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On O^*-representability and C^*-representability of $*$-algebras.

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Abstract

A characterization of the $*$-subalgebras of L(H) analogues to Choi and Effros characterization of abstract operator systems is presented. An internal characterization of the C^*-representability of bounded $*$-algebras is obtained and for a large class of $*$-algebras A, C^*-representability is proved to be equivalent to the condition that the equation $x^*x = 0$ has only zero solution in $M_n(A)$ for all $n \geq 1$. Sufficient conditions for the O^*-representability of a $*$-algebra in terms of its Göbner basis are given. These conditions are generalization of the unshrinkability of monomial $*$-algebras introduced by C. Lance and P. Tapper. The applications to $*$-doubles, monomial $*$-algebras etc. are presented.

KEYWORDS: $*$-algebra, C^*-algebra, O^*-algebra, A^*-algebra, Banach $*$-algebra, noncommutative Gröbner basis, Hilbert space, faithful representation, algebraically admissible cone.

1 Introduction

This paper concerns with one aspect of the theory of $*$-algebras: the conditions for a $*$-algebra to be faithfully represented by operators on a Hilbert space.

The term "algebra of unbounded operators" admits different interpretations. In present work this term means O^*-algebra ([15, p.36]), i.e. a $*$-subalgebra of the algebra of linear operators acting on a pre-Hilbert space. Let E denote a pre-Hilbert space and H a

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Hilbert space which is the completion of E. The $*$-algebras of linear operators acting on these spaces are denoted by L(E) and L(H). Let A be a $*$-algebra over complex numbers. In this paper we study conditions for the existence of an embedding of A into L(E) and L(H). In the first case, it is equivalent to A being $*$-isomorphic to a O^*-algebra, such algebras will be called O^*-representable. In the second case A is isomorphic to a pre-C^*-algebra and we will say (following C. Lance and P.Tapper [7]) that A is C^*-representable.

If A is embedded in L(E) and every operator $a \in A$ is bounded then one can extend each $a \in A$ to an operator acting on H and thus obtain an inclusion $A \hookrightarrow$ L(H). In the general case A will be represented by unbounded operators on H such that the intersection of their domains is dense.

The celebrated Gelfand-Naimark theorem characterizes closed $*$-subalgebras of L(H) in terms of the norm on a $*$-algebra. There are also characterizations of such subalgebras in terms of orders on the set of self-adjoint elements [14]. The noncomplete subalgebras of L(H) are less well studied. A characterization of pre-C^*-algebras inside the class of normed $*$-algebras is given by G. Allan (see [4, p. 281]).

Our characterizations of C^*-representability given in Teorems 2 and 4 are significantly different from the ones cited above. We do not require any additional structure on the $*$-algebra. These characterizations are consequences of Theorem 1. The latter is analogues to the Choi and Effors characterization of abstract operator systems. The conditions of Theorem 2 could be considered as a generalization of a simple necessary condition of the C^*-representability of a $*$-algebra A that the equation $x^*x = 0$ has only zero solution in $M_n(A)$ for all $n \geq 1$. Such algebras are called completely positive. We also prove (see Corollaries 3 and 4) that for a large class of $*$-algebras complete positivity is also sufficient for C^*-representability. We also present several examples which show that the condition of complete positivity is not sufficient in general. As an application of the obtained results to Banach $*$-algebras we present a characterization of A^*-algebras in Theorem 6.

The literature on the O^*-representability of finitely presented $*$-algebras consists so far only of isolated classes of examples. In [9], the author proved that a monomial $*$-algebra is O^*-representable if and only if in the minimal defining set of monomial relations of the form $w_j = 0$ where w_j is a word, all w_j are unshrinkable. It should be noted that Lance and Tapper [13, 7] conjectured that such $*$-algebras are C^*-representable. This is still an open problem. In Section 3 we introduce a larger class of O^*-representable $*$-algebras which we call non-expanding (see Definition 6). This class is a generalization of monomial $*$-algebras. The main novelty of our approach is that we use the notion of Gröbner basis to define this class and use methods of Gröbner bases theory to establish O^*-representability and derive further results.

The sufficient conditions of non-expendability obtained in Section 4 allowed one to show that several known classes of $*$-algebras fall in the class of non-expanding $*$-algebras. Thus their representability could be treated from a unified point of view. These sufficient conditions are algorithmically verifiable for $*$-algebras given by a finite number of generators and relations.

2 Representability by bounded operators.

In this section several characterizations of representability of a $*$-algebra by bounded operators acting on a Hilbert space H are presented.

If a $*$-algebra A is $*$-isomorphic to a subalgebra of a C^*-algebra $\mathcal{A}$ then by the Gelfand-Naimark theorem A is also $*$-isomorphic to a subalgebra of L(H) and thus can be faithfully represented by bounded operators on H. Such $*$-algebra is called C^*-representable (see [7]).

Firstly we will present a criterion of C^*-representability in terms of *algebraically admissible cones*. Let A_{sa} denote the set of self-adjoint elements in A. The following definition was introduced in [11].

Definition 1. *Given a $*$-algebra A with unit e, we say that a subset $C \subset A_{sa}$ is algebraically admissible cone if*

(i) C is a cone in A_{sa} and $e \in C$;

(ii) $C \cap (-C) = \{0\}$;

(iii) $xCx^* \subseteq C$ for every $x \in A$;

The assumptions of the C^*-representability criterion given in Theorem 1 are the same as in Choi and Effros characterization of abstract operator systems [3], however we do not require any additional structure on the matrices over a $*$-algebra to exist and the matrix order is replaced with the order given by an algebraically admissible cone.

With a cone C we can associate a partial order $\geq_C$ on the real vector space A_{sa} given by the rule $a \geq_C b$ if $a - b \in C$. Henceforth we will suppress subscript C if it will not lead to ambiguity. Some elementary properties of this order which will be frequently used are given in the following.

Lemma 1. *The following properties hold.*

(a) $x^*x \in C$ *for every* $x \in A$*, in particular* $a^2 \in C$ *for* $a \in A_{sa}$.

(b) For $\lambda \in \mathbb{R}_+$ *and* $a \geq b$ *in* A_{sa} $\lambda a \geq \lambda b$ *and* $-\lambda b \geq -\lambda a$.

(c) If $a \geq b$ *and* $b \geq c$ *then* $a \geq c$.

(d) If $a \geq b$ *and* $c \in A_{sa}$ *then* $a + c \geq b + c$.

(e) If $a \geq b$ *and* $c \geq d$ *then* $a + c \geq b + d$.

(f) If $a \geq b$ *and* $x \in A$ *then* $x^*ax \geq x^*bx$.

Recall that an element $u \in A_{sa}$ is called an *order unit* for A_{sa} provided that for every $x \in A_{sa}$ there exists a positive real r such that $ru+x \in C$. An order unit u is called *Archimedean* if $ru+x \in C$ for all $r > 0$ implies that $x \in C$. A $*$-algebra is called *positive* if for every $x \in A$ the equality $x^*x = 0$ implies $x = 0$.

Our first characterization of C^*-representability is given in the following theorem.

Theorem 1. *A* $*$*-algebra* A *with unit* e *is* C^**-representable if and only if* A *is positive and there is an algebraically admissible cone on* A *such that* e *is an Archimedean order unit.*

The proof of the theorem will be divided into a sequence of lemmas.

Lemma 2. *Let* A *be a* $*$*-algebra with algebraically admissible cone* C *and unit* e *which is an order unit. The function* $\|\cdot\|$ *defined as*

$$\|a\| = \inf\{r > 0 : re \geq a \geq -re\} = \inf\{r > 0 : re \pm a \in C\}$$

is a seminorm on the $\mathbb{R}$*-space* A_{sa}*. Moreover* $\|x^*ax\| \leq \|x^*x\|\|a\|$ *for every* $x \in A$ *and* $a \in A_{sa}$.

Proof. If $re \geq a \geq -re$ then, by Lemma 1, for $\lambda > 0$ we have $\lambda re \geq \lambda a \geq -\lambda re$ and for $\lambda < 0$ we have $\lambda re \leq \lambda a \leq -\lambda re$. Hence $\|\lambda a\| = |\lambda|\|a\|$. To prove the subadditivity of $\|\cdot\|$ take arbitrary a and b in A. If $r_1 e \geq a \geq -r_1 e$ and $r_2 e \geq b \geq -r_2 e$ then, by Lemma 1, $(r_1+r_2)e \geq a+b \geq -(r_1+r_2)e$. Hence $\|a+b\| \leq \|a\|+\|b\|$.

From $re \geq a \geq -re$, by Lemma 1, it follows that $rx^*x \geq x^*ax \geq -rx^*x$ for every $x \in A$. For every $\varepsilon > 0$ we will have $(\|x^*x\|+\varepsilon)e \geq x^*x \geq -(\|x^*x\|+\varepsilon)e$. Thus $r(\|x^*x\|+\varepsilon)e \geq x^*ax \geq -r(\|x^*x\|+\varepsilon)e$. Letting $\varepsilon \to 0$, we obtain $\|x^*ax\| \leq \|x^*x\|\|a\|$. □

Lemma 3. *Let A be a $*$-algebra with algebraically admissible cone C and with unit e which is an Archimedean order unit. For $x \in A$ define $|x| = \sqrt{\|x^*x\|}$. Then*

1. $|\lambda x| = (\lambda\overline{\lambda})^{1/2}|x|$ *for every* $\lambda \in \mathbb{C}$ *and* $x \in A$;
2. $|xy| \leq |x||y|$ *for every* x, y *in* A;
3. $\|a\| \leq |a|$ *for every* $a \in A_{sa}$.

Proof. The first statement is trivial.

