, we get an $G$-equivariant inclusion of the vector space $W$ into the coordinate ring $k[G/H]$. In particular, using assumption (2), we deduce that there is a dominant $\lambda' \in \operatorname{Pic}(X)$ and a subset $\Theta' \subset \tilde{\Sigma}(\lambda')$ containing $\lambda'$, such that, as a $G$-module,

$$W \simeq \bigoplus_{\mu'\in\Theta'} V_{\mu'}\,.$$

Also, by Assumption (1), we have that $\mathcal{H}$ preserves the line spanned by $v$, so that $W$ is stable under the action of $\mathcal{P}$ and $W \subset M^{\mathcal{U}}$, where $\mathcal{U}$ is the unipotent radical in $\mathcal{P}$. Since $v$ spans $M$ as a $\mathcal{G}$-module, we deduce that indeed $W = M^{\mathcal{U}}$. Since the $\mathcal{G}$-module $M$ is irreducible if and only if the $G$-module $M^{\mathcal{U}}$ is irreducible, using our description of $\operatorname{Pic}(\mathcal{X})$, we deduce that there is a subset $\Theta \subset \operatorname{Pic}(\mathcal{X})$ mapped bijectively onto $\Theta'$ by the map $r : \operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(X)$ such that, as a $\mathcal{G}$-module,

$$M \simeq \bigoplus_{\mu\in\Theta} M_\mu\,.$$

Let $\lambda \in \Theta$ be the unique element mapped to $\lambda'$ by $r$. Given a dominant $\mu' = \lambda' - \sum a_i\tilde{\alpha}_i$, we set $\rho(\mu') = \lambda - \sum a_i\tilde{\beta}_i$. Notice that the set $\Theta(\lambda)$ of all $\rho(\mu')$ for $\mu'$ as above, coincides with the set of highest weights of the irreducible components of the $\mathcal{G}$-module $\Gamma(\mathcal{X}, L_\lambda)$.

DEFINITION 3.7. The variety $\mathcal{X}(\Theta)$ is the cone over the orbit $\mathcal{G} \cdot v$, i.e., letting $G_m$ act on $M$ by homotheties, we define

$$\mathcal{X}(\Theta) = \overline{(\mathcal{G} \times G_m) \cdot v}\,.$$

In the following theorem we summarize the properties of the varieties $\mathcal{X}(\Theta)$.

THEOREM 3.8. (1) *$\Theta = \rho(\Theta')$. In particular, $\Theta \subset \Theta(\lambda)$.*

(2) *The variety $\mathcal{X}(\Theta)$ depends, up to isomorphism of $\mathcal{G}$-varieties, only on $\Theta$ and not on the choice of a specific vector $v$ satisfying Assumptions* 3.6.

(3) *The variety $\mathcal{X}(\Theta)$ is normal with rational singularities if and only if $\Theta = \Theta(\lambda)$.*

(4) *For a general $\Theta \subset \Theta(\lambda)$, $\mathcal{X}(\Theta(\lambda))$ is the normalization of $\mathcal{X}(\Theta)$. In particular, $\mathcal{X}(\{\lambda\})$ is normal if and only if $\lambda'$ is minuscule for the restricted root system.*

The theorem above can be extended as follows. Suppose $G = G_1 \times G_2 \times \cdots \times G_n$, with involution $\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$. Let $M_1, \ldots, M_n$ be $\mathcal{G}$-modules and let $v_1, \ldots, v_n$ be vectors with $v_i \in M_i$ each of which satisfying Assumptions 3.6. Assume further that for each $i = 1, \ldots, n$ and $j \neq i$, $G_j$ fixes $v_i$. By what we have already seen, for each $i$, $M_i$ is a highest weight module of highest weight $\lambda_i$, and we get a subset $\lambda_i \in \Theta_i \subset \mathrm{Pic}(\mathcal{X})$ such that $M_i \simeq \bigoplus_{\mu \in \Theta_i} M_{\mu_i}$. Denote by $V$ the subspace in $M = M_1 \oplus \cdots \oplus M_n$ spanned by the vectors $v_1, \ldots, v_n$. We now define $\mathcal{X}(\Theta_1, \ldots, \Theta_n)$ as the closure of $\mathcal{G} \cdot V \subset M$. One then obtains

