

ON NUMERICAL DIE SWELL CALCULATION

M.J. CROCHET AND R. KEUNINGS

Unité de Mécanique Appliquée, Université Catholique de Louvain, Bâtiment Stévin, Place du Levant, 2, B-1348, Louvain-la-Neuve, (Belgium)

(Received April 22, 1981)

Summary

A few recent theoretical papers have been devoted to the effect of the elastic properties of a fluid upon die swell. A rapid survey might suggest that the range of Weissenberg numbers covered by the calculations and the values of swelling ratio and exit pressure loss depend upon the numerical algorithm. In the present note, we show the importance of mesh refinement on the analysis; we also show that mixed and displacement techniques lead to slightly different results.

1. Introduction

Several results on die swell have been obtained since 1974 by means of the finite element method. The Newtonian creeping jet was investigated by Nickell et al. [1], while inertia and surface tension effects were incorporated into the previous method by Reddy and Tanner [2]. Shear-thinning effects were studied by Tanner et al. [3] who considered the exit flow of a power-law fluid.

Elastic effects have been considered only recently. Reddy and Tanner [4] studied swelling of a sheet of second-order fluid, while Chang et al. [5] were able to apply the collocation and Galerkin method to slit and circular die swell of a generalized Maxwell fluid. Crochet and Keunings [6] studied slit, circular and annular die swell of an upper convected Maxwell fluid. More recently, Coleman [7] obtained results on slit die swell, while Viriyayutha-

korn and Caswell [8] have solved the problem of circular die swell of a fluid of the integral type.

The methods used in [4] to [8] fall essentially into two categories: [4], [5] and [8] use an extension of the classical *displacement* (or $u-v-p$) finite element method, while [6] and [7] utilize a *mixed* finite element method.

Creeping flow of a Maxwell fluid in the absence of surface tension depends upon a single parameter which we choose to be $\lambda\dot{\gamma}_w$, i.e. the product of the relaxation time and the wall shear rate in the upstream fully developed flow. A synoptic study of the results contained in [4] to [8] reveals that all available techniques fail to converge beyond some value of $\lambda\dot{\gamma}_w$.

Coleman [7] obtained the highest value allowing convergence, i.e. 1.25; on the other hand, Chang et al. [5] obtained the highest swelling ratio (although for a moderate value of 0.6 for $\lambda\dot{\gamma}_w$). It is striking to observe that these authors used very coarse meshes for obtaining their results. In particular, Chang et al. [5] use for the analysis of slit die a mesh which is only one element wide and four elements long, with Hermite cubic interpolating functions for velocity components and pressure.

The present note collects numerical experiments showing that mesh effects have a dominant influence in die swell calculation. We will show that the mixed method used by Crochet and Keunings [6] allows us to consider the same relatively high value of $\lambda\dot{\gamma}_w$ as Coleman [7], provided we use a similarly coarse mesh. We will also use an extended ($u-v-p$) method [9] for showing that the swelling ratio depends upon the coarseness of the mesh when it is calculated with that method.

2. Mixed methods

Let \mathbf{T} denote the extra-stress tensor, p the pressure, \mathbf{v} the velocity vector, \mathbf{L} the velocity gradient tensor, \mathbf{D} the rate of deformation tensor, ρ the specific mass, μ a constant shear viscosity, λ a relaxation time, and \mathbf{f} a body force per unit volume. The conservation of mass, momentum balance and constitutive equations for steady flow of an upper convected Maxwell fluid are given as follows,

$$\operatorname{div} \mathbf{v} = 0, \quad (1)$$

$$-\operatorname{grad} p + \nabla \cdot \mathbf{T} + \mathbf{f} = \rho \mathbf{v} \cdot \nabla \mathbf{v}, \quad (2)$$

$$\mathbf{T} + \lambda \overset{\nabla}{\mathbf{T}} = 2\mu \mathbf{D}, \quad (3)$$

where ∇ is the gradient operator, and

$$\overset{\nabla}{\mathbf{T}} = \mathbf{v} \cdot \nabla \mathbf{T} - \mathbf{L} \mathbf{T} - \mathbf{T} \mathbf{L}^T. \quad (4)$$

