Pacific Journal of Mathematics

ON NORMALITY OF THE CLOSURE OF A GENERIC TORUS ORBIT IN G/P

ROMUALD DABROWSKI

Volume 172 No. 2 February 1996

PACIFIC JOURNAL OF MATHEMATICS
Vol. 172, No. 2, 1996

ON NORMALITY OF THE CLOSURE OF A GENERIC TORUS ORBIT IN G/P

ROMUALD DABROWSKI

In this paper we consider generic orbits for the action of a maximal torus T in a connected semisimple algebraic group G on the generalized flag variety G/P, where P is a parabolic subgroup of G containing T. The union of all generic T-orbits is an open dense (possibly proper, if P is not a Borel subgroup) subset of the intersection of the big cells in G/P. We prove that the closure of a generic T-orbit in G/P is a normal equivariant T-embedding (whose fan we explicitely describe). Moreover, the closures of any two generic T-orbits are isomorphic as equivariant T-embeddings.

1. Introduction.

Let G be a connected semisimple algebraic group over an algebraically closed field k of arbitrary characteristic. As usual, let B^+ denote a fixed Borel subgroup of G, T a maximal torus in B^+, $\Gamma(T)$ the character group of T, B the opposite to B^+, Φ the corresponding root system in an euclidian space $(E, (\ ,\ ))$, Φ_+ the set of positive roots relative to B^+, Δ the set of simple roots in Φ_+, s_α the reflection about the linear subspace of E perpendicular to root α, W the Weyl group of Φ generated by the reflections $s_\alpha, \alpha \in \Phi_+$ (W can also be naturally identified with $N_G(T)/T$), and R the root lattice in E.

Let P be a fixed parabolic subgroup containing B. Let Δ_P be the set of simple roots α such that $s_\alpha \in W_P = N_P(T)/T$. Then the map $P \to \Delta_P$ is a bijection between the set of all parabolic subgroups containing B and the power set of Δ (see e.g. [**B**, Proposition 14.18]). We denote by S^P the subsemigroup of the root lattice generated by all positive roots which are not sums of simple roots in Δ_P.

We will be concerned with T-orbits of points in the projective variety G/P. Let λ be an integral dominant weight (with respect to Φ_+) whose stabilizer in W is W_P Then λ extends to a character of P (we will also call it λ), inducing a line bundle $\mathcal{L}^\lambda$ on G/P. We let $V(\lambda)$ denote the Weyl G-module

$$H^0(G/P, \mathcal{L}^\lambda) = \{f \in k[G] | f(xy) = \lambda^{-1}(y) f(x) \text{ for all } x \in G, y \in P\}$$

of global sections of $\mathcal{L}^\lambda$ (see e.g. [**J**, Sec. 5.8, p. 84]).

Let Π_λ denote the set of weights of $V(\lambda)$ for the action of T. Let $\mathcal{A}_\lambda$ denote the set of weights of $V(\lambda)$ listed with multiplicity. For each $\mu \in \mathcal{A}_\lambda$, we pick a corresponding weight vector (function) f_μ so that $\{f_\mu | \mu \in \mathcal{A}_\lambda\}$ is a basis of $V(\lambda)$. Functions $f_\mu, \mu \in \mathcal{A}_\lambda$, are called the Plücker coordinates in G/P. By abuse of language we use f_μ to denote any Plücker coordinate of a given weight μ. Let $x = u.P$ be an element of G/P. We let $\Pi_\lambda(x)$ denote the set of weights $\mu \in \Pi_\lambda$ such that at least one of the Plücker coordinates f_μ does not vanish at u. It is easy to see that $\Pi_\lambda(x)$ depends on x and λ only (not on the choice of the Plücker coordinates). It turns out that $\lambda - \Pi_\lambda \subseteq S^P$. Hence by W-invariance of Π_λ, $\lambda - w\Pi_\lambda(x) \subseteq S^P$, for any $x \in G/P$ and $w \in W$. Intuitively, a torus orbit $Tx \subset G/P$ can be called generic if sufficiently many Plücker coordinates of x do not vanish. The following definition makes this requirement precise.

