

ON ω -LIMIT SETS AND ATTRACTION OF NON-AUTONOMOUS DISCRETE DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we study ω -limit sets and attraction of non-autonomous discrete dynamical systems. We introduce some basic concepts such as ω -limit set and attraction for non-autonomous discrete system. We study fundamental properties of ω -limit sets and discuss the relationship between ω -limit sets and attraction for non-autonomous discrete dynamical systems.

1. Introduction

Throughout this paper, \mathbb{N} denotes the natural number set and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let X be a topological space, $f_n : X \rightarrow X$ for each $n \in \mathbb{N}$ be a continuous map and $f_{1,\infty}$ be the sequence $(f_1, f_2, \dots, f_n, \dots)$. The pair $(X, f_{1,\infty})$ is referred to as a non-autonomous discrete dynamical system [12]. If X is compact, then $(X, f_{1,\infty})$ is called a compact non-autonomous system. Define

$$f_1^n := f_n \circ f_{n-1} \circ \dots \circ f_2 \circ f_1 \text{ for all } n \in \mathbb{N},$$

and $f_1^0 := id_X$, the identity on X . In particular, when $f_{1,\infty}$ is a constant sequence (f, \dots, f, \dots) , the pair $(X, f_{1,\infty})$ is just classical discrete dynamical system (autonomous discrete dynamical system) (X, f) . The orbit initiated from $x \in X$ under $f_{1,\infty}$ is defined by the set

$$\gamma(x, f_{1,\infty}) = \{x, f_1(x), f_1^2(x), \dots, f_1^n(x), \dots\}.$$

Its long-term behaviors are determined by its limit sets.

In past ten years, a large number of papers have been devoted to dynamical properties in non-autonomous discrete systems. Kolyada and Snoha [12] gave definition of topological entropy in non-autonomous discrete systems, Kolyada, Snoha and Trofimchuk [13] discussed minimality of non-autonomous dynamical systems, Kempf [11] and Canovas [5] studied ω -limit sets in non-autonomous

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discrete systems respectively. Krabs [14] discussed stability in non-autonomous discrete systems, Huang, Wen and Zeng ([9, 10]) studied topological pressure and pre-image entropy of non-autonomous discrete systems, Shi and Chen [21] and Oprocha and Wilczynski [19] discussed chaos in non-autonomous discrete systems.

The ω -limit sets give fundamental information about the asymptotic behavior of a dynamical system and its concept for classical discrete dynamical system (autonomous discrete dynamical system) was introduced by Block and Coppel ([1, 3]). The attraction is an important property in dynamical system, for example, asymptotically stable set [3] and attractor ([8, 20, 22]) belong to the problem of attraction. In recent years, Mimna and Steele [17] discussed ω -limit sets and asymptotically stable sets for semi-homeomorphisms, Aniello and Steele [2] discussed the stability of ω -limit sets, Oprocha [18] studied asymptotically stable sets in continuous dynamical systems and Braga and Souza [4] studied attraction for semigroup actions. There are classes of dynamical systems for which the behavior of trajectories of sets seems in some sense much simpler than that of trajectories of points. For example, Marzocchi and Necca [15] gave the definition of ω -limit set which describes the long-term behavior of trajectories of sets. Let (X, f) be an autonomous discrete system and let B be a nonempty subset of X . Define $\omega(B, f)$ as the set of limit points of the orbit $\gamma(B, f)$, i.e., $\omega(B, f) = \bigcap_{m \in \mathbb{Z}_+} \overline{\gamma_m(B, f)}$, where $\gamma_m(B, f)$ denotes the positive orbit through B starting at time m .

Motivated by the idea of Marzocchi and Necca's concept of ω -limit set, in this paper we give the concepts of ω -limit set and attraction for non-autonomous discrete system. The definition of ω -limit set describes the long-term behavior of trajectories of sets but not points. Our purpose is to study the fundamental properties of ω -limit sets and attraction for non-autonomous discrete dynamical systems, e.g., the set operations of ω -limit set (Proposition 3.1) and attraction is preserved by the conjugated systems (Theorem 3.1). In particular, we give a sufficient condition for ω -limit set of non-autonomous discrete system is a nonempty compact set. Also, we discuss the relationship between the attraction and ω -limit sets for non-autonomous discrete systems in regular spaces (Theorem 4.2).

2. Preliminaries

Definition 2.1. Let $(X, f_{1, \infty})$ be a non-autonomous discrete system. For every $B \subseteq X$ and $m \in \mathbb{Z}_+$, the set $\gamma_m(B, f_{1, \infty}) = \bigcup_{x \in B} \{f_1^n(x) : n \geq m\}$ is called positive orbit through B starting at time m . If $B = \{x\}$, we will write $\gamma_m(x, f_{1, \infty})$ instead of $\gamma_m(\{x\}, f_{1, \infty})$. If $m = 0$, we will omit time index.