Finite element methods for solving this system of non-linear partial differential equations consist in selecting an approximation \mathbf{v}^* for the velocity field and p^* for the pressure; some methods also use an approximation \mathbf{T}^* for the extra-stress tensor. These approximations are given as follows,

$$\begin{aligned}\mathbf{v}^* &= \sum_{j=1}^L \mathbf{v}^j \psi_j, & p^* &= \sum_{j=1}^N p^j \pi_j, \\ \mathbf{T}^* &= \sum_{j=1}^M \mathbf{T}^j \phi_j,\end{aligned}\tag{5}$$

where \mathbf{v}^j , p^j , \mathbf{T}^j are nodal values and ψ_j , π_j , ϕ_j are shape functions. Most available techniques apply Galerkin method for solving eqns. (1) and (2); we write

$$\langle \pi_j; \operatorname{div} \mathbf{v}^* \rangle = 0, \quad 1 \leq j \leq N,\tag{6}$$

$$\langle (\nabla \psi_j)^T; -p^* \mathbf{I} + \mathbf{T}^* \rangle + \langle \psi_j; \rho \mathbf{v}^* \cdot \nabla \mathbf{v}^* \rangle = \mathbf{F}_j, \quad 1 \leq j \leq L,\tag{7}$$

where $\langle ; \rangle$ denotes the scalar product in the L^2 space over the domain of integration, \mathbf{F}_j is the nodal force resulting from volume and contact forces, while an integration by parts has been performed for obtaining eqn. (7).

In [6] and [7], the system is closed by applying the Galerkin method to eqn. (3), i.e.

$$\langle \phi_j; \mathbf{T}^* + \lambda \overset{\nabla}{\mathbf{T}}^* - 2\mu \mathbf{D}^* \rangle = \mathbf{o}, \quad 1 \leq j \leq M;\tag{8}$$

the resulting algorithm follows the philosophy of mixed variational principles in solid mechanics.

A modified mixed method used in [5] and [8] consists in splitting the extra-stress as follows,

$$\mathbf{T} = 2\mu \mathbf{D} + \boldsymbol{\Sigma}.\tag{9}$$

Eqn. (7) becomes

$$\begin{aligned}\langle (\nabla \psi_j)^T; -p^* \mathbf{I} + 2\mu \mathbf{D}^* + \boldsymbol{\Sigma}^* \rangle \\ + \langle \psi_j; \rho \mathbf{v}^* \cdot \nabla \mathbf{v}^* \rangle = \mathbf{F}_j, \quad 1 \leq j \leq L.\end{aligned}\tag{10}$$

In [8], $\boldsymbol{\Sigma}$ is calculated on the basis of the integral representation of eqn. (3), while in [5] it is obtained by solving eqn. (8). In [9], $\boldsymbol{\Sigma}$ is replaced by $-\lambda \overset{\nabla}{\mathbf{T}}$ in view of eqn. (3), and eqn. (10) becomes

$$\begin{aligned}\langle (\nabla \psi_j)^T; -p^* \mathbf{I} + 2\mu \mathbf{D}^* \rangle - \lambda \langle (\nabla \psi_j)^T; \overset{\nabla}{\mathbf{T}}^* \rangle \\ + \langle \psi_j; \rho \mathbf{v}^* \cdot \nabla \mathbf{v}^* \rangle = \mathbf{F}_j, \quad 1 \leq j \leq L.\end{aligned}\tag{11}$$

When λ vanishes, this finite element algorithm then reduces to the classical *displacement*, or *u-v-p* technique.

In the next section, we will compare results obtained on the basis of the following methods:

MIX1: eqns. (6),(7),(8); $\psi_j = \phi_j \in P_2, \pi_j \in P_1$

MIX2: eqns. (6),(11),(8); $\psi_j = \phi_j \in P_2, \pi_j \in P_1$

MIX3: eqns. (6),(7),(8); $\psi_j = \phi_j \in P_1, \pi_j \in P_0^+$;

P_1 and P_2 denote the spaces of first and second degree complete polynomials over triangles, while P_0^+ denotes the space of discontinuous functions with a constant value over a rectangle formed by two triangles. Methods MIX1 and MIX2 use Newton-Raphson procedure for solving the non-linear system in the nodal values while MIX3 uses an alternate iterative scheme (see [7]).

The boundary conditions used in the present work are similar to those used in [6] with the exception of boundary conditions on the stress components. It may be shown that the nodal values of the shear stress component must vanish on a plane of symmetry; moreover, numerical experiments have confirmed the need of imposing fully developed values of the extra-stress components in the entry section.