Definition 1.1. Let x be an element of G/P. Then the torus orbit $Tx \subset G/P$ is called *generic* if and only if $\{w\lambda | w \in W\} \subseteq \Pi_\lambda(x)$, and for each $w \in W$, the semigroup generated by $\lambda - w\Pi_\lambda(x)$ is S^P (that is, the maximal semigroup that $\lambda - w\Pi_\lambda(x)$ can generate).

We will show that this definition does not depend on the choice of λ. It turns out that $\Pi_\lambda(x) = \Pi_\lambda$ implies Tx is generic. Therefore generic orbits exist since there are points in G/P at which all Plücker coordinates do not vanish. We will also prove that in the case of G/B, Tx is generic if and only if x belongs to $\bigcap_{w \in W/W_P} wB^+.P$.

The aim of this note is to prove that the closure of a generic T-orbit in G/P is a normal equivariant T-embedding. We can then use the general theory of equivariant torus embeddings (see e.g. [**K**, **Oda1**]) to show that the closures of any two generic orbits are isomorphic (as equivariant T-embeddings). We prove this by identifying the fan describing the isomorphism class of these T-embeddings.

Remark. We point out that if $P \neq B$, the definition of generic T-orbit given here differs from the one used in [**F-H**, Remark 1, p. 257]. There, an orbit Tx is called "generic" if and only if x belongs to the non-degenerate stratum $Z = \bigcap_{w \in W/W_P} wB^+.P$ in the stratification of G/P introduced in [**G-S**] (note that in [**F-H**] B is the "positive" Borel subgroup, while here B denotes the "negative" Borel subgroup). It is easy to see that the set of all $x \in G/P$ with Tx generic in the sense of Definition 1.1 is an open subset of Z. It is proved in [**G-S**, Section 5.1, Proposition 1] that if k is the field of complex numbers then the image under the moment map of the closure of each torus orbit contained in Z is the convex hull of $\{w\lambda | w \in W\}$. In [**F-H**] the general theory of torus embeddings is used to study the closure of Tx in

G/P for $x \in Z$. It appears however that normality of these varieties, required in the theory, has not been proved (as pointed out in [**Oda2**, Section 2.6]). Also, contrary to what is claimed in [**F-H**], two T-orbits in Z may have nonisomorphic closures in G/P (see the example below).

Example. Let $\mathbf{C}$ denote the field of complex numbers. Let q be a nondegenerate quadratic form on $V = \mathbf{C}^5$, and let $G = SO(q)$ be the subgroup of determinant one linear transformations of V, preserving q. Then G is a connected, semisimple, rank 2 algebraic group over $\mathbf{C}$, and V is an irreducible representation of G. Let L be a fixed isotropic line for q (that is $q(v) = 0$ for all $v \in L$), and let $P \subset G$ be the stabilizer of L. Then P is a parabolic subgroup of G, and G/P is naturally isomorphic to the smooth quadric hypersurface Q in the complex projective space $\text{Proj}(V)$ given by the homogeneous equation $q(x) = 0$. For brevity, we will equate G/P with Q. Let $\{e_1, e_2, e_3, e_4, e_4\}$ be the standard basis of V and let $q(x) = x_1x_3 + x_2x_4 - 2x_5^2$, where $[x_1, x_2, x_3, x_4, x_5]$ are the coordinates of $x \in V$ relative to the standard basis. We let $L = \mathbf{C}e_1$. Then the maximal torus contained in P is $T = \{\text{diag}(t_1, t_2, 1/t_1, 1/t_2, 1) | t_i \in \mathbf{C} \setminus \{0\}, i = 1, 2\}$. Here, the Plücker coordinates in $Q = G/P$ are just the standard homogeneous coordinates in $\text{Proj}(V)$. Clearly, $L_1 = \mathbf{C}[1, 1, -1, 1, 0]$ and $L_2 = \mathbf{C}[1, 1, 1, 1, 1]$ are q-isotropic. Also, the T-orbits of L_1 and L_2 are "generic" in the sense of [**F-H**], but only TL_2 is generic in the sense of the Definition 1.1. Also, $\Pi(L_1) \neq \Pi(L_2) = \Pi$, where Π denotes the set of weights of V. This directly contradicts Lemma 13 in [**F-H**]. Let $X_i = \overline{TL_i}, \quad i = 1, 2$, where the closure is taken in Q (or in $\text{Proj}(V)$, since Q is closed in $\text{Proj}(V)$). It is easy to see that X_1 is isomorphic to $\mathbf{C}P^1 \times \mathbf{C}P^1$. On the other hand X_2 is the singular closed subvariety of $\text{Proj}(V)$ given by homogeneous equations $x_1x_3 = x_5^2$, $x_2x_4 = x_5^2$ (the singular points of X_2 are $[1:1:0:0:0]$, $[1:0:0:1:0]$, $[0:1:1:0:0]$, and $[0:0:1:1:0]$). Therefore, the example shows that two T-orbits "generic" in the sense of [**F-H**] may not have isomorphic closures in G/P.