Definition 2.2. Let $(X, f_{1, \infty})$ be a non-autonomous discrete system and let B be a nonempty subset of X . Define $\omega(B, f_{1, \infty})$ as the set of limit points of the orbit $\gamma(B, f_{1, \infty})$, i.e., $\omega(B, f_{1, \infty}) = \bigcap_{m \in \mathbb{Z}_+} \overline{\gamma_m(B, f_{1, \infty})}$, where $\overline{\gamma_m(B, f_{1, \infty})}$

denotes the closure of $\gamma_m(B, f_{1,\infty})$. If $B = \{x\}$, we write $\omega(x, f_{1,\infty})$ instead of $\omega(\{x\}, f_{1,\infty})$.

Remark 1. Let $(X, f_{1,\infty})$ be a non-autonomous discrete system and X be a compact metric space. Then by [7] we may define $\hat{\omega}(B, f_{1,\infty})$ as follows:

$$\hat{\omega}(B, f_{1,\infty}) = \{A \subseteq X : \text{there exists } n_k \rightarrow \infty \text{ such that } A = Lt_{k \rightarrow \infty} f_1^{n_k}(B)\},$$

where Lt is for the operation of going to topological limit ($Lt_{k \rightarrow \infty} Y_k = Y$ if $d_H(Y, Y_k) \rightarrow 0$ as $k \rightarrow \infty$ with d_H standing for the Hausdorff metric given by $d_H(A, B) = \inf\{\delta > 0 : A \subseteq U_\delta(B), B \subseteq U_\delta(A)\}$, where U_δ is the δ -neighborhood of a set). If $B = \{x\}$, then we will write $\hat{\omega}(x, f_{1,\infty})$ instead of $\hat{\omega}(\{x\}, f_{1,\infty})$. Furthermore,

$$\hat{\omega}(x, f_{1,\infty}) = \{y \in X : \text{there exists } n_k \rightarrow \infty \text{ such that } y = \lim_{k \rightarrow \infty} f_1^{n_k}(x)\}.$$

By Definition 2.2, the sets $\omega(B, f_{1,\infty}) \subseteq X$ and $\hat{\omega}(B, f_{1,\infty})$ may be empty, but by Ref. [7], $\hat{\omega}(B, f_{1,\infty})$ does not belong to the phase space X and is a subset of the set 2^X of all closed subsets of X . If X is a compact metric space, then 2^X is a compact metric space from Michael [16] and Engelking [6]. Furthermore, we have $\omega(B, f_{1,\infty}) \neq \emptyset$, $\hat{\omega}(B, f_{1,\infty}) \neq \emptyset$ and $\omega(x, f_{1,\infty}) = \hat{\omega}(x, f_{1,\infty})$.

Definition 2.3 ([13]). Let $(X, f_{1,\infty})$ be a non-autonomous discrete system. Set $A \subseteq X$ is said to be invariant if $f_1^n(A) \subseteq A$ for every $n \in \mathbb{N}$.

For an autonomous system (X, f) , by Block and Coppel [3], if X is a compact space, then $\omega(x, f)$ is invariant for every $x \in X$. However, for a non-autonomous system $(X, f_{1,\infty})$, we have $\omega(B, f_{1,\infty})$ cannot be invariant for any $B \subseteq X$.

Example 2.1. Let $X = [0, 1]$, $f_n : [0, 1] \rightarrow [0, 1]$ be a sequence of continuous maps and

$$f_n(x) = \begin{cases} 1 - \frac{1}{n+1} & \text{for } x = \frac{1}{n} \text{ and } n \text{ even,} \\ \frac{1}{n+1} & \text{for } x = 1 - \frac{1}{n} \text{ and } n \text{ odd,} \\ 0, & \text{otherwise} \end{cases}$$

for every $n \in \mathbb{N}$. Then $\omega(\{0\}, f_{1,\infty})$ is not invariant.

From the definition of $f_n(x)$, we have

$$f_1^n(0) = \begin{cases} \frac{1}{n+1} & \text{for } n \text{ odd,} \\ \frac{n}{n+1} & \text{for } n \text{ even.} \end{cases}$$

Hence, $\omega(\{0\}, f_{1,\infty}) = \omega(0, f_{1,\infty}) = \{0, 1\}$. As $f_1^1(\omega(0, f_{1,\infty})) = f_1^1(\{0, 1\}) = \{0, \frac{1}{2}\}$. Therefore, $\omega(\{0\}, f_{1,\infty})$ is not invariant.

Definition 2.4 ([21]). Let $(X, f_{1,\infty})$ be a non-autonomous discrete system. $f_{1,\infty}$ is said to be k -periodic discrete system if there exists $k \in \mathbb{N}$ such that $f_{n+k}(x) = f_n(x)$ for every $x \in X$ and $n \in \mathbb{N}$.