THEOREM 3.9. (1) *The variety $\mathcal{X}(\Theta_1, \ldots, \Theta_n)$ is normal with rational singularities if and only if $\Theta_i = \Theta(\lambda_i)$ for each $i = 1, \ldots, n$.*

(2) *For a general sequence $\Theta_1, \ldots, \Theta_n$ with $\Theta_i \subset \Theta(\lambda_i)$, the normalization of the variety $\mathcal{X}(\Theta_1, \ldots, \Theta_n)$ is given by $\mathcal{X}(\Theta(\lambda_1), \ldots, \Theta(\lambda_n))$. In particular the variety $\mathcal{X}(\{\lambda_1\}, \ldots, \{\lambda_n\})$ is normal if and only if $\lambda_i'$ is minuscule for the restricted root system of $(G_i, \sigma_i)$ for each $i = 1, \ldots, n$.*

**4. Examples.** In this Section we are going to illustrate a number of examples of varieties of the form $\mathcal{X}(\Theta)$.

EXAMPLE 4.1. Let $0 < h \leq n$ be integers, take $\mathcal{G} = \mathrm{SL}(n)$ and let $\mathcal{P}$ be the parabolic of $\mathcal{G}$ whose Levi factor has semisimple part $G = \mathrm{SL}(h) \times \mathrm{SL}(n-h)$. Consider the involution $\sigma(\mathsf{A}, \mathsf{B}) = ({}^t\mathsf{A}^{-1}, \mathsf{B})$ on $G$, whose fixed point subgroup $H$ in $G$ is clearly $\mathrm{SO}(h) \times \mathrm{SL}(n-h)$. In the $\mathcal{G}$ module $V_{2\omega_1}$ of $n \times n$ symmetric matrices, take the matrix

$$\mathsf{M} = \begin{pmatrix} I_h & 0 \\ 0 & 0 \end{pmatrix}.$$

Denote by $O_h$ the orbit of $\mathsf{M}$ under the action of $G \times G_m$ (with $G_m$ acting by homotheties). Consider the $\mathcal{G}$ equivariant morphism $\bigwedge^r : V_{2\omega_1} \to V_{2\omega_r}$ mapping each matrix to its $r$-exterior power.

Remark that the closure of $\bigwedge^r(O_h)$ is the variety $\mathcal{X}(\{2\omega_r\})$. We can now apply our Theorem 3.8 and conclude that this variety is normal, since $2\omega_r$ is minuscule for the restricted root system of $\sigma$.

Now let $\mathsf{X} = (x_{i,j})_{1\le i,j\le n}$ with $x_{i,j} = x_{j,i}$ be a symmetric matrix of indeterminate. In the polynomial ring $S = K[x_{i,j}]_{i,j}$ consider the subring $S_r$ generated by the determinants of the $r\times r$ minors of $\mathsf{X}$. Let $I_{h+1}$ be the ideal generated by the determinants of the $(h+1)\times(h+1)$ minors of $\mathsf{X}$ and denote by $S_{r,h}$ the image of $S_r$ modulo $I_{h+1}$. Our construction clearly implies that $S_{r,h}$ is the coordinate ring of $\mathcal{X}(\{2\omega_r\})$, hence it is an integrally closed domain (recall that the ideal $I_{h+1}$ is prime [6]) with rational singularities. A similar statement holds for non symmetric matrices (see [2]).

In particular, if $r = 1$, we obtain a (very complicated) proof of the normality of the determinantal varieties for symmetric matrices.

EXAMPLE 4.2. In a similar fashion, using the symplectic involution instead of the orthogonal involution, we get the following result, the details of whose proof we leave to the reader.