2. Weights of Weyl G-modules.

We will need the following notation. For any additive set A of real numbers and any subset Y of E, let AY denote the set of all linear combinations of elements in Y with coefficients in A. By definition, a semigroup S contained in a lattice L in E is saturated in L if and only if

$$L \cap \mathbf{Q}_+ S = S$$

(see [**K**, Chapter 1, Section 1]). Equivalently, S is saturated in L if and only if for any positive integer m, $m\mu \in S$ and $\mu \in L$ imply $\mu \in S$.

Proposition 2.1. *S^P is saturated in R.*

Proof. Let Φ_+^P denote the set of positive roots which are not linear combinations of roots in Δ_P. Then $S^P = \mathbf{Z}_+\Phi_+^P$. Suppose that S^P is not saturated in R. Let $\mu \in R$ be an element of minimal height among the elements of $\mathbf{Q}_+\Phi_+^P$ which are not elements of S^P. Then $\mu = \mu_1 + \mu_2$, with

$$\mu_1 = \sum_{\beta \in M} m_\beta \beta$$

where $M \subseteq \Phi_+^P$, m_β are positive integers, and

$$\mu_2 = \sum_{\alpha \in N} n_\alpha \alpha$$

where $N \subseteq \Delta_P$, and n_α are positive integers. From the above decompositions of μ we choose one with μ_2 of minimal height. Since the sum of any two roots with negative scalar product is again a root, minimality of μ_2 implies that

$$(\alpha, \beta) \geq 0,$$

for all $\alpha \in N, \beta \in M$. Take any simple root α in N, such that $(\mu_2, \alpha) > 0$. Consider $\nu = s_\alpha(\mu) \in R$. Since elements of Φ_+^P are permuted by s_α, ν belongs to $\mathbf{Q}_+\Phi_+^P$ but not to S^P. This is a contradiction, since $\mathrm{ht}(\nu) < \mathrm{ht}(\mu)$ and μ was assumed to be of minimal height among the root lattice elements in $\mathbf{Q}_+\Phi_+^P$, not in S^P. □

Let $V(\lambda), \lambda, \Pi_\lambda$ be as in the introduction. The following proposition lists some basic properties of Π_λ.

Proposition 2.2.
(i) $\lambda - \Pi_\lambda$ *coincides with the set of root lattice points in the convex hull of* $\{\lambda - w\lambda | w \in W\}$.
(ii) S^P *is generated by* $\lambda - \Pi_\lambda$. *If* $P = B$ *and* λ *is the sum of fundamental weights then* S^P *is generated by* $\{\lambda - w\lambda, w \in W\}$.

Remark. Part (i) is well known, but were not able to locate an appropriate reference.