Let $(X, f_{1,\infty})$ be a k -periodic discrete system for a $k \in \mathbb{N}$. Define $g := f_k \circ f_{k-1} \circ \cdots \circ f_1$, we say that (X, g) is induced an autonomous discrete system by k -periodic discrete system $(X, f_{1,\infty})$.

Finally, we need some topological definitions and properties.

Definition 2.5 ([6]). Let X be a topological space. The family $\{Y_i\}_{i \in I}$ has the finite intersection property if, for every finite subset J of I , the intersection $\bigcap_{j \in J} Y_j$ is a nonempty set.

By Engelking [6], a topological space X is compact if and only if any family of closed subsets of X satisfying the finite intersection property has a nonempty intersection.

Theorem 2.1 ([6]). Let X be a regular space and let K be a compact set in X and C be a closed set in X with $K \cap C = \emptyset$. Then there exist two open sets U and V in X , with $K \subseteq U, C \subseteq V$ and $U \cap V = \emptyset$.

Corollary 2.1. Let X be a regular space and let K be a compact set in X and U be a neighborhood of K . Then there exists a closed neighborhood V of K with $V \subseteq U$.

Remark 2. Since $\{x\}$ is compact set, every neighborhood of a point in regular space contains a closed neighborhood.

3. Fundamental properties of the ω -limit sets and attraction

In this section, we give the definition of attraction and discuss fundamental properties of the ω -limit set and attraction for non-autonomous discrete system.

Proposition 3.1. Let $(X, f_{1,\infty})$ be a non-autonomous discrete system and $A, B \subseteq X$. The following properties hold:

- (1) if $A \subseteq B$, then $\omega(A, f_{1,\infty}) \subseteq \omega(B, f_{1,\infty})$;
- (2) $\omega(A \cap B, f_{1,\infty}) \subseteq \omega(A, f_{1,\infty}) \cap \omega(B, f_{1,\infty})$;
- (3) $\omega(A \cup B, f_{1,\infty}) = \omega(A, f_{1,\infty}) \cup \omega(B, f_{1,\infty})$.

Proof. (1) Since $A \subseteq B$, then for every $m \in \mathbb{Z}_+$, $\gamma_m(A, f_{1,\infty}) \subseteq \gamma_m(B, f_{1,\infty})$. Furthermore, we have

$$\overline{\gamma_m(A, f_{1,\infty})} \subseteq \overline{\gamma_m(B, f_{1,\infty})} \quad \text{and} \quad \bigcap_{m \in \mathbb{Z}_+} \overline{\gamma_m(A, f_{1,\infty})} \subseteq \bigcap_{m \in \mathbb{Z}_+} \overline{\gamma_m(B, f_{1,\infty})}.$$

Hence, $\omega(A, f_{1,\infty}) \subseteq \omega(B, f_{1,\infty})$.

(2) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by above (1), we have

$$\omega(A \cap B, f_{1,\infty}) \subseteq \omega(A, f_{1,\infty}) \quad \text{and} \quad \omega(A \cap B, f_{1,\infty}) \subseteq \omega(B, f_{1,\infty}).$$

Hence, $\omega(A \cap B, f_{1,\infty}) \subseteq \omega(A, f_{1,\infty}) \cap \omega(B, f_{1,\infty})$.

(3) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by above (1), $\omega(A, f_{1,\infty}) \subseteq \omega(A \cup B, f_{1,\infty})$ and $\omega(B, f_{1,\infty}) \subseteq \omega(A \cup B, f_{1,\infty})$. Hence, $\omega(A, f_{1,\infty}) \cup \omega(B, f_{1,\infty}) \subseteq \omega(A \cup B, f_{1,\infty})$.

To verify $\omega(A, f_{1,\infty}) \cup \omega(B, f_{1,\infty}) \supseteq \omega(A \cup B, f_{1,\infty})$, we suppose by contradiction that there exists $x \in \omega(A \cup B, f_{1,\infty})$ such that $x \notin \omega(A, f_{1,\infty}) \cup \omega(B, f_{1,\infty})$. Since $x \notin \omega(A, f_{1,\infty}) = \bigcap_{m \in \mathbb{Z}_+} \overline{\gamma_m(A, f_{1,\infty})}$, there exists $m_1 \in \mathbb{Z}_+$ such that $x \notin \overline{\gamma_{m_1}(A, f_{1,\infty})}$. Furthermore, there exists a neighborhood U of x such that for every $y \in U$, $y \notin \overline{\gamma_{m_1}(A, f_{1,\infty})}$. Similarly, $x \notin \omega(B, f_{1,\infty}) = \bigcap_{m \in \mathbb{Z}_+} \overline{\gamma_m(B, f_{1,\infty})}$, there exists $m_2 \in \mathbb{Z}_+$ such that $x \notin \overline{\gamma_{m_2}(B, f_{1,\infty})}$. Furthermore, there exists a neighborhood V of x such that for every $y \in V$, $y \notin \overline{\gamma_{m_2}(B, f_{1,\infty})}$. Take $m = \max\{m_1, m_2\}$. We have $\gamma_m(A, f_{1,\infty}) \subseteq \gamma_{m_1}(A, f_{1,\infty})$ and $\gamma_m(B, f_{1,\infty}) \subseteq \gamma_{m_2}(B, f_{1,\infty})$. Hence, $U \cap \gamma_m(A, f_{1,\infty}) = \emptyset$ and $V \cap \gamma_m(B, f_{1,\infty}) = \emptyset$. Since $U \cap V$ is a neighborhood of x and $\gamma_m(A, f_{1,\infty}) \cup \gamma_m(B, f_{1,\infty}) = \gamma_m(A \cup B, f_{1,\infty})$, it follows that $(U \cap V) \cap \gamma_m(A \cup B, f_{1,\infty}) = \emptyset$. This is a contradiction because $x \in \omega(A \cup B, f_{1,\infty})$. \square