For x, y in A, by Lemma 2, we have $\|(xy)^*xy\| = \|y^*(x^*x)y\| \leq \|y^*y\|\|x^*x\|$. Hence $|xy| \leq |x||y|$.

Clearly, for every $\alpha \in \mathbb{R}$, $\alpha \pm a \in A_{sa}$. Hence, by Lemma 1, $(\alpha \pm a)^2 \in C$. Thus

$$-(\alpha^2 + a^2) \leq 2\alpha a \leq \alpha^2 + a^2,$$

and for $\alpha = \|a\|$ one has

$$-(\|a\|^2 + a^2) \leq 2\|a\|a \leq \|a\|^2 + a^2.$$

If $a^2 \leq \varepsilon$ then

$$-(\|a\|^2 + \varepsilon) \leq 2\|a\|a \leq \|a\|^2 + \varepsilon.$$

Consequently, $\|2 \cdot \|a\| \cdot a\| \leq \|a\|^2 + \varepsilon$ and, thus, $\|a\|^2 \leq \varepsilon$. Letting $\varepsilon \searrow \|a^2\|$ we obtain that $\|a\|^2 \leq \|a^2\|$. Therefore, $\|a\| \leq |a|$. □

Lemma 4. *Let A be a $*$-algebra with algebraically admissible cone and unit e which is an Archimedean order unit. Then $|\cdot|$ is a seminorm on A satisfying C^*-axiom, i.e. $|x^*x| = |x|^2$ for every $x \in A$.*

Proof. First we will prove that $|x^*| = |x|$ for every $x \in A$. For this it suffices to show that $|x^*| \le |x|$. In fact, if this is true then $|x| = |(x^*)^*| \le |x^*|$. By definition $|x^*|^2 = \|xx^*\|$. Since xx^* is self-adjoint, $\|xx^*\| \le |xx^*|$ by Lemma 3. Thus $|x^*|^2 \le |xx^*| \le |x||x^*|$. If $|x^*| = 0$ then $0 \le |x|$ and the required inequality holds, otherwise we have $|x^*| \le |x|$.

For every $x \in A$ by Lemma 3 we have $|x^*x| \le |x||x^*| = |x|^2$ and $|x|^2 = \|x^*x\| \le |x^*x|$. Thus $|x|^2 = |x^*x|$.

Applying the previous equality to a self-adjoint element a we obtain $|a|^2 = |a^*a| = |a^2|$. Thus $|a^2| = |a|^2$.

We will prove that $|x+y| \le |x| + |y|$. For every $x \in A$ one has $\|x^2 + x^{*2}\| \le 2\|x^*x\|$. Indeed, since $x + x^*$ is self-adjoint we have $(x+x^*)^2 \ge 0$, i.e

$$x^2 + x^{*2} + xx^* + x^*x \ge 0.$$

From this it follows that $x^2 + x^{*2} \ge -\{x, x^*\}$ where $\{x, x^*\} = xx^* + x^*x$. Since $i(x - x^*)$ is also self-adjoint we have $-(x-x^*)^2 \ge 0$. Thus $\{x, x^*\} \ge x^2 + x^{*2}$ and therefore $-\{x, x^*\} \le x^2 + x^{*2} \le \{x, x^*\}$. Hence

$$\begin{aligned} \|x^2 + x^{*2}\| &\le \|\{x, x^*\}\| = \|xx^* + x^*x\| \\ &\le \|xx^*\| + \|x^*x\| = |x|^2 + |x^*|^2 \\ &= 2|x|^2 = 2\|xx^*\|. \end{aligned}$$

We will prove the following.

$$\|x^* + x\| \le 2\|x^*x\|^{1/2} = 2|x|. \tag{1}$$

Indeed, for self-adjoint a by Lemma 3 $\|a\|^2 \le \|a^2\|$ and

$$\begin{aligned} \|x + x^*\|^2 &\le \|(x+x^*)^2\| \\ &= \|x^2 + x^{*2} + xx^* + x^*x\| \\ &\le \|x^2 + x^{*2}\| + \|xx^* + x^*x\| \\ &\le 2\|x^*x\| + \|x^*x\| + \|xx^*\| \\ &= 4\|x^*x\|. \end{aligned}$$

Thus $\|x^* + x\| \le 2|x|$. We will prove that $\|x^*y + y^*x\| \le 2|x||y|$. Indeed, the substitution x^*y instead of x in (1) implies $\|x^*y + y^*x\| \le 2|x^*y| \le 2|x||y|$.

The inequality $|x+y| \leq |x|+|y|$ follows from the following estimates:

$$\begin{array}{rcl} |x+y|^2 & = & \|(x+y)^*(x+y)\| \\ & = & \|x^*x + y^*y + x^*y + y^*x\| \\ & \leq & \|x^*x\| + \|y^*y\| + \|x^*y + y^*x\| \\ & \leq & |x|^2 + |y|^2 + 2|x||y| \\ & = & (|x| + |y|)^2. \end{array}$$

□

Proof of Theorem 1. To prove the statement of the theorem it is sufficient to show that the norm $|\cdot|$ defined in Lemma 3 is a C^*-norm on A. In view of Lemma 4 we need only to prove that $|x| = 0$ implies $x = 0$ for every $x \in A$. Assume that $|x| = 0$, i.e. $\inf\{r > 0 : re \geq x^*x \geq -re\} = 0$. Thus $re \pm x^*x \in C$ for every $r > 0$. Since e is Archimedean we have that $\pm x^*x \in C$. As $C \cap (-C) = \{0\}$ we conclude that $x^*x = 0$. The positiveness of A implies $x = 0$.

If A is C^*-representable then A can be identified with a unital subalgebra of a C^*-algebra $\mathcal{A}$. We can define then $C = \mathcal{A}_+ \cap A$. Using well know properties of the cone of positive elements in a C^*-algebra one can easily show that C is an algebraically admissible cone and e is an Archimedean order unit. □

The main drawback of the characterization given in Theorem 1 is that it requires some additional structure on a $*$-algebra. So our next objective is to give an intrinsic characterization of C^*-representability using the algebraic structure of involuting algebra alone. It is turn out to be possible under the assumption of boundedness which is an algebraic version of a well known notion of $*$-boundedness.

Recall that a $*$-algebra A is called *$*$-bounded* if for every $a \in A$ there is constant C_a such that for every $*$-representation $\pi : A \to B(H)$ we have $\|\pi(a)\| \leq C_a$.

Definition 2. *An element $a \in A_{sa}$ is called positive if $a = \sum_{i=1}^n a_i^* a_i$ for some $n \geq 1$ and $a_i \in A$ for $1 \leq i \leq n$. The set of positive elements in A will be denoted by A_+.*

It is easy to check that the cone A_+ on a unital $*$-algebra A is an algebraically admissible cone. To formulate our next result we will need some definitions from the theory of ordered algebras ([14]).

Definition 3. *Let A be a unital $*$-algebra.*

1. *An element $a \in A_{sa}$ is bounded if there is $\alpha \in \mathbb{R}_+$ such that $\alpha e \geq a \geq -\alpha e$.*
2. *An element $x = a + ib$ with $a,\ b \in A_{sa}$ is bounded if so are the elements a and b.*
3. *The algebra A is bounded if all its elements are bounded.*

We will collect some useful facts about bounded elements in the following Lemma. They can be found in [14, proposition 1, p. 196]:

Lemma 5. *Let A be a unital $*$-algebra then*

1. *the set of all bounded elements in A is a $*$-subalgebra in A;*
2. *an element $x \in A$ is bounded if and only if xx^* is bounded;*
3. *if A is generated by a set $\{s_j\}_{j\in J}$ such that each $s_j s_j^*$ is bounded then A is bounded.*

For example, an algebra A generated by isometries (i.e., elements satisfying relation $s^*s = e$) or projections (i.e., elements satisfying relation $p^* = p = p^2$) is bounded. One can easily prove that a bounded $*$-algebra A is $*$-bounded and thus there exists its universal enveloping C^*-algebra $C^*(A)$.

Recall the definition of $*$-radical introduced by Gelfand and Naimark (see [4, (30.1)]).

Definition 4. *For a $*$-algebra A the $*$-radical is the set $\mathrm{R}^*(A)$ which is the intersection of the kernels of all topologically irreducible $*$-representations of A by bounded operators on Hilbert spaces.*

It is known that $\mathrm{R}^*(A)$ is equal to the intersection of the kernels of all $*$-representations (see for example [4, Theorem (30.3)]). Clearly the factor algebra $A/\mathrm{R}^*(A)$ of a $*$-bounded algebra A is C^*-representable. As a direct corollary of Theorem 1 we obtain the following theorem proved in the author earlier paper [8].

Theorem 2. *Let A be a bounded $*$-algebra then the following holds.*

1. *$|x|$ coincides with the norm of the universal enveloping C^*-algebra $C^*(A)$ of $x \in A$, i.e. $|x| = sup_\pi \|\pi(x)\|$ where π runs*

over all $*$-representations of A by bounded operators on Hilbert spaces. Thus

$$\sup_{\pi} \|\pi(x)\|^2 = \inf_{f \in A_+} \{(xx^* + f) \cap \mathbb{R}e\}.$$

Moreover, $\|a\| = |a|$ for self-adjoint $a \in A$.