Let $\mathsf{X} = (x_{i,j})_{1\le i,j\le n}$ with $x_{i,j} = -x_{j,i}$ be a antisymmetric matrix of indeterminates. In the polynomial ring $S = K[x_{i,j}]_{i,j}$ consider the subring $S_r$ generated by the pfaffians of sizes $2r \times 2r$ of principal minors of $\mathsf{X}$. Let $I_{h+1}$ be the ideal generated by the pfaffians of the $2(h+1)\times 2(h+1)$ principal minors of $\mathsf{X}$ and denote by $S_{r,h}$ the image of $S_r$ modulo $I_{h+1}$. Then the ring $S_{r,h}$ is an integrally closed domain (also in this case where $I_{h+1}$ is a prime ideal [6]) with rational singularities.

In particular, for $r = 1$ we get the normality of the pfaffian varieties for antisymmetric matrices.

EXAMPLE 4.3. Consider the vector space of $n\times n$ symmetric matrices of trace 0. This is the representation of $\mathsf{SO}(n) = \{\mathsf{A} \mid \mathsf{A}^t\mathsf{A} = \mathsf{I},\ \det\mathsf{A} = 1\}$ of highest weight $2\omega_1$. Take the matrix

$$\mathsf{M}_h = \begin{pmatrix} (n-h)\mathsf{I}_h & 0 \\ 0 & -h\mathsf{I}_{n-h} \end{pmatrix}.$$

The stabilizer $H$ of this matrix is isogenous to $\mathsf{SO}(h)\times\mathsf{SO}(n-h)$. It follows that $H$ is the subgroup fixed by the involution given by conjugation with the matrix

$$g = \begin{pmatrix} \mathsf{I}_h & 0 \\ 0 & -\mathsf{I}_{n-h} \end{pmatrix}.$$

The closure $Y_h$ of the orbit of the matrix $\mathsf{M}_h$ under the action of $G\times G_m$ is our variety $\mathcal{X}(\{2\omega_1\})$.

If $h = 1$, the restricted root system is of type $\mathsf{A}_1$ and the Picard group is generated by $\omega_1$ so that $2\omega_1$ is not minuscule. Hence $Y_1$ is not normal.

If $1 < h < \lfloor n/2\rfloor$, the restricted root system is of type $\mathsf{B}_h$ and $\tilde\omega_1 = 2\omega_1$ but $\tilde\omega_1$ is not minuscule for the restricted root system. Hence also in this case $Y_h$ is not normal.

If $h = n/2$ (so $n$ is even), the restricted root system is of type $\mathsf{D}_n$, $\tilde\omega_1 = 2\omega_1$ and $\tilde\omega_1$ is minuscule for the restricted root system. Hence $Y_{n/2}$ is normal.

EXAMPLE 4.4. We can apply our theory to the study of the normality of the closure of nilpotent orbits of height equal to 2 as Domingo Luna pointed out to us.

Let $\mathcal{G}$ be a simple algebraic group over an algebraically closed field of characteristic zero and let $\mathfrak{G}$ be its Lie algebra. Let $\mathcal{O}$ be a nilpotent adjoint orbit, let $e \in \mathcal{O}$ be a non zero element and consider an $\mathfrak{sl}(2)$-triple $(e, h, f)$. Choose a maximal toral subalgebra $\mathfrak{T}$ of $\mathfrak{G}$ containing $h$ and a Borel subalgebra $\mathfrak{B}$ containing $\mathfrak{T}$ and $e$. Let $\alpha_1, \alpha_2, \dots$ be the simple roots defined by the choice of $\mathfrak{T}$ and $\mathfrak{B}$.