We give an example to show that the inclusion in (2) of Proposition 3.1 can be strict.

Example 3.1. Let $X = [0, 1]$ and $f_n : [0, 1] \rightarrow [0, 1], f_n(x) = e^{-n}x$ for every $n \in \mathbb{N}$ and $x \in [0, 1]$. Let $A = [0, \frac{1}{3}]$ and $B = [\frac{1}{2}, 1]$. Then $\omega(A \cap B, f_{1,\infty}) \subsetneq \omega(A, f_{1,\infty}) \cap \omega(B, f_{1,\infty})$.

Since $A \cap B = [0, \frac{1}{3}] \cap [\frac{1}{2}, 1] = \emptyset$, then $\omega(A \cap B, f_{1,\infty}) = \emptyset$. For $x \in [0, 1]$ and $n \in \mathbb{N}$, we have $f_1^n(x) = e^{-\frac{n(n+1)}{2}}x$. Hence, $\omega(A, f_{1,\infty}) = \{0\}$ and $\omega(B, f_{1,\infty}) = \{0\}$. Furthermore, we have $\omega(A, f_{1,\infty}) \cap \omega(B, f_{1,\infty}) = \{0\}$. Therefore, $\omega(A \cap B, f_{1,\infty}) \subsetneq \omega(A, f_{1,\infty}) \cap \omega(B, f_{1,\infty})$.

Definition 3.1. Let $(X, f_{1,\infty})$ be a non-autonomous discrete system and A and B be two subsets of X . A $f_{1,\infty}$ -attracts B if for every open set U containing A there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(B, f_{1,\infty}) \subseteq U$. When the reference to $f_{1,\infty}$ is evident, we will omit this dependence.

$(X, f_{1,\infty})$ is said to have attraction if there exist two subsets A and B of X such that A attracts B .

It is clear that A attracts B if and only if for every open set U containing A there exists $m \in \mathbb{Z}_+$ such that for every $n \geq m$ it is true $f_1^n(B) \subseteq U$.

Example 3.2. Let $X = [0, 1]$ and $f_n : [0, 1] \rightarrow [0, 1], f_n(x) = e^{-n}x$ for every $n \in \mathbb{N}$ and $x \in [0, 1]$. Then $(X, f_{1,\infty})$ has attraction.

Take $A = [0, \frac{1}{2}]$ and $B = [0, \frac{1}{4}]$. Let U be any open set of X with $A \subseteq U$. We will show that A attracts B . Since for every $n \in \mathbb{Z}_+$, we have $f_n(B) \subseteq B$. Hence, $f_1^n(B) \subseteq B$ for every $n \in \mathbb{N}$. Moreover, $B \subseteq A$. Furthermore, we have $f_1^n(B) \subseteq A$ for every $n \in \mathbb{Z}_+$. This means that $f_1^n(B) \subseteq U$ for every $n \in \mathbb{Z}_+$. Hence, $\gamma_n(B, f_{1,\infty}) \subseteq U$ for every $n \in \mathbb{Z}_+$. This shows A attracts B . Therefore, $(X, f_{1,\infty})$ has attraction.

Definition 3.2 ([23]). Let $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ be two non-autonomous discrete systems and let $h : X \rightarrow Y$ be a continuous map and

$$g_n(h(x)) = h(f_n(x)) \text{ for all } n \in \mathbb{N} \text{ and } x \in X.$$

- (1) If $h : X \rightarrow Y$ is a surjective map, then $f_{1,\infty}$ and $g_{1,\infty}$ are said to be topologically semi-conjugate.
- (2) If $h : X \rightarrow Y$ is a homeomorphism, then $f_{1,\infty}$ and $g_{1,\infty}$ are said to be topologically conjugate.