Proof. We first observe that the weights of the Weyl module $V(\lambda)$ (λ integral dominant) are independent of the characteristic of k. This follows from the fact that character formulas for Weyl modules are the same in each characteristic. Therefore we can assume, that $\mathrm{char}(k) = 0$.

Part (i). Let C denote the convex hull of $\{w\lambda | w \in W\}$ and we let $\Pi = (\lambda + R) \cap C$. We have to prove that $\Pi = \Pi_\lambda$. It is a known fact that $\Pi_\lambda \subset \lambda + R$.

Therefore it is enough to show that Π_λ is contained in C. Suppose that this is not the case, and let μ be a weight in Π_λ, not in C. Assume also, that μ is a maximal such weight in the usual order in E relative to Φ_+. Since both Π_λ and C are $W-$invariant, we must have $s_\alpha(\mu) \leq \mu$ for all positive roots α. Hence μ is dominant. Since μ is not the highest weight λ, there must be a positive root α and a positive integer m such that $\mu_1 = \mu + m\alpha \in \Pi_\lambda$. Then by maximality of μ, μ_1 (hence also $s_\alpha(\mu_1)$) is in C. A straightforward computation shows that μ belongs to the line segment connecting μ_1 and $s_\alpha(\mu_1)$. This is a contradiction, since we have assumed that μ is not in C.

We are left with showing that Π is contained in Π_λ. An easy argument by induction on the length function in W, shows that for any $w \in W$, $\lambda - w\lambda$ is a sum of roots in Φ_+^P. Therefore Π is contained in $\lambda - \mathbf{Z}_+\Phi_+$. It is proved in [**H**, Proposition, p. 114] that the elements of Π_λ are exactly the weights whose W-orbit is contained in $\lambda - \mathbf{Z}_+\Phi_+$. Hence $\Pi \subseteq \Pi_\lambda$, as required.

Part (ii) We have observed in the proof of Part (i) that $\lambda - C$ is contained in convex cone spanned by Φ_+^P. Therefore

$$\lambda - \Pi_\lambda = R \cap (\lambda - C) \subseteq S^P$$

since S^P is saturated in R. The opposite inclusion holds since $\Phi_+^P \subseteq \lambda - \Pi_\lambda$. This follows from the fact that weigths of irreducible $G-$representations (in characteristic 0) satisfy the following property: for any positive root α, and a positive integer n, if μ and $\mu - n\alpha$ are weights of the representation, so are $\mu - q\alpha$ for any $q, 0 \leq q \leq n$ (see e.g. [**H**, Sec. 21.3, Prop.]). One applies this property to λ and $s_\alpha(\lambda)$, where $\alpha \in \Phi_+^P$.

The second claim of Part(ii) follows since $\Delta \subseteq \{\lambda - w\lambda | w \in W\}$ if λ is the sum of fundamental weights. □

3. Generic orbits of T in G/P.

Let $x \in G/P$ and let X denote the the closure of Tx in G/P. For any $w \in W$, let

$$Y_w = \{y.P | f_{w\lambda}(y) \neq 0\} = \{y.P | w\lambda \in \Pi_\lambda(y.P)\}$$

and

$$X_w = Y_w \cap X.$$

It is well known that each Y_w is an affine space which is open in G/P and whose coordinate ring is generated by functions $f_\mu / f_{w\lambda}, \mu \in \mathcal{A}_\lambda$. Moreover, the union of $Y_w, w \in W$ is G/P. Let $T_x = \{t \in T | tx = x\}$ and $T^x = T/T_x$. We have the following proposition

Proposition 3.1. *Let $x \in G/P$.*

(i) *Tx is open in X and it is isomorphic to T^x. Therefore, X is an equivariant T^x−embedding in the sense of* [**K**].