The numbers $\alpha_1(h), \alpha_2(h), \dots$ uniquely determine the orbit $\mathcal{O}$, moreover they are non-negative integer numbers less or equal to 2. Let $\theta$ be the highest root for the chosen simple system and define the height of $\mathcal{O}$ as $\mathrm{height}(\mathcal{O}) \doteq \theta(h)$. The height does not depend on the various choices we have made (see [4]); furthermore, $\mathcal{O}$ is spherical (i.e., it has a dense orbit under a Borel subgroup of $\mathcal{G}$) if and only if $\mathrm{height}(\mathcal{O}) \leq 3$ (see [13]). (Notice that this last condition is equivalent to say that $\mathcal{O}$ is $\{0\}$ or it has height equal to 2 or to 3, see again [4].)

Assume for now on that $\mathrm{height}(\mathcal{O}) = 2$. For $n \in \mathbf{Z}$ let $\mathfrak{G}_n = \{x \in \mathfrak{G} \mid [h, x] = nx\}$ and notice that, being $h$ semisimple and $\theta(h) = 2$, we have $\mathfrak{G} = \bigoplus_{-2\leq n\leq 2} \mathfrak{G}_n$. Let $\mathcal{P}$ be the parabolic subalgebra $\bigoplus_{0\leq n\leq 2} \mathfrak{G}_n$ and notice that $\mathfrak{L} = \mathfrak{G}_0$ is a Levi factor of $\mathcal{P}$ and $\mathfrak{U} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$ is the nilpotent radical of $\mathfrak{P}$. Call $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{U}$ the subgroups of $\mathcal{G}$ whose Lie algebras are respectively $\mathfrak{P}$, $\mathfrak{L}$ and $\mathfrak{U}$. Let $\bar{G} = \mathcal{L}/Z(\mathcal{L})$ and let $G$ be the simply connected cover of $\bar{G}$: we have a morphism from $\bar{G}$ onto the derived subgroup $\mathcal{L}'$ of $\mathcal{L}$. In particular, we regard each representation of $\mathcal{G}$ as a representation of $G$ and we identify the Lie algebra $\mathfrak{g}$ of $G$ with $\mathcal{L}'$ through this morphism.

Consider the morphism $SL(2) \to \mathcal{G}$ induced by the $\mathfrak{sl}(2)$-triple $(e, h, f)$ and the image $w$ in $\mathcal{G}$ of the element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in SL(2)$. Observe that $w$ defines an involution of $\mathfrak{L}$ and this induces an involution $\sigma$ of $\bar{G}$ and of $G$. We denote by $H$ the normalizer in $G$ of the subgroup of $\sigma$-fixed points. We want to apply the discussion of Section 3 to deduce the following result originallly proved in [12],

PROPOSITION 4.5. *If $\mathcal{O}$ is a nilpotent adjoint orbit of height* 2, *then the closure of $\mathcal{O}$ is normal.*

PROOF. We want to apply Theorem 3.8 to the group $\mathcal{G}$, the parabolic $\mathcal{P}$, the module $M = \mathfrak{G}$ and the vector $v = e$ (using the notation of the previous Section 3). As in the preliminary discussion of that Theorem we must check that Assumptions 3.6 are verified for such choices. Next we will show that $\mathcal{X}(\{\theta\}) = \bar{\mathcal{O}}$ and that $\theta$ is a spherical $\sigma$-minuscule weight and conclude the proof.

In order to show that Assumptions 3.6 are verified, we denote by $r$ the line spanned by $e$ and claim that:

(i) $\mathfrak{G}_2$ is an irreducible representation of $G$ of highest weight $\theta$;
(ii) $[\mathfrak{U}, e] = 0$;
(iii) $Z(L) \cdot r = r$;
(iv) $H \cdot r = r$.

Indeed, (ii), (iii) and (iv) imply Assumption 3.6 (1), while (i) implies Assumption 3.6 (2).