2. *The null-space of $|\cdot|$ which is $\mathrm{R}^*(A)$ consists of those x such that for every $\varepsilon > 0$ there are $x_1, \ldots, x_n$ in A satisfying the equality*

$$x^*x + \sum_{j=1}^{n} x_j^* x_j = \varepsilon e. \tag{2}$$

3. *A is C^*-representable if and only if $\mathrm{R}^*(A) = \{0\}$.*

Proof. Since every x in A is bounded there are real $\alpha > 0$ and $x_1, \ldots, x_m$ in A such that

$$xx^* + \sum_{i=1}^{m} x_i x_i^* = \alpha. \tag{3}$$

If π is a representation of A by bounded operators then $\|\pi(xx^*)\| \leq \alpha$. Thus $\sup_\pi \|\pi(x)\|^2 \leq \inf \alpha$, where π runs over all $*$-representations of A and infimum is taken over all α as in (3). Therefore $|x| \geq \sup_\pi \|\pi(x)\|$ for all $x \in A$. The converse inequality also holds since the right-hand side is the maximal pre-C^*-norm. This proves the universal property of the pre-norm $|\cdot|$. Obviously its null-space is $\mathrm{R}^*(A)$. By Lemma 3, $\|a\| \leq |a|$ for every self-adjoint $a \in A$. But inequality $-\alpha e \leq a \leq \alpha e$ implies that $-\alpha \mathrm{I} \leq \pi(a) \leq \alpha \mathrm{I}$ for every $*$-representation π and identity operator I. Hence $\|\pi(a)\| \leq \alpha$. From this follows $|a| \leq \|a\|$ and, consequently, $|a| = \|a\|$.

Thus we only have to prove that the null-space of $|\cdot|$ is the set of all x such that for every $\varepsilon > 0$ there are $x_1, \ldots, x_n$ in A such that (2) is fulfilled. As in the proof of Theorem 1 the null-space is the set of x such that $\inf\{r > 0 : re \geq x^*x \geq -re\} = 0$. But by definition of the order $re - x^*x \geq 0$ if there $x_1, \ldots, x_n \in A$ such that $re - x^*x = x_1^* x_1 + \ldots x_n^* x_n$ which proves (2) and the theorem. □

As a corollary of the above theorem we obtain the following description of the elements positive in every representation.

Corollary 1. *Let A be a bounded $*$-algebra. An element $a \in A_{sa}$ has the property that $\pi(a) \geq 0$ for each $*$-representation π of A in* L(H) *if and only if for every $\varepsilon > 0$ there are $x_1, \ldots, x_n \in A$ such that $a + \varepsilon = \sum_{j=1}^n x_j x_j^*$.*

Proof. Clearly, given $a \in A$, $\tau(a) \geq 0$ for every $*$-representation τ of A in L(H) if and only if $\pi(a) \geq 0$ for universal representation π of A. Since every representation could be factored through the universal representation π, $|x| = \|\pi(x)\|$ for all $x \in A$. Here $|\cdot|$ is the norm as in Theorem 2. A self-adjoint operator $\pi(a)$ is positive if and only if $\|C\mathrm{I} - a\| \leq C$ where $C = \|\pi(a)\|$ and I is the identity operator. Thus assuming $\pi(a) \geq 0$ we have, by Theorem 2, that $||a| - a| \leq |a|$ and hence $\||a| - a\| \leq |a|$. Consequently, $|a| - a \leq |a| + \varepsilon$ for every $\varepsilon > 0$. Which means that $a + \varepsilon$ can be written as $\sum_{j=1}^n x_j x_j^*$ for some $x_j \in A$. The converse statement is obvious. □

It is a well known fact that for a finite dimensional $*$-algebra A the necessary and sufficient conditions for C^*-representability is that A is positive, i.e. the equation $x^*x = 0$ has only zero solution in A. For an infinite dimensional $*$-algebra A the above condition is not sufficient since there are positive (even commutative) $*$-algebras such that $M_2(A)$ is not positive (see [4, Example (32.6)]). This motivates the following definition.

Definition 5. *A $*$-algebra A is called completely positive if $M_n(A)$ is positive for every $n \geq 1$.*

We will prove below that for a large class of $*$-algebras the complete positivity is equivalent to C^*-representability. However, we will also present examples of completely positive algebras which are not C^*-representable.

Consider the inductive limit $M_\infty(\mathbb{C}) = \lim(M_n(\mathbb{C}), \phi_n)$ where

$$\phi_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

is an embedding of $M_n(\mathbb{C})$ into $M_{n+1}(\mathbb{C})$. It is clear that A is completely positive if and only if $A \otimes M_\infty(\mathbb{C})$ is positive. Since the $*$-algebra $M_\infty(\mathbb{C})$ is not unital and is not finitely generated we prefer to replace it with the Teoplitz $*$-algebra $\mathcal{T} = \mathbb{C}\langle u, u^* | u^*u = e\rangle$ in the above characterization of complete positivity.

Theorem 3. *For a $*$-algebra A the following conditions are equivalent.*

1. *A is completely positive.*
2. *For every $n \geq 1$ the equation $x_1^* x_1 + \ldots x_n^* x_n = 0$ has only zero solution $x_1 = \ldots = x_n = 0$ in A.*
3. *$A \otimes \mathcal{T}$ is positive.*

Proof. If $x_1^* x_1 + \ldots x_n^* x_n = 0$ for some $x_1, \ldots, x_n \in A$ then for a matrix $C \in M_n(A)$ with the first row equal to $(x_1, \ldots, x_n)$ and the rest rows being zero we have $CC^* = 0$. Thus (1) implies (2). If for some non-zero matrix $D \in M_n(A)$ we have $DD^* = 0$ and j-row is not-zero then considering (j, j)-entry in DD^* we have $d_{j1} d_{j1}^* + \ldots + d_{j1} d_{j1}^* = 0$. Thus (2) is equivalent to (1).

It is easy to see that the element $p = e - uu^*$ is a projection in $\mathcal{T}$ and the elements $e_{ij} = u^{i-1} p (u^*)^{j-1}$ for $i, j \leq n$ satisfy the matrix units relations and thus generate an algebra isomorphic to $M_n(\mathbb{C})$. From this it follows that $A \otimes \mathcal{T}$ contains a subalgebra isomorphic to $A \otimes M_\infty(\mathbb{C})$. Hence the condition that $A \otimes \mathcal{T}$ is positive implies that A is completely positive.

We prove now the converse statement. Assume that A is completely positive. Since the relation $u^* u - e$ constitutes a Gröbner basis for $\mathcal{T}$ the set $\{u^k u^{*l} | k \geq 0, l \geq 0\}$ forms a linear basis for $\mathcal{T}$. Thus arbitrary $x \in A \otimes \mathcal{T}$ can be written in the form $\sum_{i=1,j=1}^n a_{i,j} \otimes u^i u^{*j}$, where $a_{i,j} \in A$. Using the relation $u^* u = e$ we obtain

$$x^* x = \sum_{i \leq k} a_{i,j}^* a_{k,l} \otimes u^j u^{k-i} u^{*l} + \sum_{i' > k'} a_{i',j'}^* a_{k',l'} \otimes u^{j'} u^{k'-i'} u^{*l'} =$$

$$= \sum_{s=1}^n \sum_{l=1}^n \left[\sum_{j=1}^s \sum_{k=s-j+1}^n a_{j+k-s,j}^* a_{k,l} + \sum_{r=1}^l \sum_{i=l-r+1}^n a_{i,s}^* a_{i+r-l,r} \right] u^s u^{*l}.$$

Thus $x^* x = 0$ would imply that for every $1 \leq s, l \leq n$:

$$\sum_{j=1}^s \sum_{k=s-j+1}^n a_{j+k-s,j}^* a_{k,l} + \sum_{r=1}^l \sum_{i=l-r+1}^n a_{i,s}^* a_{i+r-l,r} = 0. \qquad (4)$$

For $s = 1$ and $l = 1$ we have $\sum_{k=1}^n a_{k,1}^* a_{k,1} + \sum_{i=1}^n a_{i,1}^* a_{i,1} = 0$. Since A is completely positive we have $a_{k,1} = 0$ for all $1 \leq k \leq n$. We will prove that $a_{k,t} = 0$ for all k using an induction on t. We have already check the base of the induction. So assume that $a_{k,m} = 0$ for all k and prove that $a_{k,m+1} = 0$. Setting $s = l = m + 1$ in (4) and using the induction hypothesis we obtain

$$\sum_{k=1}^{n} a_{k,m+1}^* a_{k,m+1} + \sum_{i=1}^{n} a_{i,m+1}^* a_{i,m+1} = 0.$$

Since A is completely positive we get $a_{k,m+1} = 0$ for all $1 \leq k \leq n$ which proves our induction claim and the theorem. $\square$

One can easily show that complete positivity is preserved under taking sub-direct products, direct limits and taking subalgebras. It also preserved under making extensions, i.e. if J is a $*$-ideal in A which, considered as $*$-algebra, is completely positive and such that A/J is also completely positive then A itself is completely positive. Indeed, if $\sum_{j=1}^{n} x_j^* x_j = 0$ in A then passing to the factor algebra A/J and using its complete positivity we obtain that x_j are elements from J. Using completely positivity of J we conclude that $x_j = 0$ for all j.

It is an open question whether the tensor product $A \otimes B$ of two completely positive $*$-algebras is completely positive. Even if we impose a stronger requirement of O^*-representability on B and require A to be completely positive we are unable to prove that $A \otimes B$ is completely positive. Hoverer, it can be easily checked that a tensor product of two of two O^*-representable algebras is O^*-representable.

There is a priori possibility to obtain new necessary conditions of C^*-and O^*-representability of $*$-algebra A by taking a tensor products of $A \otimes B$ with some representable algebra B and requiring this product to be positive. Our conjecture is, although, that we obtain no new necessary condition in this way.

Using Theorem 3 we can simplify the conditions of Theorem 2 in the following way.