(ii) *$\{X_w | w \in W, w\lambda \in \Pi_\lambda(x)\}$ is a covering of X by T-invariant open affine subsets of X. The coordinate ring of $X_w, w\lambda \in \Pi_\lambda(x)$, is the subalgebra of $k[T^x] = k[\Gamma(T^x)]$ generated by $\Pi_\lambda(x) - w\lambda$.*

(iii) *Let $w \in W$ be such that $w\lambda \in \Pi(x)$. Then*

$$T_x = \{t \in T | \mu(t) = 1 \ for \ \ all \ \ \mu \in w\lambda - \Pi(x)\},$$

Proof. The fist part of (i) follows from the fact the map $t \to tx$ is a separable morphism from T onto an open subvariety Tx of X whose fibers are the cosets of T_x in T (the morphism is separable since it is the composition of the inclusion of T in G with the quotient map from G to G/P).

Part (ii) follows, since for each $w \in W$ such that $w\lambda \in \Pi_\lambda(x)$, X_w can be viewed as a closed T-invariant subvariety of the affine space Y_w. Hence the coordinate ring of X_w is generated by the restrictions to X_w of functions $f_\mu/f_{w\lambda}, \mu \in \mathcal{A}_\lambda$.

Part(iii). Suppose that $w \in W$ satisfies $w\lambda \in \Pi(x)$. Then $x \in X_w$. Clearly, $t \in T_x$ if and only if t fixes all elements of X_w (or equivalently, t fixes all regular functions on X_w). Therefore the desired formula for T_x follows from the description of the coordinate ring of X_w given in (ii). □

Before we state a corollary of Proposition 3.1, we need to introduce the following notation. Let R^P denote the subgroup of the root lattice generated by S^P. One can show that $R^P = R$ if Φ is an irreducible system. If Φ a union of irreducible root systems $\Phi_j, j \in J$, then R^P is the root lattice of the root system

$$\cup\{\Phi_j | \Phi_j \cap S^P \neq \emptyset\}.$$

Let

$$T_P = \bigcap_{\nu \in R^P} \ker(\nu).$$

Note that if $R^P = R$, then T_P is coincides with the center of G.

Corollary. (Suggested by the referee.)

(i) *The stabilizer of each generic torus orbit is T_P. Moreover, T_P it is the smallest subgroup of T among the T-stabilizers of elements of G/P.*

(ii) *(Partial converse of* (i)*). If $x \in G/P$ is such that Tx is contained in the nondegenerate stratum Z, $\overline{Tx}$ is normal and $T_x = T_P$, then Tx is generic.*

Proof. Part (i) follows from Proposition 3.1 (iii). Suppose that Tx satisfies the assumptions of (ii). Let S^x denote the semigroup generated by $\lambda - \Pi_\lambda(x)$. We have to show that $S^x = S^P$. Since $T_x = T_P$, one has

$$\bigcap_{\nu \in R^P} \ker(\nu) = \bigcap_{S^x} \ker(\nu)$$

by Proposition 3.1 (iii). Therefore R^P is generated by S^x as a subgroup of $\Gamma(T)$. Assumed normality of $\overline{Tx}$ implies that S^x is saturated in R^P. On the other hand $\{\lambda - w\lambda | w \in W\} \subset S^x$ since Tx is assumed to be generic. Hence $S^x = S^P$ since both semigroups are saturated in R^P and $\mathbf{Q}_+ S^x = \mathbf{Q}_+ S^P$ by Proposition 2.2. □

From now on we assume for simplicity that $R^P = R$ (equivalently, S^P contains at least one root from each irreducible component of Φ). Let $W^P \subseteq W$ be a fixed set of representatives of W/W_P. Let D denote the fundamental chamber $\{\nu \in E | (\mu, \alpha) \geq 0 \text{ for all } \alpha \in \Delta\}$. We are now ready to state the main result of this paper.

Theorem 3.2. *Let $x \in G/P$ be such that $Tx \subset G/P$ is generic. Let $X = \overline{Tx}$. Then:*

(i) *X is a normal variety (hence by* [**K**, Theorem 14, page 52], *also Cohen-Macaulay with rational singularities).*

(ii) *The fan corresponding to X consists of the cones*

$$C_w = -w \bigcup_{z \in W_P} zD, \quad w \in W^P$$

together with their faces. In particular, the closures of any two generic orbits in G/P are isomorphic as T-equivariant embeddings.