We begin proving $Z_{\mathfrak{L}}(e) = \mathfrak{L}^\sigma$. The decomposition of the $SL(2)$-module $\mathfrak{G}_{-2} \oplus \mathfrak{L} \oplus \mathfrak{G}_2$ into isotypical components is given by $(V \otimes \mathfrak{G}_2) \oplus Z_{\mathfrak{L}}(e)$, where $V$ is the three-dimensional

$SL(2)$-module, and $\mathfrak{G}_2$ and $Z_{\mathfrak{L}}(e)$ are considered as trivial $SL(2)$-module. Now let $V_0$ be the zero weight space of $V$ and notice that $w$ acts by multiplication by $-1$ on $V_0$. So we have a $\sigma$-equivariant morphism $\mathfrak{L} \simeq (V_0 \otimes \mathfrak{G}_2) \oplus Z_{\mathfrak{L}}(e)$. Hence we conclude $\mathfrak{L}^\sigma = Z_{\mathfrak{L}}(e)$ proving our claim.

So, in particular, the Lie algebra $\mathfrak{h} \doteq [\mathfrak{L}, \mathfrak{L}]^\sigma$ of $H$ commutes with $e$.

Now notice that $[\mathfrak{G}_i, \mathfrak{G}_j] \subset \mathfrak{G}_{i+j}$. From this we derive at once (ii) and also that any highest weight vector for $G$ in $\mathfrak{G}_2$ is also a highest weight vector for $\mathcal{G}$. This shows that $\mathfrak{G}_2$ has $x_\theta$ (the root vector of weight $\theta$) as its unique highest weight vector. Hence $\mathfrak{G}_2$ is a spherical (irreducible) $G$-representation, and (i) and (iv) follow.

Now also (iii) follows, since the vector fixed by $\mathfrak{h}$ is unique up to scalar ($\mathfrak{G}_2$ being irreducible) and $Z(L)$ commutes with $\mathfrak{h}$. This finishes the proof that the Assumptions 3.6 are fulfilled.

Notice that the orbit $\mathcal{O} = \mathcal{G} \cdot e$ is already a cone, since $[h, e] = 2e$, and it is now clear that its closure is $\mathcal{X}(\{\theta\})$, using the notation of the previous Section. So we are now in a position to apply Theorem 3.8, and the normality of $\mathcal{O}$ follows once we show that the spherical $G$-representation $\mathfrak{G}_2$ is $\sigma$-minuscule.

We choose the set of positive roots for $\mathfrak{g}$ with torus $\mathfrak{T} \cap \mathfrak{g}$ according to the Borel $\mathcal{B}$ of $\mathcal{G}$.

Suppose now that $\mathfrak{G}_2$ has a vector, not multiple of $x_\theta$, which is also a dominant weight vector with respect to $\mathfrak{g}$. We can assume that it is a root vector $x_\beta$. Notice that, since $x_\beta \in \mathfrak{G}_2$, we have obtained $\beta$ by subtracting from $\theta$ simple roots in the root system of $\mathfrak{g}$. So $\beta$ is dominant also for $\mathfrak{G}$.

In particular, if $\mathfrak{g}$ is simply laced, this is not possible, and hence $\mathfrak{G}_2$ is minuscule as a $\mathfrak{g}$-module, which is also $\sigma$-minuscule.

If $\mathfrak{g}$ is not simply laced, then this forces $\beta = \bar{\theta}$ the highest short root for $\mathfrak{G}_0$. Also, the support of $\theta - \bar{\theta}$ (i.e., the set of simple roots appearing with nonzero coefficient in the expression of $\theta - \bar{\theta}$ as a sum of simple roots) is contained in the root system of $\mathfrak{g}$. We conclude now by analyzing the remaining possibilities (we use the numbering convention as in Bourbaki [1]):

*Case* $\mathsf{B}_l$, $l \geq 3$: We have $\theta - \bar{\theta} = \alpha_2 + \cdots + \alpha_l$, and $\theta = \omega_1$. So $\mathfrak{g}$ is the semisimple part of the Levi factor whose root system is generated by $\alpha_2, \ldots, \alpha_l$ and $\mathfrak{G}_2$ is the trivial representation.