Theorem 4. *Let A be a bounded unital $*$-algebra and $\mathcal{T}$ be the Teoplitz $*$-algebra. Then A is C^*-representable if and only if every $x \in A \otimes \mathcal{T}$ with the property that for every $\varepsilon > 0$ there exists $y \in A \otimes \mathcal{T}$ such that $xx^* + yy^* = \varepsilon(e - uu^*)$ is zero.*

Proof. To prove that A is C^*-representable it is suffices to prove that $\mathrm{R}^*(A) = \{0\}$. If $x \in \mathrm{R}^*(A)$ then, by Theorem 2, for every $\varepsilon > 0$ there are $x_1, \ldots, x_n \in A$ such that $xx^* + \sum_{j=1}^{n} x_j x_j^* = \varepsilon e$. Consider

$n \times n$-matrices X and C with coefficients in A such that the first row of X is $(x, 0, \dots, 0)$ and the first row of C is $(x_1, x_2, \dots, x_n)$ and all other rows of X and C are equal to zero.

Since the subalgebra B_n of $\mathcal{T}$ with basis e_{ij} is isomorphic to $M_n(\mathbb{C})$. One can identify B_n with $M_n(\mathbb{C})$ and consider the algebra $M_n(A) \simeq A \otimes M_n(\mathbb{C})$ as a subalgebra of $A \otimes \mathcal{T}$. Moreover, after this identification one has $XX^* + CC^* = \varepsilon(e - uu^*)$. Thus $X = 0$ and, consequently, $x = 0$.

The necessity of the conditions of the theorem follows easily from the fact that $\mathcal{T}$ is C^*-representable and thus its tensor product with any C^*-representable algebra A is also C^*-representable. □

Corollary 2. *Each bounded completely positive $*$-algebra A has a non-trivial representation in $B(H)$.*

Proof. Assume that $|e| = 0$. Then there are $x_1, \dots, x_m \in A$ such that $e + x_1 x_1^* + \dots + x_m x_m^* = \frac{1}{2}e$. Therefore $\sum_{j=1}^m x_j x_j^* + yy^* = 0$ where $y = \frac{1}{\sqrt{2}}e$, which contradicts the complete positivity of A. Hence $|e| \neq 0$. For the universal representation π of A, which is a faithful representation of the enveloping C^*-algebra $C^*(A)$, we have $\pi(e) \neq 0$. □

The assumptions of the previous corollary can be weakened. Recall that an ideal I of a $*$-algebra A is called *endomorphically closed* if $f(I) \subseteq I$ for every $*$-endomorphism $f : A \to A$. An algebra A is called *endomorphically simple* if it has only trivial endomorphically closed $*$-ideals. We will say that a $*$-ideal J of A is *square root closed* if for every elements $x_1, \dots, x_n \in A$ equality $\sum_{j=1}^n x_j x_j^* \in J$ implies that $x_j \in J$. This is equivalent to A/J being completely positive.

Corollary 3. *Let A be a bounded unital $*$-algebra without non-trivial endomorphically closed and square root closed $*$-ideals. Then A is C^*-representable if and only if A is completely positive.*

Proof. The necessity is obvious. Since the $*$-radical of a $*$-algebra is an endomorfically closed and a square root closed $*$-ideal which, by the previous corollary, does not coincide with A, it must be zero. □

Corollary 4. *If a unital bounded algebra A is a direct sum of endomorfically simple $*$-algebras A_n, then A is C^*-representable if and only if A is completely positive.*

Proof. Let π_n be the canonical $*$-homomorphism $A \to A_n$. By Lemma 5, for any $a \in A$, there are elements $a_j \in A$ and $c \in \mathbb{R}$ such that $ce - a^*a = \sum_{i=1}^n a_i^* a_i$. Thus $ce - \pi_n(a)\pi_n(a)^*$ is a positive element of A_n. Hence $|\pi_n(a)\pi_n(a)^*| < c$. Since π_n is surjective A_n is bounded by Lemma 5. The previous corollary then imply that each A_n is C^*-representable and hence the same is true for their direct sum A. □

Theorem 5. *A bounded $*$-algebra A is C^*-representable if and only if there are mappings $F : A_+ \to \mathbb{R}$ and $G : A_+ \to \mathbb{R}$ such that*

1. $F(aa^*) > 0$ for each $a \neq 0$

2. $G(\sum_{i=1}^n a_i a_i^*) \geq F(a_j a_j^*)$ for arbitrary elements $a_1, \dots, a_n \in A$ and $1 \leq j \leq n$.

3. $\lim_{\varepsilon \to 0+} G(\varepsilon e) = 0$ for $\varepsilon \in \mathbb{R}$.

Proof. If A is not C^*-representable, then there is a nonzero $x \in \mathrm{R}^*(A)$. By Theorem 8, for each $\varepsilon > 0$ one can find $x_1, \dots, x_l \in A$ such that $xx^* + \sum_{i=1}^l x_i x_i^* = \varepsilon e$ and thus $G(\varepsilon e) \geq F(xx^*)$. From this we obtain $F(xx^*) = \lim_{\varepsilon \to 0} G(\varepsilon e) = 0$ contrary to the condition 1 of the theorem.

If A is C^*-representable then there is pre-C^*-norm $\|\cdot\|$ on A. Put $G(x) = F(x) = \|x\|$. For each positive x in A, $F(x) = \sup s(x)$ where supremum is taken over all states on the enveloping C^*-algebra $C^*(A)$. For every state s we have $s(\sum_i x_i x_i^*) \geq s(x_j x_j^*)$ and, taking supremum, we obtain $G(\sum_i x_i x_i^*) \geq F(x_j x_j^*)$ □

Recall that a Banach $*$-algebra $(\mathcal{B}, \|\cdot\|)$ is called to be an A^*-algebra provided there exists a second norm $\rho(\cdot)$, not necessarily complete, which satisfies $\rho(xy) \leq \rho(x)\rho(y)$ and $\rho(x)^2 = \rho(x^*x)$ for all $x, y \in A$ (see [4, p.77]). The second norm is called auxiliary. As an application to Banach $*$-algebras we will get the following.

Theorem 6. *Let $(\mathcal{B}, \|\cdot\|)$ be a unital Banach $*$-algebra. Then the following are equivalent.*

1. *$\mathcal{B}$ is C^*-representable.*

2. *$\mathcal{B}$ is A^*-algebra.*

3. *There is function $f : \mathcal{B}_+ \to \mathbb{R}_+$ such that $f(x) = 0$ implies that $x = 0$ and for arbitrary $x_1, \ldots, x_n$ in $\mathcal{B}$ and every $1 \leq j \leq n$*

$$\|\sum_{i=1}^{n} x_i x_i^*\| \geq f(x_j x_j^*).$$

Proof. If $\mathcal{B}$ is C^*-representable then it can be identified with a $*$-subalgebra of a C^*-algebra $\mathcal{A}$ with norm $|\cdot|$. Then by definition $\mathcal{B}$ is a A^*-algebra with auxiliary norm $|\cdot|$.

Let $(\mathcal{B}, \|\cdot\|)$ be an A^*-algebra with auxilary norm $|\cdot|$ by [4, corollary (23.6)] there exists constant $\beta > 0$ such that $|x| \leq \beta\|x\|$ for all $x \in \mathcal{B}$. Thus for arbitrary $x_1, \ldots, x_n$ in $\mathcal{B}$ and $1 \leq j \leq n$ we will have

$$\|\sum_{i=1}^{n} x_i x_i^*\| \geq \frac{1}{\beta}|\sum_{i=1}^{n} x_i x_i^*| \geq \frac{1}{\beta}|x_j x_j^*|.$$

Hence we can take $f(x) = \frac{1}{\beta}|x|$ to see that (3) is fulfilled.

To prove that (3) implies (1) note that by [4, Proposition (22.6)] every element of $\mathcal{B}$ is a linear combination of unitary elements. Hence $\mathcal{B}$ is a bounded $*$-algebra. If we take $G(x) = \|x\|$ and $F(x) = f(x)$ then, by theorem 5, $\mathcal{B}$ is C^*-representable. □

We apply now Theorem 8 to the group $*$-algebras. Let G be a discrete group and $\mathbb{C}[G]$ its group $*$-algebra. Elements of $\mathbb{C}[G]$ could be considered both as a formal linear combinations of elements of G with complex coefficients and as a functions from G to $\mathbb{C}$ with finite support. Let P denote the set $\{\sum_{j=1}^{n} f_j f_j^* | n \in \mathbb{N}, f_j \in \mathbb{C}[G]\}$ which is a subset of the set of positive definite functions on G with compact support. Considered as a positive definite function element $\phi \in P$ give rise to a cyclic representation π_ϕ in a Hilbert space with cyclic vector ξ such that $\phi(s) = (\pi_\phi(s)\xi, \xi)$ for every $s \in G$. By [5, Lemma 14.1.1] for every $f \in \mathbb{C}[G]$ and $\phi \in P$ we have that $\|\pi_\phi(f)\| \leq \|\lambda(f)\|$ where λ denote left regular representation of $\mathbb{C}[G]$. Since $\delta_e \in P$ and $\pi_{\delta_e} = \lambda$, $\sup_{\phi \in P}\|\pi_\phi(f)\| = \|\lambda(f)\|$. Thus the set P, from one side, define the norm of the reduced group C^*-algebra $C^*_{red}(G)$ and, from the other side, by the next corollary it also defines the norm of group C^*-algebra $C^*(G)$.