3. Slit die swell: mixed methods

The swelling of a jet at the exit of slit and circular dies was studied in [6] with method MIX1; the amount of elasticity of the flow was characterized by a non-dimensional number $\lambda\dot{\gamma}_w$, which is the product of the relaxation time and the maximum shear rate on the wall in the fully developed upstream flow. The Weissenberg number W used by other authors is equal to $\lambda\bar{v}/h$, where \bar{v} is the average velocity and h is the half-width of the die; one finds easily that, for slit dies, $\lambda\dot{\gamma}_w = 3W$. It was found that the iterative technique does not converge for values of $\lambda\dot{\gamma}_w$ higher than 0.75; even for that value, we found that the stress and velocity fields lacked the smoothness which is usually encountered for such value of $\lambda\dot{\gamma}_w$ in other flows. In a recent paper [7], Coleman has obtained a solution for slit die swell up to a value of $\lambda\dot{\gamma}_w$ equal to 1.25 with the use of method MIX3. Coleman imputes this success to a modified iterative technique which enjoys a larger radius of convergence than the Newton-Raphson method.

Although using a modified iterative technique was necessary for reaching a value of 1.25 with method MIX3, we do not believe that lack of convergence is in general due to a poor choice of iterative algorithm. Rather, we want to show by means of numerical experiments that the maximum value of $\lambda\dot{\gamma}_w$ allowing convergence is related to the coarseness of the mesh used in the

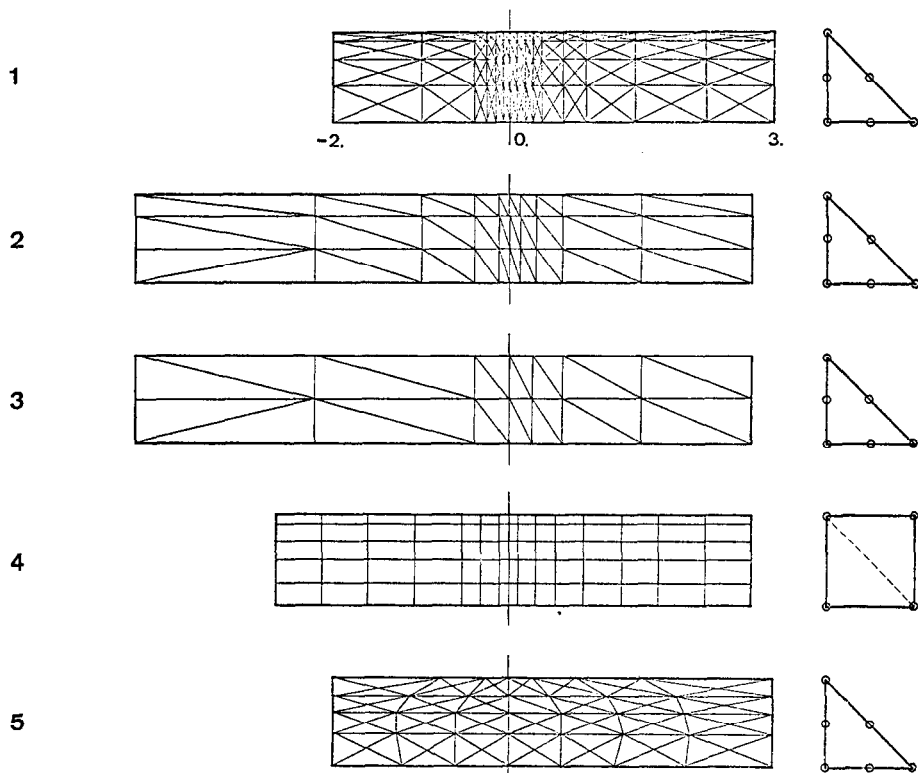


Fig. 1. Finite element meshes used by various authors: 1, Crochet and Keunings [6]; 2,3, present paper; 4, Coleman [7]; 5, Reddy and Tanner [4].

finite element analysis. The reason is that a fine mesh in the neighborhood of the edge enhances the stress singularity while a coarse mesh smoothes the stress field; the mixed method, in its present form, imposes stress continuity which is incompatible with the singularity at the edge.