Example 3.3. Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ with $f_n(x) = nx$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, where \mathbb{R} is a real line, and $g_n : S^1 \rightarrow S^1$ with $g_n(e^{i\theta}) = e^{in\theta}$ for all $n \in \mathbb{N}$, where S^1 is the unite circle. Define $h : \mathbb{R} \rightarrow S^1$ by $h(x) = e^{2\pi ix}$. It can be easily verified that h is a continuous surjective map and $h \circ f_n = g_n \circ h$. Therefore, $(\mathbb{R}, f_{1,\infty})$ and $(S^1, g_{1,\infty})$ are topologically semi-conjugate.

Theorem 3.1. *Let $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ be two non-autonomous discrete systems and let $h : X \rightarrow Y$ be a semi-conjugate map. If $(X, f_{1,\infty})$ has attraction, then $(Y, g_{1,\infty})$ has attraction.*

Proof. Since $(X, f_{1,\infty})$ has attraction, then there exist two sets A and B in X such that A $f_{1,\infty}$ -attracts B . Moreover, $h : X \rightarrow Y$ is a continuous map, thus $h(A), h(B) \in Y$. We will prove that $h(A)$ $g_{1,\infty}$ -attracts $h(B)$.

Let U be an open set of Y such that $h(A) \subseteq U$. Then $h^{-1}(U)$ is an open set of X and $A \subseteq h^{-1}(U)$. Since A $f_{1,\infty}$ -attracts B , there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(B, f_{1,\infty}) \subseteq h^{-1}(U)$. Furthermore, we have $h(\gamma_m(B, f_{1,\infty})) \subseteq h(h^{-1}(U))$. Moreover, h is a semi-conjugate map, i.e., h is a surjective map and $g_k(h(x)) = h(f_k(x))$ for every $k \in \mathbb{N}$ and $x \in X$. Furthermore,

$$\begin{aligned} h(\gamma_m(B, f_{1,\infty})) &= h\left(\bigcup_{x \in B} \{f_1^n(x) : n \geq m\}\right) = \bigcup_{x \in B} \{h(f_1^n(x)) : n \geq m\} \\ &= \bigcup_{x \in B} \{g_1^n(h(x)) : n \geq m\} = \bigcup_{x \in h(B)} \{g_1^n(x) : n \geq m\}. \end{aligned}$$

Hence, $h(\gamma_m(B, f_{1,\infty})) = \gamma_m(h(B), g_{1,\infty})$. Furthermore, we have

$$\gamma_m(h(B), g_{1,\infty}) \subseteq U.$$

This shows $h(A)$ $g_{1,\infty}$ -attracts $h(B)$. Therefore, $(Y, g_{1,\infty})$ has attraction. □

Corollary 3.1. *Let $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ be topologically conjugate. Then $(X, f_{1,\infty})$ has attraction if and only if $(Y, g_{1,\infty})$ has attraction.*

Theorem 3.2. *Let $(X, f_{1,\infty})$ be a k -periodic discrete system and (X, g) be its induced autonomous discrete system, where $g = f_k \circ f_{k-1} \circ \dots \circ f_1$. If $(X, f_{1,\infty})$ has attraction, then (X, g) has attraction.*

Proof. Since $(X, f_{1,\infty})$ has attraction, there exist two sets A and B in X such that A $f_{1,\infty}$ -attracts B . Furthermore, for every open set U of X containing A , there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(B, f_{1,\infty}) \subseteq U$. We will prove A g -attracts B .

As $(X, f_{1,\infty})$ is a k -periodic discrete system and $g = f_k \circ f_{k-1} \circ \dots \circ f_1 = f_1^k$, we have $f_{n+k}(x) = f_n(x)$ for every $x \in X$. Furthermore, $g^m(x) = (f_1^k)^m(x) = f_1^{mk}(x)$. Moreover, for every $x \in X$, $\{g^n(x) : n \geq m\} =$

$\{g^m(x), g^{m+1}(x), \dots\} = \{f_1^{mk}(x), f_1^{(m+1)k}(x), \dots\}$, thus, $\{g^n(x) : n \geq m\} \subseteq \{f_1^n(x) : n \geq m\}$. Hence, $\bigcup_{x \in B} \{g^n(x) : n \geq m\} \subseteq \bigcup_{x \in B} \{f_1^n(x) : n \geq m\}$, i.e., $\gamma_m(B, g) \subseteq \gamma_m(B, f_{1,\infty})$. Furthermore, we have $\gamma_m(B, g) \subseteq U$. This shows A g -attracts B . Therefore, (X, g) has attraction. \square

4. Main results

Theorem 4.1. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system, where X is a regular topological space. Let K be a compact set of X and $B \subseteq X$. Then K $f_{1,\infty}$ -attracts B if and only if for every closed neighborhood V of K , there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(B, f_{1,\infty}) \subseteq V$.*

Proof. Necessity. Let V be a closed neighborhood of K , then $K \subseteq \text{int}(V)$, where $\text{int}(V)$ denotes the interior of V . Since K $f_{1,\infty}$ -attracts B and $\text{int}(V)$ is an open set containing K , there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(B, f_{1,\infty}) \subseteq \text{int}(V)$. Furthermore, we have $\gamma_m(B, f_{1,\infty}) \subseteq V$.