Corollary 5. *Let $\|\cdot\|$ denote the norm on $C^*(G)$. Then for every $f \in \mathbb{C}[G]$ the following formula holds*

$$\|f\|^2 = \inf_{\phi \in P}\{(\phi + ff^*) \cap \mathbb{R}e\}.$$

Proof. Clearly P is the set of positive elements of $*$-algebra $\mathbb{C}[G]$. For every $f \in \mathbb{C}[G]$ norm $\|f\|$ is the norm of universal enveloping C^*-algebra of $\mathbb{C}[G]$ and consequently, by Theorem 8, $\|f\|^2 = \inf_{\phi\in P}\{(\phi + ff^*) \cap \mathbb{R}e\}$. $\square$

Since G is amenable if an only if reduced norm is equal to universal enveloping norm for every $f \in \mathbb{C}[G]$ we obtain the following.

Corollary 6. *A discrete group G is non-amenable if an only if there exists $f \in \mathbb{C}[G]$ and $\varepsilon > 0$ such that for every $g \in \mathbb{C}[G]$ element $\frac{\|fg\|_2}{\|g\|_2} + \varepsilon$ can not be presented in the form $ff^* + \sum_{j=1}^{n} f_j f_j^*$ for some $f_j \in \mathbb{C}[G]$. Here $\|g\|_2^2 = \sum_{k=1}^{m} |\alpha_k|^2$ for the element $g = \sum_{k=1}^{m} \alpha_k w_k$ with $\alpha_k \in \mathbb{C}$ and distinct $w_k \in G$.*

In the following example we present a completely positive bounded $*$-algebra which is not C^*-representable. The definitions of the Gröbner basis, the set of basis words BW and operator $\mathrm{R_S}$ used below could be found in the appendix.

Example 1.

Consider $*$-algebra given by generators and relations

$$A = \mathbb{C}\big\langle a,\ x | a^*a = qaa^*, xx^* + aa^* = e\big\rangle$$

where $0 < q < 1$. Clearly, A is bounded. It can be easily checked that the set $S = \{a^*a - qaa^*, xx^* - aa^* - e\}$ is a Gröbner basis of A. Thus the set BW consisting of the words containing no subword a^*a or xx^* forms a linear basis for A. For arbitrary z in $\mathbb{C}\big\langle a,\ x\big\rangle$ the element $\mathrm{R_S}(z)$ could be written as $\sum_{i=1}^{n} \alpha_i u_i x^{k_i}$, where u_i does not end with x, $k_i \geq 0$, $\alpha_i \neq 0$ and $u_i \in BW$ for all $1 \leq i \leq n$.

Let t be the minimal length of the words $u_i x^{k_i}$. Put $J = \{j : |u_j| = t\}$. Denote by $F(z)$ the sum of those α_i with $i \in J$ such that $u_i x^{k_i} = ww^*$ for some word w. We will prove that $F(zz^*) = \sum_{j\in J} |\alpha_j|^2$. Indeed,

$$\mathrm{R_S}(u_i x^{k_i} x^{*k_j} u_j^*) =$$
$$\begin{cases} -u_i(\sum_{1\leq s\leq k_i} x^{k_i-s} aa^* x^{*k_j-s}) u_j^* + \mathrm{R_S}(u_i u_j^*), & \text{if } k_i = k_j \\ -u_i(\sum_{1\leq s\leq min(k_i,k_j)} x^{k_i-s} aa^* x^{*k_j-s}) u_j^*, & \text{if } k_i \neq k_j \end{cases}$$

The sum $u_i(\sum_{1\leq s\leq min(k_i,k_j)} x^{k_i-s} aa^* x^{*k_j-s}) u_j^*$ contains no words of length t. Thus computing $F(zz^*)$ it is sufficient to consider only the sum $-u_i(\sum_{1\leq s\leq k_i} x^{k_i-s}aa^*x^{*k_j-s})u_j^* + \mathrm{R_S}(u_iu_j^*)$. Since both u_i and u_j do not end with x the element $\mathrm{R_S}(u_iu_j^*)$ is a monomial of length $|u_i|+|u_j|$. Thus, if some monomial $\mathrm{R_S}(u_iu_j^*)$ in $\mathrm{R_S}(zz^*)$ has minimal length (which is equal to $2t$) then $i,\ j \in J$ (in particular $|u_i| = |u_j|$). Equality $u_iu_j^* = ww^*$ implies $u_i = u_j$. Indeed, if u_i ends with a or with x^* or word u_j ends with a^* or with x^* then $\mathrm{R_S}(u_iu_j^*)$ is just $u_iu_j^*$ (as in free $*$-algebra). Thus using equality $u_iu_j^* = ww^*$ we can conclude that $u_i = u_j$. Otherwise, write $u_i = v_ia^{*k}$ and $u_j = v_ja^m$ where v_i does not end with a^* and v_j does not end with a. Thus $\mathrm{R_S}(u_iu_j^*) = q^{km}v_ia^ma^{*k}v_j^*$. If $m > k$ then, since $u_iu_j^* = ww^*$, we have $v_ia^{m_1} = w$ and $a^{m-m_1}a^{*k}v_j^* = w^*$, for some $1 \leq m_1 < m$. This is a contradiction since w ends with a and a^* simultaneously. Similarly if $m < k$ then $w = v_ia^ma^{*k_1}$ and $w^* = a^{*(k-k_1)}v_j^*$, for some $(1 \leq k_1 < k)$. We obtain that w ends with a and a^* which is again a contradiction. Thus $m = k$ and $w = v_ia^k = v_ja^k$. So $v_i = v_j$ and $u_i = u_j$. We have proved so far that $u_iu_j^* = ww^*$ implies that $u_i = u_j$. From this it easily follows that $F(zz^*) = \sum_{j\in J} |\alpha_j|^2$. Obviously $F(aa^*) > 0$ if $a \neq 0$ and

$$F(\sum_{i=1}^{n} a_ia_i^*) \geq \min_i F(a_ia_i^*),$$

end clearly $F(\varepsilon e) = \varepsilon$ for $\varepsilon \in \mathbb{R}$. Thus A is completely positive $*$-algebra. If π is a representation of A in Hilbert space then

$$\|\pi(aa^*)\| = \|\pi(a^*a)\| = q\|\pi(aa^*)\|,$$

which implies that $\|\pi(aa^*)\| = 0$. Thus A is not C^*-representable.

We will end this section by a few remarks on C^*-representability of finite dimensional algebras. Since C^*-representability for finite dimensional $*$-algebras is equivalent to positiveness it is natural to consider C^*-representability of their direct limits and inverse limits. It is routine to check that positiveness is also equivalent to C^*-representability for inductive limits of finite dimensional $*$-algebras. The case of inverse limits is much more complicated and there is no simple answer up to now. We will content ourself in this paper by presenting the following example of completely positive $*$-algebra which has a separating family of $*$-homomorphisms into finite dimensional $*$-algebras but which is not faithfully representable even in pre-Hilbert space.

Example 2.

Consider a $*$-algebra $A = \mathbb{C}\langle a | a^*a = qaa^* \rangle$, where $0 < q < 1$ which can be identified with the subalgebra generated by a in the algebra $\mathbb{C}\langle a,\ x | a^*a = qaa^*, xx^* + aa^* = e \rangle$ from Example 1. Algebra A is completely positive as a subalgebra in a completely positive $*$-algebra. It is clearly not C^*-representable. We claim that A has residual family of homomorphism with finite dimensional images. Denote by J_k the $*$-ideal generated by a^k. Since $S = \{a^*a - qaa^*\}$ is a Gröbner basis for A we have that the set of all words in a and a^* that contain no subword a^*a is a linear basis for A. Thus $\cap_{k\geq 3} J_k = \{0\}$ and, obviously, A/J_k is finite dimensional $*$-algebra linearly generated by $a^n a^{*m}$ where $m < k,\ n < k$. This proves our claim.

3 Generalization of unsrinkability condition and Gröbner bases. O^*-representability.

C. Lance and P. Tapper (cf. [7, 13]) studied C^*-representability of $*$-algebras A_w generated by x and x^* with one monomial defining relation $w = 0$ where $w = x^{\alpha_1} x^{*\beta_1} \ldots x^{\alpha_k} x^{*\beta_k}$. They conjectured that A_w is C^*-representable if and only if the word w is *unshrinkable*, i.e. w can not be presented in the form d^*du or ud^*d where u and d are words and d is non-empty. A very appealing feature of this conjecture is that being true it gives a condition of C^*-representability of a monomial $*$-algebras in terms of its defining relations. It is significantly different from other characterizations which require some additional structures on a $*$-algebra to be present. In [9] the author proved that a monomial $*$-algebra is O^*-representable if and only if the defining relations are unshrinkable words. In this section we will introduce a much more general class of $*$-algebras which is defined by imposing some conditions on the set of defining relations (see Definition 6). For this class we will prove O^*-representability. We also show that several unrelated, at first glance, classes of $*$-algebras fall in this class.

Denote by F_* a free associative algebra with generators $x_1, \ldots, x_m$, $x_1^*, \ldots, x_m^*$. We do not incorporate the number of generators in the notations explicitly since it will be always clear from the context. Algebra F_* is a $*$-algebra with involution given on generators by

$(x_j)^* = x_j^*$ for all $j = 1, \dots, m$. Forgetting about involution we get a free associative algebra with $2m$ generators F_{2m}. The algebra F_* is a semigroup algebra of a semigroup W of all words in generators $x_1, x_2, \dots, x_m,\ x_1^*, x_2^*, \dots, x_m^*$.

We have compiled all necessary prerequisites from Gröbner basis theory of noncommutative associative algebras in the appendix. Below we will explain how this theory will be applied for $*$-algebras.