*Case* $\mathsf{C}_l$, $l \geq 2$: We have $\theta - \bar{\theta} = \alpha_1$ and $\theta = 2\omega_1$. In this case $\mathfrak{g}$ is the semisimple part of a Levi factor whose root system contains $\alpha_1$. In particular, $\mathfrak{g}$ need not to be simple but certainly the root $\alpha_1$ is contained in a factor $\mathfrak{g}_1$ of type $\mathsf{A}$ of the Lie algebra $\mathfrak{g}$: i.e., $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1$ simple and of type $\mathsf{A}$. Notice now that, being the highest weight of $\mathfrak{G}_2$ equal to $2\omega_1$, $\mathfrak{g}_2$ acts trivially on $\mathfrak{G}_2$. In particular, since $Z_{\mathfrak{g}}(e) = \mathfrak{g}^\sigma$, the involution leaves this part fixed and $\mathfrak{G}_2$ is a spherical representation of weight $2\omega_1$ of $\mathfrak{g}_1$. Finally, for groups of type $\mathsf{A}$ there is only one involution that has the representation of highest weight $2\omega_1$ has a spherical representation and this is the orthogonal involution for which $2\omega_1$ is a $\sigma$-minuscule spherical weight.

*Case* $\mathsf{F}_4$: We have $\theta - \bar{\theta} = \alpha_1 + \alpha_2 + \alpha_3$ and $\theta = \omega_1$. So $\mathfrak{g}$ is the semisimple part of the Levi factor whose root system is generated by $\alpha_1, \alpha_2, \alpha_3$. Hence $\mathfrak{G}_2$ is the representation of highest weight $\omega_1$ of a groups of type $\mathsf{B}_3$: a minuscule representation and in particular a $\sigma$-minuscule representation.

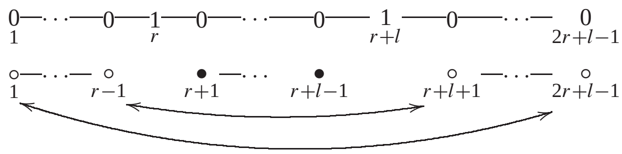
*Case* $\mathsf{G}_2$: We would have $\theta - \bar{\theta} = \alpha_1 + \alpha_2$ and so $\mathfrak{g} = \mathfrak{G}$, which is not possible. □

REMARK 4.6. We can complete the analysis of the proof given above by describing for each nilpotent orbit of height 2 the associated involution.

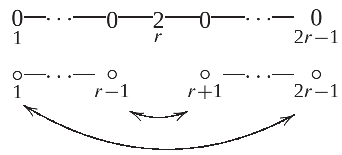
This is easily done as follows. From the Dynkin-Kostant diagram we deduce the Levi factor $\mathfrak{L}$ and from the expression of $\theta$ we deduce which representation is the spherical representation $\mathfrak{G}_2$. Now from the classification of the involutions, it is easy to see that in each case there is only one involution that has the representation $\mathfrak{G}_2$ as a faithful spherical representation. (With being faithful here we mean that the stabilizer of the spherical vector $e$ in $\mathfrak{G}_2$ is exactly $\mathfrak{h}$ and not bigger.) We have to make this remark because the Levi factor is in general not simple.

We list in the table below the result of this analysis. For each nilpotent orbit of height 2 we report the Lie algebra $\mathfrak{G}$, the module $\mathfrak{G}_2$ and two diagrams: the first is the Dynkin-Kostant diagram and the second one is the Satake diagram of the involution $\sigma$ on $\mathfrak{g}$. For the classical cases we give also the partition related to the nilpotent orbit.