Fig. 1 shows five meshes designed for calculating slit die swell, while Table 1 gives the associated numbers of degrees of freedom. MESH1 was

TABLE 1

Number of degrees of freedom for five finite element meshes

	Velocities	Pressure	Stresses	Total
MESH1	1038	140	1557	2735
MESH2	294	44	441	779
MESH3	150	24	225	399
MESH4	192	75	288	555
MESH5	494	68	—	562

used in [6], while MESH2 and MESH3 were designed for comparison with MESH4 used by Coleman [7]. MESH5 was used by Reddy and Tanner [4] for obtaining results which will be discussed in Section 4. The mesh used by Chang et al. in [5] contained 120 degrees of freedom in velocity components and pressure.

The maximum values of $\lambda\dot{\gamma}_w$ allowing convergence in slit die swell with a mixed method are as follows:

MESH1 and MIX1: 0.75 MESH2 and MIX1: 0.85
 MESH3 and MIX1: 1.25 MESH4 and MIX3: 1.25

It is obvious that MIX1 with MESH3, and MIX3 with MESH4, with almost the same number of degrees of freedom, allow us to reach the same maximum value of $\lambda\dot{\gamma}_w$, despite the use of the Newton–Raphson method in MIX1.

Significant numbers arising from die swell calculations are the value of the swelling ratio S_w , and the quotient of the exit pressure loss δp to the shear stress τ_w at the wall in the upstream flow. Fig. 2 shows the values of S_w and $\delta p/\tau_w$ as a function of $\lambda\dot{\gamma}_w$ for the four cases we have just described. It is surprising to see that the swelling ratios obtained with method MIX1 depend very little upon the mesh, although we observe a significant difference between results obtained with MIX1 and with MIX3 based on different approximating functions. On the other hand, we observe on Fig. 2 that methods MIX1 and MIX3 exhibit good agreement for the exit pressure losses.

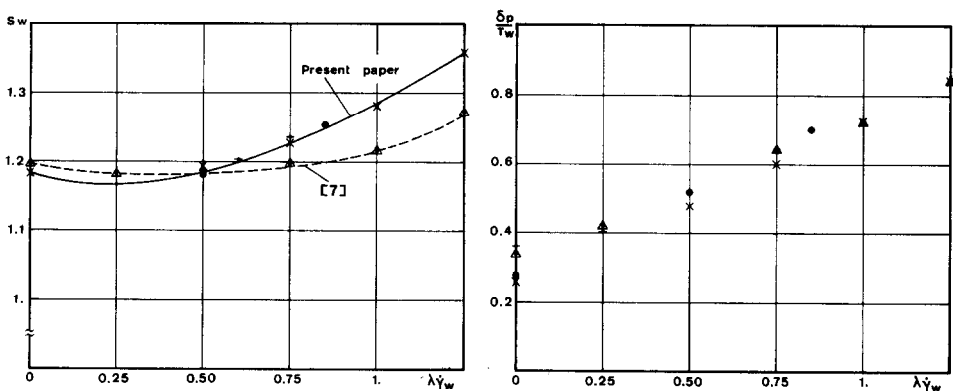


Fig. 2. Swelling ratio and exit pressure loss as a function of the non-dimensional shear rate $\lambda\dot{\gamma}_w$; mixed methods; ●, Crochet and Keunings [6]; ○, present paper, MESH2, MIX1; ×, present paper, MESH3, MIX1; △, Coleman [7].

4. Slit die swell: alternate mixed method and displacement method

The mixed method MIX1 described in Section 2 reduces, in the linear case, to an application of the Reissner–Hellinger–Washizu variational principle; it is not used in general for solving Newtonian problems since the velocity field obtained with the displacement (or $u-v-p$) method is smoother, and also cheaper to obtain. Method MIX2 extends the $u-v-p$ technique (described in [9]) by adding a stress field which is C^0 continuous over the domain of integration. Fig. 3 shows *Newtonian* contour lines obtained on MESH2 with methods MIX1 and MIX2 for the streamlines, the pressure, the longitudinal velocity component and the shear stress. These results are typical of observations made on other problems:

- (i) the streamlines are usually smooth since the stream function integrates the velocity field;
- (ii) the velocity field is smoother for MIX2 ($u-v-p$) than for MIX1 (mixed);
- (iii) the pressure contours are almost identical.