A set $S \subseteq F$ of defining relations of an associative algebra A is called a Gröbner basis if it is closed under compositions (see Appendix). A Gröbner basis of a $*$-algebra A is a Gröbner basis of A considered as an associative algebra. We need to put some extra requirements on a Gröbner basis to make it "compatible" with the involution. The main requirement we impose is a generalization of the notion of unshrinkability of the word (see Definition 6 below). A set $S \subseteq F_*$ is called *symmetric* if the ideal $\mathcal{I}$ generated by S in F_* is a $*$-subalgebra of F_*. In particular, S is symmetric if $S^* = S$.

For the notations $u \prec w$, $\mathrm{R_S}(w)$, BW and order on W used below we refer the reader to the appendix.

Definition 6. *A symmetric subset $S \subseteq F_*$ closed under compositions is called non-expanding if for every $u, v, w \in BW$ such that $u \neq v$ and $ww^* \prec \mathrm{R}_S(uv^*)$ the inequality $w < \sup(u, v)$ holds, i.e. $w < u$ or $w < v$. If in addition for every word $d \in BW$ the word dd^* also belongs to BW then S is called strictly non-expanding.*

A $*$-algebra A is called (strictly) non-expanding if it possesses a Gröbner basis GB which is (strictly) non-expanding.

Lemma 6. *A symmetric closed under compositions subset $S \subseteq F_*$ is non-expanding if and only if for every $u, v \in BW$ such that $u > v$ and $|u| = |v|$ the property $uu^* \prec \mathrm{R}_S(uv^*)$ does not hold.*

Proof. Let for some $u, v, w \in BW$, $ww^* \prec \mathrm{R}_S(uv^*)$. Then $ww^* \leq uv^*$ and therefore $|w| \leq \frac{|u|+|v|}{2}$. If $|u| \neq |v|$ then $|w| < \max(|u|, |v|)$ and, consequently, $w < \sup(u, v)$. We can assume, henceforth, that $|u| = |v|$. Then $ww^* \leq uv^*$ implies that $w \leq u$. If $u < v$ then, clearly, $w < v$. If $u > v$ then by the assumptions of the Lemma $uu^* \not\prec \mathrm{R}_S(uv^*)$ and, hance, $w < u$ which proves the statement of the Lemma completely. ☐

Let $G \subseteq W_n$ and $T = [1, n] \cap \mathbb{Z}$ is an interval of positive integers with $n = |G|$. An *enumeration* of G is a bijection $\phi : G \to T$

such that $u > v$ implies $\phi(u) > \phi(v)$. It is easy to check that enumerations exist for any given G.

Let $\mathrm{H} : F_* \to F_*$ be a linear operator defined by the rule $\mathrm{H}(uu^*) = u$ for $u \in W$ and $\mathrm{H}(v) = 0$ if v is not of the form uu^* for some word u.

Fix a set $S \subseteq F_*$ closed under compositions, an enumeration $\phi : BW \to \mathbb{N}$ of the corresponding linear basis and a sequence of positive real numbers $\xi = \{a_k\}_{k\in\mathbb{N}}$. Define a linear functional $T_\xi^\phi : K \to \mathbb{C}$ by putting $T_\xi^\phi(u) = a_{\phi(u)}$ for every word $u \in BW$, where K denotes the linear span of BW. Let $n = |BW|$ which can be infinite and V denote a vector space over $\mathbb{C}$ with a basis $\{e_k\}_{k=1}^n$.

Definition 7. *Define $\langle \cdot, \cdot \rangle_\xi$ to be a sesquilinear form on V defined by the following rules*

$$\langle e_i, e_i \rangle_\xi = a_i,$$

and

$$\langle e_i, e_j \rangle_\xi = T_\xi^\phi \circ \mathrm{H} \circ \mathrm{R}_S(uv^*),$$

where $\phi(u) = i$, $\phi(v) = j$, $u, v \in BW$.

The definition is correct since u and v as above are unique.

Theorem 7. *If S is strictly non-expanding then there exists a sequence $\xi = \{a_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that the sesquilinear form $\langle \cdot, \cdot \rangle_\xi$ is positively defined.*

Proof. Let $g_{ij} = \langle e_i, e_j \rangle_\xi$ for $i,\ j \in \mathbb{N}$ and let $G = (g_{ij})_{1\le i,j\le\infty}$ denote the Gram matrix. We will use Silvester's criterion to show, by induction on m, that a_m can be chosen such that principal minor $\Delta_m > 0$. For $m = 1$ put $a_1 = 1$ then $\Delta_1 = 1 > 0$. Assume that $a_1, \dots, a_{m-1}$ are chosen such that $\Delta_1 > 0, \dots, \Delta_{m-1} > 0$.

By definition if $u \in BW$, then u^*u is also in BW. Thus by Definition 7 we have $\langle e_{\phi(u)}, e_{\phi(u)} \rangle_\xi = a_{\phi(u)}$. Take some $i \le m$ and $j \le m$ with $i \ne j$ and find unique $u, v \in BW$ such that $i = \phi(u)$, $j = \phi(v)$. Then $uv^* = \sum_k \alpha_k w_k$ for unique $\alpha_k \in \mathbb{C}$ and $w_k \in BW$. Clearly $\langle e_{\phi(u)}, e_{\phi(u)} \rangle_\xi$ is $\sum_k \alpha_k a_{\phi(h_k)}$ where the sum is taken over those k for which w_k is of the form $w_k = h_k h_k^*$ for some word h_k. Since S is non-expanding we have that $h_k < \sup(u, v)$. Hence g_{ij} is a polynomial in variables $a_1, \dots, a_{m-1}$. Decomposing determinant Δ_m by the m-th row we obtain $\Delta_m = \Delta_{m-1} a_m + p_m(a_1, \dots, a_{m-1})$ for some polynomial $p_m \in \mathbb{C}[a_1, \dots, a_{m-1}]$. Since $\Delta_{m-1} > 0$ it is clear that a_m can be chosen such that $\Delta_m > 0$. This completes the inductive proof. □

The space K is obviously isomorphic to V via the map $u \to e_{\phi(u)}$. Thus the inner product $\langle \cdot, \cdot \rangle_\xi$ on V gives rise to an inner product on K which will be denoted by the same symbol. It is a routine to check that $\langle u, v \rangle_\xi = \alpha(P(u \diamond v^*))$, where $P : F_* \to F_*$ is the projection on the linear span of positive words $W_+ = W \cap F_{*+}$, $\alpha : K \to \mathbb{C}$ is a linear functional and $\diamond$ is the operation defined in the appendix. Let $z \mapsto L_z$ denote the right regular representation of $A = F_*/\mathcal{I}$, i.e. $L_z(f) = fz$ for any z, $f \in A$.

Theorem 8. *Let $S \subseteq F_*$ be strictly non-expanding and let $\mathcal{I}$ be the ideal generated by S in F_*. Then the right regular representation L of the $*$-algebra $A = F_*/\mathcal{I}$ on a pre-Hilbert space $(K, \langle \cdot, \cdot \rangle_\xi)$ is a faithful $*$-representation.*

Proof. The representation stated in the theorem is associated by the GNS construction with the positive functional $\alpha(P(\cdot))$ on A. Thus it is a $*$-representation. Indeed, as in the GNS construction the set $N = \{a \in A | \alpha(P(aa^*)) = 0\}$ is a right ideal in A. We can define an inner product on A/N by the usual rule $\langle a+N, b+N \rangle = \alpha(P(a^*b))$. It is easy to verify that the right multiplication operators define a $*$-representation of A on pre-Hilbert A/N. The only difference with classical GNS construction is that this representation could not be, in general, extended to the completion of A/N.

We will show that this representation is faithful. Take any $f = \sum_{i=1}^n c_i w_i \in A$, where $c_i \in \mathbb{C}, w_i \in BW$. Without loss of generality consider w_1 to be the greatest word among w_j. Then $L_f(w_1^*)$ contains element $w_1 w_1^*$ with coefficient c_1. Hence $L_f \neq 0$. □

As a straightforward corollary of Theorem 8 we obtain the following.

Corollary 7. *Every strictly non-expanding $*$-algebra is O^*-representable.*

4 Sufficient conditions of strictly non-expendability. Examples.

In this section we will show that the class of strictly non-expanding $*$-algebras contains several known classes of $*$-algebras. To accomplish this we introduce below several other classes of $*$-algebras (see Definition 8, Corollary 8, and Theorem 10) and prove that they are contained in the class of non-expanding $*$-algebras. The definition given below may look complicated but, in fact, it is much easier to verify its conditions than the conditions of non-expanding $*$-algebra. A more thorough look reveals that the conditions of Definition 8 and in the theorems in this section are algorithmically verifiable. In the end of the section we will present some concrete examples.

We call a subset $S \subseteq F$ *reduced* if for every $s \in S$ and any word $w \prec s$ no word $\hat{s'}$ with $s' \in S$ is contained in w as a subword. If S is closed under compositions then S being reduced is equivalent to $\mathrm{R_S}(s) = s$ for every $s \in S$. If the set S is closed under compositions then one can obtain reduced set S' closed under compositions generating the same ideal by replacing each $s \in S$ with $\mathrm{R}_S(s)$.

Definition 8. *A symmetric reduced subset $S \subseteq F_*$ is called strictly appropriate if it is closed under compositions and for every $s \in S$ and every word $u \prec s$ such that $|u| = \deg(s)$ the following conditions hold.*

1. *The word u is unshrinkable.*
2. *If $u \neq \hat{s}$, $\hat{s} = ab$, and $u = ac$ for some words b, c and nonempty word a then for any $s_1 \in S$ such that there is word $w \prec s_1$, $w \neq \hat{s}_1, |w| = |\hat{s}_1|$ either word $\hat{s}_1$ does not contain u as a subword or $\hat{s}_1$ and u do not form a composition in such a way that $\hat{s}_1 = d_1 a d_2$ and $u = a d_2 d_3$ with some nonempty words d_1, d_2, d_3.*

A $*$-algebra A is called strictly appropriate $*$-algebra if it possesses a strictly appropriate Gröbner basis.