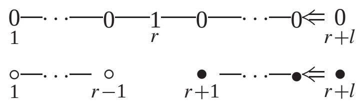
(1.1) $\mathfrak{sl}(2r+l),\ l \geq 1\ :\ \mathfrak{G}_2 \simeq \mathrm{End}(\boldsymbol{k}^r),\ \pi = 1^l 2^r$



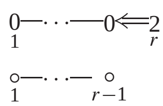

(1.2) $\mathfrak{sl}(2r)\ :\ \mathfrak{G}_2 \simeq \mathrm{End}(\boldsymbol{k}^r),\ \pi = 2^r$



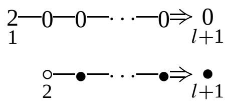

(2.1) $\mathfrak{sp}(2(r+l)),\ l \geq 1\ :\ \mathfrak{G}_2 \simeq S^2(\boldsymbol{k}^r),\ \pi = 1^{2l} 2^r$



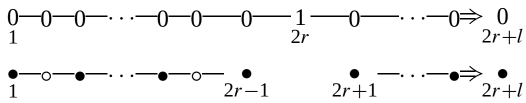

(2.2) $\mathfrak{sp}(2r)\ :\ \mathfrak{G}_2 \simeq S^2(\boldsymbol{k}^r),\ \pi = 2^r$



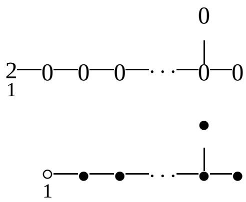
(3.1) $\mathfrak{so}(2l+3), l \geq 2 \, : \, \mathfrak{G}_2 \simeq \boldsymbol{k}^{2l+1}, \; \pi = 1^{2l}3^1$



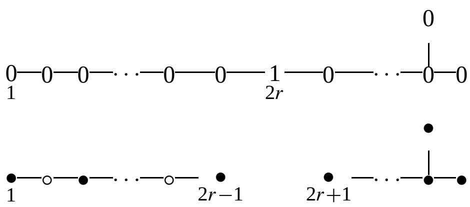
(3.2) $\mathfrak{so}(4r+2l+1) \, : \, \mathfrak{G}_2 \simeq \Lambda^2\boldsymbol{k}^{2r}, \; \pi = 1^{2l+1}2^{2r}$



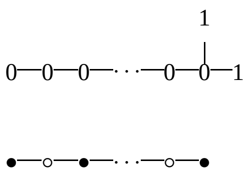
(4.1) $\mathfrak{so}(2l+4) \, : \, \mathfrak{G}_2 \simeq \boldsymbol{k}^{2l+2}, \; \pi = 1^{2l+1}3^1$



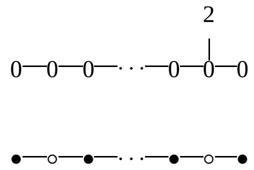
(4.2) $\mathfrak{so}(4r+2l), \; l \geq 2 \, : \, \mathfrak{G}_2 \simeq \Lambda^2\boldsymbol{k}^{2r}, \; \pi = 1^{2l}2^{2r}$



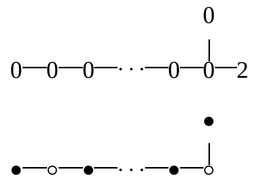
(4.3) $\mathfrak{so}(4r+2) \, : \, \mathfrak{G}_2 \simeq \Lambda^2\boldsymbol{k}^{2r}, \; \pi = 1^2 2^{2r}$



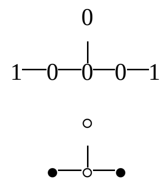
(4.4) $\mathfrak{so}(4r) \,:\, \mathfrak{G}_2 \simeq \Lambda^2 \boldsymbol{k}^{2r},\ \pi = (2^{2r})^I$



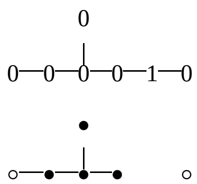
(4.5) $\mathfrak{so}(4r) \,:\, \mathfrak{G}_2 \simeq \Lambda^2 \boldsymbol{k}^{2r},\ \pi = (2^{2r})^{II}$