Proof. Part (i). By [**K**, Theorem 6, p. 24] a general equivariant T-embedding is a normal variety if and only if it admits a covering by open affine T-stable subvarieties whose coordinate rings are generated by semigroups saturated in $\Gamma(T)$. Hence Part(ii) follows from Propositions 3.1 and 2.1.

Part(ii) follows, since the dual cone of S^P is $\bigcup_{z \in W_P} zD$, and by Proposition 3.1(ii) the coordinate ring of X_w, $w \in W$, is $k[-wS^P]$. □

The following theorem shows that Definition 1.1 of a generic torus orbit does not depend on the choice of the Weyl module $V(\lambda)$.

Theorem 3.3. *Let $x \in G/P$. The following statements are equivalent.*

(i) *There exist an integral dominant weight λ whose stabilizer in W is W_P, such that for any $w \in W$, the semigroup generated by $\lambda - w\Pi_\lambda(x)$ is S^P.*

(ii) *For each integral dominant weight* λ *whose stabilizer in* W *is* W_P, *and each* $w \in W$, *the semigroup generated by* $\lambda - w\Pi_\lambda(x)$ *is* S^P.

(iii) *There exists an integral dominant weight* λ *whose stabilizer in* W *is* W_P, *such that* $\Pi_\lambda(x) = \Pi_\lambda$.

Proof. Clearly, (ii) implies (i). Also, by Proposition 2.2, (iii) implies (i). We have to prove that if (i) holds, so do (ii) and (iii). Let $X = \overline{Tx}$ and let $X_w, w \in W$ be as in Theorem 3.1. Since the coordinate ring of X_w does not depend on the choice of a Plücker embedding, Theorem 3.1(ii) implies that (ii) follows from (i).

It remains to prove that (i) implies (iii). Let $x \in G/P$ and let λ be as in (i). For any integral dominant weight μ whose stabilizer in W is W_P, let $\mathcal{L}^\mu$ denote the corresponding line bundle on G/P. Let $\mathcal{L}^\mu_X$ denote the pullback of $\mathcal{L}^\mu$ to $X = \overline{Tx}$. Since X contains an open, dense T-orbit, every weight of $H^0(X, \mathcal{L}^\mu_X)$ under the natural T-action has multiplicity one. Therefore the dimension of the image of the restriction map

$$H^0(G/P, \mathcal{L}^\mu) \to H^0(X, \mathcal{L}^\mu_X)$$

is $\sharp(\Pi_\mu(x))$. We observe that line bundle $\mathcal{L}^\mu_X$ is ample. This is because the piecewise linear function on E corresponding to $\mathcal{L}^\mu_X$ (see [**F-H**, Theorem 2]) is strictly upper convex. Then the description of the fan of X given in Theorem 3.2(iii), [**Oda1**, Theorem 2.13 and Corollary 2.9], and Proposition 2.2 (i) imply that

$$\dim \; H^0(X, \mathcal{L}^\mu_X) = \sharp(\Pi_\mu).$$

Since $\mathcal{L}^\lambda$ is ample there exists a positive integer q such that the restriction map

$$H^0(G/P, \mathcal{L}^{q\lambda}) \to H^0(X, \mathcal{L}^{q\lambda}_X)$$

is surjective. Hence $\Pi_{q\lambda}(x) = \Pi_{q\lambda}$ as required. □

It is easy to see that Theorem 3.3 and Proposition 2.2 imply:

Corollary. *Let* $x \in G/B$. *Then* Tx *is generic if and only if* $x \in \bigcap_{w\in W} wB^+.B$ *(i.e. it is "generic" in the sense of* [**F-H**]*). Moreover, if* xT *is generic then* $X = \overline{Tx}$ *is smooth.*

Remark. Smoothness of the closure of a generic torus orbit in G/B is well known (we do not know however, to whom this fact should be attributed).

Final remarks and questions.