We will use the following simple combinatorial facts proved in [8, Lemma 2]. For every two words u and v in $*$-semigroup W such that $uv^* = ww^*$ for some word w either $u = v$ or $v = udd^*$ for some $d \in W$ or $u = vcc^*$ for some $c \in W$ depending on whether $|u| = |v|$ or $|u| < |v|$ or $|u| > |v|$.

If $S = S^*$ is a closed under composition subset of F_* such that $\hat{s}$ is unshrinkable for every $s \in S$ then $u \in BW$ if and only if $uu^* \in BW$.

In the following theorem for a word $w \in W$ of even length $w = w_1 w_2, |w_1| = |w_2|$ we will denote $\mathrm{H}_0(w) = w_1$.

Theorem 9. *Every strictly appropriate set $S \subseteq F_*$ is non-expanding. If in addition $S = S^*$ then S is strictly non-expanding.*

Proof. Let $u, v \in BW$ be such that $u > v$ and $|u| = |v|$.

1. If $uv^* \in BW$ then $uu^* \prec \mathrm{R}_S(uv^*)$ implies $uu^* = uv^*$ and, hence, $u = v$ which is a contradiction.

2. Now let $uv^* \notin BW$. There are words $p, q \in BW$ and element $s \in S$ such that $uv^* = p\hat{s}q$. Moreover, since $u, v \in BW$ none of them can contain $\hat{s}$ as a subword. Hence $\hat{s} = ab$ with nonempty words a and b such that $u = pa$ and $v^* = bq$. Write down $s = \alpha\hat{s} + \sum_{i=1}^{k} \alpha_k w_i + f$, where $w_i \in W$, $\alpha, \alpha_i \in \mathbb{C}$, and $\deg(f) < \deg(s)$ and $|\hat{s}| = |w_i|$ for all $i \in \{1, \ldots, k\}$. Assume that for some integer i word pw_iq belongs to BW and $pw_iq = uu^*$. If the middle of the word pw_iq comes across w_i, i.e. $\max(|p|, |q|) < |u|$, then $w_i = cd$, $u = pc$, and $w^* = dq$ with some nonempty words c, d. Hence $pc = q^*d^*$. If $|c| \leq |d|$ then $d^* = gc$ for some word g and so $w_i = cd = cc^*g^*$ which contradicts unshrinkability of w_i. If $|c| > |d|$ then $pc = q^*d^*$ implies $c = gd^*$ for some word g and we again see that $w_i = gd^*d$ is shrinkable. Thus $\max(|p|, |q|) \geq |u|$. If $|p| > |u|$ then $|u| = |p| + |a| > |u|$ which is impossible, hence $|v| = |b| + |q| > |u|$.

3. Let $uv^* = p\hat{s}q$ and $s = \alpha\hat{s} + \sum_i \alpha_i w_i + f$ as above and $uu^* \prec \mathrm{R}_S(pw_iq)$ for some i. Since $uu^* < pw_iq < uv^*$ word pw_iq begins with u. If $\hat{s} = ab$ such that $pa = u, bq = v^*$ then w_i begins with a. Therefore $\hat{s}$ and w_i begin with the same generator. Since $pw_iq \notin BW$ there is $s_1 = \alpha_1\hat{s}_1 + \sum_j \beta_j u_j + g \in S$ where $u_i \in W$, $\alpha_1, \beta_i \in \mathbb{C}$, and $\deg(g) < \deg(s_1)$ such that $pw_iq = p_1\hat{s}_1q_1$ for some words p_1, q_1. If we assume that for some j word $uu^* \prec \mathrm{R}_S(p_1u_jq_1)$ then $\mathrm{H}_0(p_1u_jq_1) = u$ since $p_1u_jq_1 < uv^*$. The word $\hat{s}_1$ can not be a subword in the first half of the word pw_iq since $\mathrm{H}_0(p_1u_jq_1) = \mathrm{H}_0(pw_iq) = u$ and assuming the contrary we see that $\hat{s}_1$ and u_j are both subwords of u in the same position, hence they must be equal $\hat{s}_1 = u_j$. The word $\hat{s}_1$ can not contain subword w_i because of condition 2 in the definition of strictly appropriateness. Obviously, $\hat{s}_1$ can not be a subword in q because $q \in BW$. Thus either w_i and $\hat{s}_1$ intersect (in the specified order) or $\hat{s}_1$ and w_i intersect in such a way that $\hat{s}_1 = d_1ad_2$ and $w_i = ad_2d_3$. But this contradicts the strictly appropriateness of S. So we have proved that S is non-expanding. The fact that for any word $g \in BW$ word gg^* lies in BW follows from the remark preceding the theorem (see also [8, lemma 2]). □

The following is a simplification of the preceding theorem which is easier to verify in examples.

Corollary 8. *Let $S \subseteq F_*$ be symmetric and closed under compositions. If for every $s \in S$ and every word $u \prec s$ such that $|u| = \deg(s)$ the word u is unshrinkable and words $\hat{s}$ and u begin with different generators then S is non-expanding. If in addition $S = S^*$ then S is strictly non-expanding.*

Example 3.

Let $\mathcal{L}$ be a finite dimensional real Lie algebra with linear basis $\{e_j\}_{j=1}^n$. Then its universal enveloping algebra $U(\mathcal{L})$ is a $*$-algebra with involution given on generators as $e_j^* = -e_j$. We claim that this $*$-algebra is non-expanding. Indeed $M = \{e_ie_j - e_je_i - [e_i, e_j], i > j\}$ is a set of defining relations for $U(\mathcal{L})$. It is closed under compositions (see example in [2] or use PBW theorem). Thus the set $S = \{e_j^* + e_j, 1 \leq j \leq n\} \cup M$ is also closed under compositions (we consider $e_1^* > e_2^* > \ldots > e_1^* > e_1 > \ldots > e_n$) since e_j^* and e_ke_l do not intersect for any j, k, l. It is easy to see that S is symmetric. Thus S is non-expanding by corollary 8. However, $S \neq S^*$ and S is not strictly non-expanding.

Theorem 10. *Let $S \subseteq F_*$ be a symmetric closed under compositions reduced subset such that the following conditions are satisfied.*

1. *For every $s \in S$ every word $w \prec s$ with $|w| = \deg(s)$ is unshrinkable.*
2. *For every s_1, $s_2 \in S$ and every word $u \prec s_1$ with $|u| = \deg(s_1)$ the words u and $\hat{s_2}$ do not form a composition.*

Then S is non-expanding. If in addition $S = S^*$ then S is strictly non-expanding.

Proof. Consider $u, v \in BW$ such that $u > v$ and $|u| = |v|$. We will prove that $uu^* \not\prec \mathrm{R}_S(uv^*)$. Assume the contrary. Then there is a sequence of words $\{q_i\}_{i=1}^n$ such that $q_1 = uv^*$, $q_n = uu^*$ and for every $1 \leq i \leq n-1$ there is $s_i \in S$ and words c_i, d_i, $u_i \in W$ such that $u_i \prec s_i$, $u_i \neq \hat{s}_i$, $|u_i| = |\hat{s}_i|$ and $q_i = c_i\hat{s}_id_i$, $q_{i+1} = c_iu_id_i$.

Let j be the greatest with the property that $\hat{s}_j$ intersects the middle of q_j. Such an index j exists because $j = 1$ satisfies this property and we are making our choice within a finite set. Clearly $j < n$ since otherwise u_{n-1} would be a subword in uu^* intersecting its middle and thus would be shrinkable, which contradicts assumption 1 of the theorem. Thus for every $i \in \{j+1, \ldots, n-1\}$ word $\hat{s}_i$ does not intersect the middle of the word $c_{i-1}u_{i-1}d_{i-1}$. But $\hat{s}_i$ could not be situated in the first half of this word because otherwise the first half of the word q_i would be strictly less than u and, consequently, $q_n < uu^*$ which is a contradiction. Thus $\hat{s}_i$ is a subword in the right half of the word q_i. If u_j and $\hat{s}_i$ does not form a composition for every $i \in \{j+1, \ldots, n-1\}$ then u_j is a subword in uu^* intersecting its middle and, thus, shrinkable. This contradicts assumption 1 of the theorem. Hence u_j and $\hat{s}_k$ intersect for some $k \in \{j+1, \ldots, n-1\}$ contrary to assumption 2 of the theorem. This proves that $uu^* \not< \mathrm{R}_S(uv^*)$ and finishes the proof of the theorem. □

Examples.

1. Let $S = \{w_j\}_{j\in\Re}$ be a symmetric set consisting of unshrinkable words. Since compositions of any two words are always zero this set is closed under compositions. The other conditions in the definition of strictly non-expanding set is obvious. Thus $*$-algebra

$$\mathbb{C}\big\langle x_1, \ldots, x_n, x_1^*, \ldots, x_n^* | w_j,\ j \in \Re\big\rangle$$

is O^*-representable.

2. Consider in more detail the simplest example of monomial $*$-algebras $A_{x^2} = \mathbb{C}\langle x, x^* | x^2 = 0, x^{*2} = 0\rangle$.

It was proved in [13] that $*$-algebra $\mathbb{C}\langle x, x^* | x^p = 0, x^{*p} = 0\rangle$ is C^*-representable for every integer $p \geq 1$. We will show that among the representations of A_{x^2} given by Theorem 8 there is a $*$-representation in bounded operators. It is an open problem for arbitrary $A_w = \mathbb{C}\langle x, x^* | w = 0, w^* = 0\rangle$ with unshrinkable word w.