Sufficiency. Let U be any open set containing K . Since X is a regular topological space and U is an open neighborhood of K , then by Corollary 2.1 there exists a closed set V of X such that $V \subseteq U$ and V is a closed neighborhood of K . Furthermore, we have $K \subseteq \text{int}(V)$. Hence, there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(B, f_{1,\infty}) \subseteq V$, implying $\gamma_m(B, f_{1,\infty}) \subseteq U$. This shows K attracts B . \square

Proposition 4.1. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system and let $A, B, F, K \subseteq X$ and K be a compact set. Then the following properties hold:*

- (1) *if $A \subseteq B$ and F attracts B , then F attracts also A ;*
- (2) *if F attracts A and B , then F attracts $A \cap B$;*
- (3) *if F attracts A and B , then F attracts $A \cup B$;*
- (4) *if X is a regular space and K attracts A , then K attracts also \bar{A} .*

Proof. (1) Since F attracts B , then for every open set U containing F there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(B, f_{1,\infty}) \subseteq U$. As $A \subseteq B$, we have $\gamma_m(A, f_{1,\infty}) \subseteq \gamma_m(B, f_{1,\infty})$, which implies $\gamma_m(A, f_{1,\infty}) \subseteq U$. Hence, F attracts A .

(2) Since $A \cap B \subseteq A$, $A \cap B \subseteq B$ and F attracts A and B , then by (1), F attracts $A \cap B$.

(3) Let U be any open neighborhood of F . Since F attracts A and B , there exist $m_1, m_2 \in \mathbb{Z}_+$ such that $\gamma_{m_1}(A, f_{1,\infty}) \subseteq U$ and $\gamma_{m_2}(B, f_{1,\infty}) \subseteq U$. Take $m = \max\{m_1, m_2\}$. Thus, $\gamma_m(A, f_{1,\infty}) \subseteq \gamma_{m_1}(A, f_{1,\infty}) \subseteq U$ and $\gamma_m(B, f_{1,\infty}) \subseteq \gamma_{m_2}(B, f_{1,\infty}) \subseteq U$. Since $\gamma_m(A \cup B, f_{1,\infty}) = \gamma_m(A, f_{1,\infty}) \cup \gamma_m(B, f_{1,\infty})$, it follows that $\gamma_m(A \cup B, f_{1,\infty}) \subseteq U$. Hence, F attracts $A \cup B$.

(4) Let V be a closed neighborhood of F . Since A is attracted by K , there exists $\bar{m} \in \mathbb{Z}_+$ such that for any $m \geq \bar{m}$ we have $f_1^m(A) \subseteq V$. Let $m \geq \bar{m}$, we have $A \subseteq (f_1^m)^{-1}(V)$ and $(f_1^m)^{-1}(V)$ is closed because it is the inverse image of a closed set through a continuous mapping. Hence, $\bar{A} \subseteq (f_1^m)^{-1}(V)$, which implies $f_1^m(\bar{A}) \subseteq V$. Furthermore, we have $\gamma_{\bar{m}}(\bar{A}, f_{1,\infty}) \subseteq V$. Therefore, if X is a regular space, then by Theorem 4.1, K attracts \bar{A} . \square

Lemma 4.1. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system and let $E \subseteq X$ and K be a compact set of X such that K attracts E . Then for every open cover of K there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(E, f_{1,\infty}) \cup K$ has a finite subcover.*

Proof. Let \mathcal{U} be an open cover of K . Since K is compact, it admits a finite subcover. Let $\{V_1, V_2, \dots, V_n\}$ be such subcover, i.e., $V_i \in \mathcal{U}$ and $K \subseteq \bigcup_{i=1}^n V_i$. Let $U = \bigcup_{i=1}^n V_i$. Since U is an open neighborhood of K and K attracts E , there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(E, f_{1,\infty}) \subseteq U = \bigcup_{i=1}^n V_i$. Hence, $\{V_1, V_2, \dots, V_n\}$ is also a finite subcover of $\gamma_m(E, f_{1,\infty})$, which implies $\{V_1, V_2, \dots, V_n\}$ is a finite subcover of $\gamma_m(E, f_{1,\infty}) \cup K$. \square

The next theorem contains the main properties of the ω -limit sets of non-autonomous discrete systems in regular spaces.

Theorem 4.2. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system, where X is a regular topological space. Let $E \subseteq X$ and K be a compact set of X such that K attracts E . Then the following properties hold:*

- (1) $\omega(E, f_{1,\infty})$ is a nonempty compact set;
- (2) if $F \subseteq X$ is a closed set, then F attracts E if and only if $\omega(E, f_{1,\infty}) \subseteq F$;
- (3) $\omega(E, f_{1,\infty})$ attracts E .