The stress field is smoother for MIX2 than for MIX1; we must point out however that with MIX2 ($u-v-p$) the stress field is a continuous representation which is calculated *a posteriori* from the velocity field. One should not conclude from the Newtonian results shown on Fig. 3 that MIX2 is superior

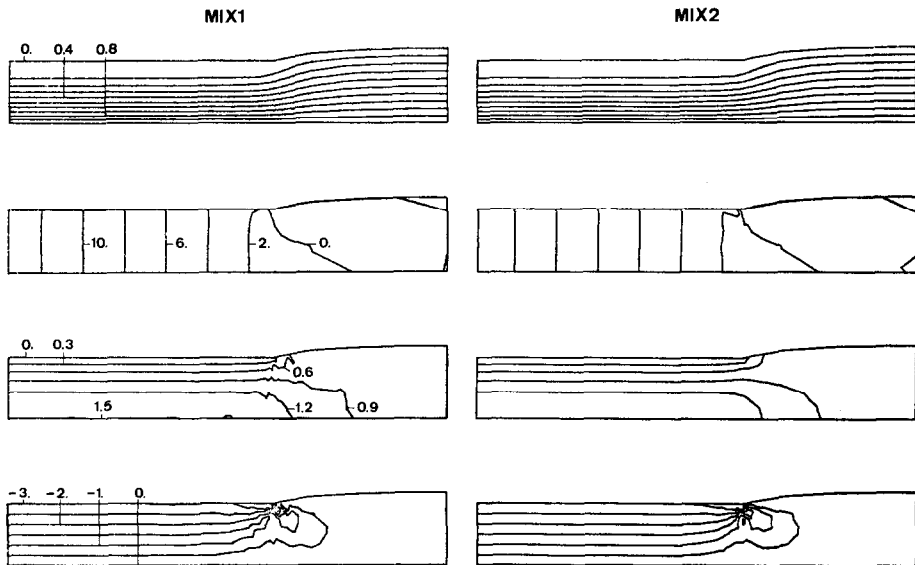


Fig. 3. Comparison between mixed and $u-v-p$ methods for the Newtonian case. Streamlines and contour lines for the pressure, the longitudinal velocity component, and the shear stress component (divided by μ).

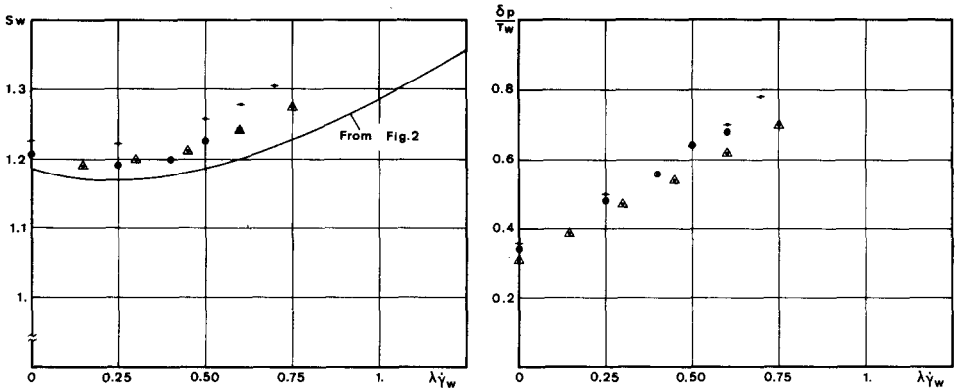


Fig. 4. Swelling ratio and exit pressure loss as a function of the non-dimensional shear rate $\lambda\dot{\gamma}_w$; $u-v-p$ methods; \odot , present paper, MESH2, MIX2; \bullet , present paper, MESH3, MIX2; \triangle , Reddy and Tanner [4].

to MIX1; while this last assertion is true for the Newtonian case, we have shown in [9] that the opposite is true in the non-Newtonian case in the absence of stress singularities.

In view of these comments, we found it instructive to solve the slit die swell problem with method MIX2 on meshes MESH2 and MESH3. These results are compared with those obtained with MIX1, and those obtained by Reddy and Tanner [4] for the flow of a second-order fluid with the $u-v-p$ method; for the low Weissenberg numbers reached at the present time, one should not indeed expect major differences between the flow of an upper convected Maxwell fluid and the flow of a second-order fluid endowed with identical viscometric functions.