1. All results about closures of T-orbit in G/P stated in [**F-H**] hold for generic orbits (in the sense of Definition 1.1) in any characteristic. This is because the arguments used in [**F-H**] are valid for normal equivariant T-embeddings, and we have shown that the closure of a generic orbit is such an embedding. We do not know however, if the results remain valid for all T-orbits in the nondegenerate stratum if $P \neq B$.

2. Let X denote the closure of a T-orbit of an element $x \in G/P$. It is not difficult to prove that if λ is an integral dominant weight whose stabilizer in W is W_P, then the line bundle $\mathcal{L}_X^\lambda$ is in fact very ample (one can use the criterion for very ampleness given in [**F**, Lemma, p. 69] or [**Oda1**, Corollary 2.9]). Then it follows from [**F**, Exercise, p. 72] that the corresponding embedding of X in $\mathrm{Proj}(H^0(X, \mathcal{L}_X^\lambda))$ is projectively normal and Cohen-Macaulay (that is, the homogeneous coordinate ring of X in $\mathrm{Proj}(H^0(X, \mathcal{L}_X^\lambda))$ is normal and Cohen-Maculay). Therefore, the embedding $X \subset \mathrm{Proj}(H^0(G, \mathcal{L}^\mu))$ is also projectively normal and Cohen-Macaulay, if the restriction map from $H^0(G/P, \mathcal{L}^\lambda)$ to $H^0(X, \mathcal{L}_X^\lambda)$ is surjective (equivalently $\Pi_\lambda(x) = \Pi_\lambda$). We do not know if this is so, if Tx is generic and $\Pi_\lambda(x) \neq \Pi_\lambda$.

3. Since the closure of any T-orbit in an equivariant normal T-embedding is normal (see [**K**, Proposition 2, p. 17]), X is normal if it is contained in the closure of a generic T-orbit. In this situation, the fan corresponding to X can be described explicitly in terms of the fan defined in Theorem 3.2 (iii) (see e.g. [**Oda2**, Section 1.1]). Since there could be non-generic orbits of maximal dimension (see the example in the introduction) not every T-orbit is contained in the closure of a generic one. The structure of the orbit is not clear. Does it have to be normal? If yes, what is its fan? Suppose that the closures of all T-orbits in G/P are indeed normal. Then the Example and the Corollary of Proposition 3.1, suggest the conjecture that the isomorphism type of $\overline{Tx}$ (as a torus equivariant embedding) is determined by two pieces of data: the stabilizer of x in T and the set $\{w \in W/W_P | x \in B^+w.P\}$.

References

[B] N. Bourbaki, *Groupes et algèbres de Lie*, Ch. 4–6, Hermann, Paris, 1968.

[F] W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, Princeton, New Jersey, 1993.

[F-H] H. Flaschka and L. Haine, *Torus orbits in G/P*, Pacific. J. Math., **149** (1991), 251–292.

[G-S] I.M. Gelfand V.V. Serganova, *Combinatorial geometries and torus strata on homogeneous compact manifolds*, Russian Math. Surveys, **42** (1987), 133–168.

[H] J.H. Humphreys, *Introduction to Lie Algebras and Representation Theory Springer-Verlag*, New York, 1972.

[J] J.C. Jantzen, *Representations of Algebraic Groups*, Academic Press, Boston, 1987.

[K] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal Embeddings* I, Lecture Notes in Math., **339** Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[Oda1] T. Oda, *Convex bodies and algebraic geometry - an introduction to the theory of toric varieties*, Springer-Verlag, Berlin-Heidelberg-New York, 1988.

[Oda2] ———, *Geometry of toric varieties*, Proc. of the Hyderabad conference on algebraic groups, Manoj Prakashan, Madras -India, 1991.

Received September 25, 1993 and revised February 16, 1994.

COLUMBIA UNIVERSITY
NEW YORK, NY 10027
E-mail address: rdab@math.columbia.edu

PACIFIC JOURNAL OF MATHEMATICS

Volume 172 No. 2 February 1996