Proof. (1) We first show that $\omega(E, f_{1,\infty})$ is a nonempty compact set. To simply the proof we divide it in some steps.

Step 1. Let $\{x_n\} \subseteq E$ be a sequence and $\{n\}$ be a positively divergent sequence. Then we have that $\{f_1^n(x_n) : n \in \mathbb{Z}_+\} \cup K$ is a compact set. In fact, let \mathcal{U} be an open cover of $\{f_1^n(x_n) : n \in \mathbb{Z}_+\} \cup K$. Since K is a compact set and K attracts E , by Lemma 4.1, for every open cover \mathcal{U} of K there exist $m \in \mathbb{N}$ and a finite subcover $\mathcal{V} \subseteq \mathcal{U}$ such that $\gamma_m(E, f_{1,\infty}) \cup K \subseteq \bigcup_{V \in \mathcal{V}} V$. Clearly, since $n \rightarrow \infty$, there exists $n \geq m$ such that $(x_n, n) \in E \times [m, +\infty)$. Therefore, \mathcal{V} is a finite subcover of $\{f_1^n(x_n) : n \geq m\} \cup K$, i.e., $\{f_1^n(x_n) : n \geq m\} \cup K \subseteq \bigcup_{V \in \mathcal{V}} V$. Since \mathcal{U} is also an open cover of $\{f_1^n(x_n) : n \in \mathbb{Z}_+\}$, there exists a finite subcover \mathcal{W} of $\{f_1^n(x_n) : 0 \leq n \leq m\}$. Further, $\mathcal{V} \cup \mathcal{W}$ is a finite subcover $\{f_1^n(x_n) : n \in \mathbb{Z}_+\} \cup K$. Hence, $\overline{\{f_1^n(x_n) : n \in \mathbb{Z}_+\} \cup K}$ is compact.

Step 2. We will show that $\bigcap_{n \in \mathbb{Z}_+} \overline{\{f_1^q(x_q) : q \geq n\}} \subseteq \omega(E, f_{1,\infty})$.

Define the closed sets $C_m = \overline{\{f_1^q(x_q) : q \geq m\}}$ for each $m \in \mathbb{Z}_+$. Since X is a regular topological space and $\{f_1^n(x_n) : n \in \mathbb{Z}_+\} \cup K$ is compact by Step 1, we have $\{f_1^n(x_n) : n \in \mathbb{Z}_+\} \cup K$ is a closed set of X . Furthermore, $C_m \subseteq \{f_1^n(x_n) : n \in \mathbb{Z}_+\} \cup K$ for each $m \in \mathbb{Z}_+$. Hence, C_m are compact sets for all $m \in \mathbb{N}$. Since $\{C_m\}$ is a decreasing sequence and C_m are compact sets for all $m \in \mathbb{N}$, so by the finite intersection property, we have $\bigcap_{m \in \mathbb{Z}_+} C_m \neq \emptyset$. Let $x \in \bigcap_{m \in \mathbb{Z}_+} C_m$. Since $q \rightarrow \infty$, so for every $m \in \mathbb{N}$ there exists $q \in \mathbb{Z}_+$ such that $q \geq m$. Since $\{x_q\}$ is a sequence in E , we have $\bigcap_{n \in \mathbb{Z}_+} C_n \subseteq \overline{\{f_1^q(x_q) : q \geq m\}} \subseteq \overline{\gamma_m(E, f_{1,\infty})}$, implying $x \in \overline{\gamma_m(E, f_{1,\infty})}$. As $m \in \mathbb{Z}_+$ is

arbitrary, which implies $x \in \bigcap_{m \in \mathbb{Z}_+} \overline{\gamma_m(E, f_{1,\infty})} = \omega(E, f_{1,\infty})$. Furthermore, we have $\bigcap_{m \in \mathbb{Z}_+} \overline{\{f_1^q(x_q) : q \geq n\}} \subseteq \omega(E, f_{1,\infty})$. Therefore, $\omega(E, f_{1,\infty}) \neq \emptyset$.

Step 3. We will prove that $\omega(E, f_{1,\infty}) \subseteq K$, which implies $\omega(E, f_{1,\infty})$ is compact.

We suppose by contradiction that $\omega(E, f_{1,\infty}) \not\subseteq K$, thus there exists $x \in \omega(E, f_{1,\infty})$ and $x \notin K$. Since X is a Hausdorff space, there exist an open neighborhood U of x and an open set V containing K such that $U \cap V = \emptyset$. Moreover, K attracts E , there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(E, f_{1,\infty}) \subseteq V$. Furthermore, we have $U \cap \gamma_m(E, f_{1,\infty}) = \emptyset$. Hence, $x \notin \overline{\gamma_m(E, f_{1,\infty})}$. As $x \in \omega(E, f_{1,\infty})$, then $x \in \overline{\gamma_n(E, f_{1,\infty})}$ for every $n \in \mathbb{Z}_+$. In particular, we take $n = m$, implying $x \in \overline{\gamma_m(E, f_{1,\infty})}$. This is a contradiction. Hence, we have $\omega(E, f_{1,\infty}) \subseteq K$. Since K is compact and $\omega(E, f_{1,\infty})$ is closed, it follows that $\omega(E, f_{1,\infty})$ is a compact set of X .