The maximum values of $\lambda\dot{\gamma}_w$ allowing convergence are as follows:

MESH2 and MIX2: 0.6,

TABLE 2

Swelling ratio and exit pressure loss for the Newtonian case obtained with different meshes, $u-v-p$ method

MESH	MESH3	MESH2	MESH5	MESH1
Degrees of freedom	174	338	562	1178
Value of S_w	1.227	1.207	1.200	1.196
Value of $\delta p / \tau_w$	0.36	0.34	0.31	0.31

MESH3 and MIX2: 0.7

MESH5, second-order fluid: 0.75;

for a second-order fluid, we have selected $\lambda = \nu_1/2\eta$ (see [4]). It appears that the maximum values of $\lambda\dot{\gamma}_w$ obtained with the $u-v-p$ technique and its extension MIX2 are lower than those obtained with method MIX1.

Fig. 4 shows the values of Sw and $\delta p/\tau_w$ as a function of $\lambda\dot{\gamma}_w$ for these three cases, together with the results of MIX1 on MESH1 which extend over the same range of $\lambda\dot{\gamma}_w$. We have also calculated the Newtonian values of Sw and $\delta p/\tau_w$ with the $u-v-p$ method on the dense mesh MESH1. Let us first observe that, even in the Newtonian case, the values of Sw and $\delta p/\tau_w$ show a significant dependence upon the number of degrees of freedom; this is best seen in Table 2. This dependence remains the same when $\lambda\dot{\gamma}_w$ increases. The swelling ratios obtained with MIX2 and MESH2 compare remarkably well with those of Reddy and Tanner [4]; although the Newtonian value differs by only 0.02 from the value found with MIX1, the discrepancy increases with $\lambda\dot{\gamma}_w$. Finally, we note that the difference between results on $\delta p/\tau_w$ obtained by Reddy and Tanner [4] and those obtained with MIX2 and MESH2 corresponds to the discrepancy between the Newtonian values obtained with these methods.

5. Conclusions

We have carried out numerical experiments on slit die swell with three finite element meshes and two different techniques, and we have compared our results with those obtained by other authors on comparable meshes.

A first conclusion is that, for a given finite element method, the maximum value of $\lambda\dot{\gamma}_w$ allowing convergence is highly dependent upon mesh refinement. A coarse mesh allows us to reach a high value of $\lambda\dot{\gamma}_w$, because the stress singularity is smoothed. A second conclusion is that, for a given mesh, the mixed method leads to higher values of $\lambda\dot{\gamma}_w$ than a $u-v-p$ method.

For a given value of $\lambda\dot{\gamma}_w$, it was found that the swelling ratio depends little upon mesh refinement when the mixed method is used; the swelling ratio appears to depend more upon mesh refinement with the $u-v-p$ method. Our comparisons show that the swelling ratio depends upon the method used for its calculation, while all available techniques show good agreement on the value of the exit pressure loss.

Finally, there is no way at the present stage of singling out 'the best method'. The range of $\lambda\dot{\gamma}_w$ covered by the available methods is so small that the differences which we have pointed out between various results may not be significant; moreover, comparisons with experiments are not possible on such a small range of Weissenberg numbers. At the present stage it seems

that further progress requires devising a numerical algorithm for treating the stress singularity, although the nature of the singularity for a Maxwell fluid is not known at the present time.

References

- 1 R.E. Nickell, R.I. Tanner and B. Caswell, *J. Fluid Mech.*, 65 (1974) 189–206.
- 2 K.R. Reddy and R.I. Tanner, *Comp. Fluids*, 6 (1978) 83–91.
- 3 R.I. Tanner, R.E. Nickell and R.W. Bilger, *Comp. Meth. in appl. Mech. Engng.*, 6 (1975) 155–174.
- 4 K.R. Reddy and R.I. Tanner, *J. Rheol.*, 22 (1978) 661–665.
- 5 P.W. Chang, Th.W. Patten and B.A. Finlayson, *Comp. Fluids*, 7 (1979) 267–293.
- 6 M.J. Crochet and R. Keunings, *J. Non-Newtonian Fluid Mech.*, 7 (1980) 199–212.
- 7 C.J. Coleman, *J. Non-Newtonian Fluid Mech.*, 8 (1981) 261.
- 8 M. Viriyayuthakorn and B. Caswell, Finite element Calculation of viscoelastic die swell, to be published.
- 9 M.J. Crochet, The flow of a Maxwell fluid around a sphere, in Gallagher (Ed.), *Finite Elements in Fluids, IV*, Wiley, Chichester, 1981, to be published.