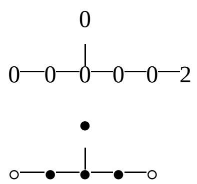
(5) $\mathsf{E}_6 \,:\, \mathfrak{G}_2 \simeq \boldsymbol{k}^8$



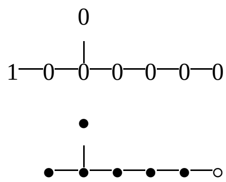
(6.1) $\mathsf{E}_7 \,:\, \mathfrak{G}_2 \simeq \boldsymbol{k}^{10}$



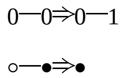
(6.2) $\mathsf{E}_7 \,:\, \mathfrak{G}_2 \simeq \boldsymbol{k}^{27}$



(7) $\mathsf{E}_8 : \mathfrak{S}_2 \simeq \boldsymbol{k}^{14}$



(8) $\mathsf{F}_4 : \mathfrak{S}_2 \simeq \boldsymbol{k}^{7}$



We hope that it is possible to develop a similar approach also for the nilpotent orbits of height equal to 3. In this case symmetric varieties should be replaced with a more general class of spherical varieties.

## REFERENCES


[ 1 ] N. BOURBAKI, Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines, Actualités Scientifiques et Industrielles 1337, Hermann, Paris, 1968.

[ 2 ] W. BRUNS, Algebras defined by powers of determinantal ideals, J. Algebra 142 (1991), 150–163.

[ 3 ] R. CHIRIVÌ AND A. MAFFEI, Projective normality of complete symmetric varieties, Duke Math. J. 122 (2004), 93–123.

[ 4 ] D. H. COLLINGWOOD AND W. M. MCGOVERN, Nilpotent orbits in semisimple Lie algebras, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.

[ 5 ] C. DE CONCINI, Normality and non normality of certain semigroups and orbit closures, Algebraic transformation groups and algebraic varieties, 15–35, Encyclopaedia Math. Sci., 132, Springer, Berlin, 2004.

[ 6 ] C. DE CONCINI AND C. PROCESI, A characteristic free approach to invariant theory, Advances in Math. 21 (1976), 330–354.

[ 7 ] C. DE CONCINI AND C. PROCESI, Complete symmetric varieties, Invariant theory (Montecatini, 1982), 1–44, Lecture Notes in Math. 996, Springer, Berlin, 1983.

[ 8 ] C. DE CONCINI AND T. A. SPRINGER, Compactification of symmetric varieties, Transform. Groups 4 (1999), 273–300.

[ 9 ] G. FALTINGS, Explicit resolution of local singularities of moduli-spaces, J. Reine Angew. Math. 483 (1997), 183–196.

[10] R. HARTSHORNE, Algebraic geometry, Grad. Texts in Math. 52, Springer, New York, 1977.

[11] S. HELGASON, Differential geometry, Lie groups, and symmetric spaces, Pure Appl. Math. 80, Academic Press, Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[12] W. HESSELINK, The normality of closures of orbits in a Lie algebra., Comment. Math. Helv. 54 (1979), 105–110.

[13] D. I. PANYUSHEV, Some amazing properties of spherical nilpotent orbits, Math. Z. 245 (2003), 557–580.

[14] T. VUST, Opération de groupes réductifs dans un type de cônes presque homogènes, Bull. Soc. Math. France 102 (1974), 317–333.

Università di Pisa
Dipartimento di Matematica
"Leonida Tonelli"
via Buonarroti n. 2, 56127 Pisa
Italy

*E-mail address*: chirivi@dm.unipi.it

Università di Roma "La Sapienza"
Dipartimento di Matematica
"Guido Castelnuovo"
Piazzale Aldo Moro n. 5, 00185 Roma
Italy

*E-mail address*: deconcin@mat.uniroma1.it

Università di Roma "La Sapienza"
Dipartimento di Matematica
"Guido Castelnuovo"
Piazzale Aldo Moro n. 5, 00185 Roma
Italy

*E-mail address*: amaffei@mat.uniroma1.it