(2) We show that, if $F \subseteq X$ is closed and F attracts E , then $\omega(E, f_{1,\infty}) \subseteq F$.

We suppose by contradiction that $\omega(E, f_{1,\infty}) \not\subseteq F$, i.e., there exists $x \in \omega(E, f_{1,\infty})$ and $x \notin F$. By the regularity property of X , there exist an open set U containing F and an open set V of x such that $U \cap V = \emptyset$. Since F attracts E , there exists $m \in \mathbb{Z}_+$ such that $\gamma_m(E, f_{1,\infty}) \subseteq U$, which implies $\gamma_m(E, f_{1,\infty}) \cap V = \emptyset$. Hence, $x \notin \overline{\gamma_m(E, f_{1,\infty})}$. As $x \in \omega(E, f_{1,\infty})$, we have $x \in \overline{\gamma_m(E, f_{1,\infty})}$. This is a contradiction. Therefore, $\omega(E, f_{1,\infty}) \subseteq F$.

Conversely, we prove that, if $\omega(E, f_{1,\infty}) \subseteq F$, then F attracts E .

We suppose by contradiction that there exists an open set U containing F such that for every $n \in \mathbb{Z}_+$, $\gamma_n(E, f_{1,\infty}) \not\subseteq U$. Hence, for every $n \in \mathbb{Z}_+$ there exist $q_n \geq n$ and $x_{q_n} \in E$ such that $f_1^{q_n}(x_{q_n}) \notin U$, we have $f_1^{q_n}(x_{q_n}) \in X \setminus U$, where $X \setminus U$ is a closed set of X . Therefore, $\overline{\{f_1^{q_m}(x_{q_m}) : m \geq n\}} \subseteq X \setminus U = X \setminus U$. Furthermore, $\bigcap_{n \in \mathbb{Z}_+} \overline{\{f_1^{q_m}(x_{q_m}) : m \geq n\}} \subseteq X \setminus U$, by the finite intersection property, we have

$$\bigcap_{n \in \mathbb{Z}_+} \overline{\{f_1^{q_m}(x_{q_m}) : m \geq n\}} \neq \emptyset.$$

From above Step 2 of (1), we have $\bigcap_{m \in \mathbb{Z}_+} \overline{\{f_1^m(x_m) : m \geq n\}} \subseteq \omega(E, f_{1,\infty})$, which implies $\bigcap_{n \in \mathbb{Z}_+} \overline{\{f_1^{q_m}(x_{q_m}) : m \geq n\}} \subseteq \omega(E, f_{1,\infty})$. Since for every $y \in \bigcap_{n \in \mathbb{Z}_+} \overline{\{f_1^{q_m}(x_{q_m}) : m \geq n\}}$, we have $y \notin U$, implying $y \notin F$. As $y \in \omega(E, f_{1,\infty})$ and $\omega(E, f_{1,\infty}) \subseteq F$, we have $y \in F$. This is a contradiction. Therefore, F attracts E .

(3) By the result of (1), $\omega(E, f_{1,\infty})$ is a nonempty closed set of X . Since $\omega(E, f_{1,\infty}) \subseteq \omega(E, f_{1,\infty})$, then by the result of (2), $\omega(E, f_{1,\infty})$ attracts E . \square

Corollary 4.1. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system, where X is a regular space. If B is a nonempty subset of X such that $\overline{\gamma(B, f_{1,\infty})}$ is a compact set, then $\omega(B, f_{1,\infty})$ is a nonempty compact set and $\omega(B, f_{1,\infty})$ attracts B .*

Proof. Since $\gamma(B, f_{1,\infty}) = \bigcup_{x \in B} \{f_1^n(x) : n \in \mathbb{Z}_+\}$, then for every $m \in \mathbb{Z}_+$, we have $f_1^m(B) \subseteq \gamma(B, f_{1,\infty})$. Furthermore, for any open neighborhood U of $\overline{\gamma(B, f_{1,\infty})}$, we have $f_1^m(B) \subseteq \overline{\gamma(B, f_{1,\infty})} \subseteq U$. Hence, $\overline{\gamma(B, f_{1,\infty})}$ attracts B . By (1) of Theorem 4.2, we have $\omega(B, f_{1,\infty})$ is a nonempty compact set. Therefore, by (3) of Theorem 4.2, $\omega(B, f_{1,\infty})$ attracts B . \square

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