It can be easily verified that BW consists of the words $u_k = x(x^*x)^k$, $v_k = x^*(xx^*)^k$, $a_m = (xx^*)^m$, $b_m = (x^*x)^m$ where $k \geq 0, m \geq 1$. Obviously BW_+ consists of the words a_m and b_m ($m \geq 1$). If z, $w \in BW$ then $zw^* \in W_+$ if and only if z and w belong simultaneously to one of the sets $\{a_k\}_{k\geq 1}$, $\{b_k\}_{k\geq 1}$, $\{u_k\}_{k\geq 0}$, $\{v_k\}_{k\geq 0}$. Moreover,

$$u_k u_t^* = a_{k+t+1}, v_k v_t^* = b_{k+t+1}, a_m a_n^* = a_{n+m}, b_m b_n^* = b_{n+m}.$$

Consider the following ordering

$$u_0 < u_1 < \ldots < a_1 < a_2 < \ldots < v_0 < v_1 < \ldots < b_1 < b_2 < \ldots.$$

Denote $\alpha(a_m) = \alpha_m$, $\alpha(b_m) = \beta_m$ then the Gram matrix of the inner product defined in theorem 7 is $\mathrm{diag}(A, A', B, B')$ where A, A', B, B'

are Hankel matrices $A = (\alpha_{i+j-1})_{ij}$, $A' = (\alpha_{i+j})_{ij}$, $B = (\beta_{i+j-1})_{ij}$, $B' = (\beta_{i+j})_{ij}$. Note that Y' obtained from Y by cancelling out the first column (here Y stands for A or B).

Thus the question of positivity of the form $\langle \cdot, \cdot \rangle$ is reduced to the question of simultaneous positivity of two Hankel matrices C and C' where the second is obtained from the first by cancelling out the first column. We will show that such matrices A, A', B, B' could be chosen to be positive and such that $B = A$ and that the representation in theorem 8 is in bounded operators.

Let $f : [0, 1] \to [0, 1]$ be continuous function $f(x) > 0$ for all $x \in [0, 1]$. Let

$$\alpha_m = \int_0^1 t^{m+1} f(t) dt$$

be the moments of the measure with density $f(t)$. It is well known that the moment matrix $A = (\alpha_{i+j-1})_{i,j=1}^n$ is positively defined. But then A' is the moment matrix of the measure with density $tf(t)$ and thus is also positively-defined. We can put $B = A$.

To prove that the representation is in bounded operators we need only to verify that the operator L_x of multiplication by x is bounded. Obviously, $xu_k = 0$ and $xa_m = 0$ for all $k \geq 0$ and $m \geq 1$. Moreover, $||xv_k||^2 = \langle a_{k+1}, a_{k+1} \rangle = \alpha_{2(k+1)}$, $||v_k||^2 = \alpha(b_{2k+1}) = \beta_{2k+1} = \alpha_{2k+1}$. Analogously, $||xb_k||^2 = \alpha_{2k+1}$ and $||b_k||^2 = \alpha_{2k}$. Thus L_x is bounded if there is a constant $c \geq 0$ such that for all $k \geq 1$

$$\alpha_{2(k+1)} \leq c\alpha_{2k-1}, \quad \alpha_{2k+1} \leq c\alpha_{2k}.$$

We have

$$\alpha_{2k} = \int_0^1 t^{2k+1} f(t) dt \leq \int_0^1 t^{2k} f(t) dt = \alpha_{2k-1}$$

and

$$\alpha_{2k+1} = \int_0^1 t^{2k+2} f(t) dt \leq \int_0^1 t^{2k+1} f(t) dt = \alpha_{2k}.$$

Thus $||L_x|| \leq 1$. This proves that A_{x^2} is C^*-representable.

3. The $*$-algebra given by the generators and relations:

$$\mathbb{C}\big\langle a_1, \ldots, a_n | a_i^* a_j = \sum_{k \neq l} T_{ij}^{kl} a_l a_k^*; i \neq j \big\rangle,$$

with $T_{ij}^{kl} = \bar{T}_{ji}^{lk}$ is strictly non-expanding by Corollary 8. Indeed, no two elements from defining relations form a composition and the greatest word of any relation begins with some a_j and all other words begin with some a_k^*. Hence this $*$-algebra is O^*-representable. Note that if the additional relations $a_i^* a_i = 1 + \sum_{k,l} T_{ii}^{kl} a_l a_k^*$ are imposed we obtain algebras allowing Wick ordering (see [6]).

4. Let $S \subset \mathbb{C}W(x_1, \ldots, x_n)$ be closed under compositions then a $*$-algebra

$$A = \mathbb{C}\langle x_1, \ldots, x_n, x_1^*, \ldots, x_n^* |\ S \cup S^* \rangle$$

is sometimes called $*$-double of $B = \mathbb{C}\langle x_1, \ldots, x_n |\ S \rangle$. By by Corollary 9 below A is non-expanding. For finite dimensional algebra B this already follows from Corollary 8. Indeed, if S satisfies additionally the property that the greatest word of every relation begins with the generator different from the beginnings of other longest words of this relation then A is strictly non-expanding by corollary 8 since $S \cup S^*$ is, clearly, closed under compositions. In particular, let B be a finite dimensional associative algebra with linear basis $\{e_k\}_{k=1}^n$. Then its "table of multiplication", i.e. the relations of the form $e_i e_j - \sum c_{ij}^k e_k = 0$, where c_{ij}^k are the structure constants of the algebra B, forms a set of defining relations S with the greatest words of length 2 and others of length 1. Thus $*$-algebra $A\mathbb{C}\langle x_1, \ldots, x_n, x_1^*, \ldots, x_n^* |\ S \cup S^* \rangle$ is the $*$-double of B. In other words, A is a free product $B_1 * B_2$, where $B_1 \simeq B_2 \simeq \overline{B}$ and involution is given on the generators by the rules $b^* = \overline{\phi(b)}$ for any $b \in B_1$ and $c^* = \overline{\phi^{-1}(c)}$ for any $c \in B_2$ with $\phi : B_1 \to B_2$ being any fixed isomorphism. The resulting $*$-algebra A does not depend on the choice of ϕ.

To deal with a general algebra B we need the following stronger result.

Theorem 11. *Let $S = S^*$ be a closed under compositions subset of a free $*$-algebra F_* with generators $x_1, \ldots, x_n$, $x_1^*, \ldots, x_n^*$ such that for any $s \in S$ the following properties holds.*

1. *$\hat{s} \in G$ or $\hat{s} \in G^*$ where $G = W(x_1, \ldots, x_n)$ is a semigroup generated by $x_1, \ldots, x_n$.*
2. *for any $u \prec s$ such that $|u| = |\hat{s}|$ words u and $\hat{s}$ both lie in the same semigroup G or G^*.*

Then S is strictly non-expanding.

Proof. Let $X = \{x_1, \ldots, x_n\}$ and $X^* = \{x_1^*, \ldots, x_n^*\}$. As always W will denote the semigroup $W(X \cup X^*)$. If some word $w = y_1 \ldots y_t$ where $y_r \in X \cup X^*$ contains subword $\hat{s}$ for some $s \in S$ then $w = p\hat{s}q$ for some words p and q in W. Let $s = \hat{s} - \sum_{i=1}^{n} \alpha_i w_i$ ($\alpha_i \in \mathbb{C}$, $w_i \in W$). The substitution rule $\hat{s} \to \bar{s}$ (see the appendix) replaces subword w with $\sum_i \alpha_i p w_i q$. The conditions of the theorem ensure that all words w_i such that $|w_i| = |\hat{s}|$ are in the same semigroup either in G or in G^*. Since decomposition $\mathrm{R}_S(w) = \sum_j \beta_j u_j$, where $u_j \in BW$, $u_j = z_1^{(j)} \ldots z_{k_j}^{(j)}$ with $z_r^{(j)} \in X \cup X^*$ $(1 \le r \le k_j)$ can be obtained by several subsequent substitutions considered above we see that for any j such that $|u_j| = |w|$ and for all $1 \le r \le t$ both generators $z_{i_r}^{(j)}$ and y_{k_r} are in the same set either X or X^*.

Let $u, v \in BW$, $u > v$ and $|u| = |v|$. Assume that $uu^* \prec \mathrm{R}_S(uv^*)$. Without loss of generality we can assume that the word $u = z_1 \ldots z_k$ ends with symbol from X, i.e. $z_k \in X$. Then $uu^* = z_1 \ldots z_k z_k^* \ldots z_1^*$. By the first part of the proof v^* begins with a generator x_l^* from the set X^*. If $uv^* \notin BW$ then there exists $s \in S$ such that $uv^* = p\hat{s}q$ for some words p and q. Since $u, v \in BW$, $\hat{s}$ intersects both u and v^*. Hence $\hat{s}$ contains $z_k x_l^*$ as a subword. This contradicts assumption 1 of the theorem. Thus $uv^* \in BW$ and $\mathrm{R}_S(uv^*) = uv^*$. Clearly, $uv^* = uu^*$ implies $u = v$. Obtained contradiction proves that S is non-expanding. Since for every $s \in S$, $\hat{s}$ is unshrinkable and $S = S^*$ we have that for any $d \in BW$ word dd^* is in BW. Thus S is strictly non-expanding. □

It could be shown using Zorn's lemma that for any algebra A and any its set of generators X there is a Gröbner basis S corresponding to X with any given inductive ordering of the generators. It is easy to check that $S \cup S^*$ satisfies assumptions of Theorem 11, thus, we have the following.

Corollary 9. *If B is a finitely generated associative algebra then its $*$-double $A = B * B$ is strictly non-expanding $*$-algebra. Hence A has a faithful $*$-representation in pre-Hilbert space.*

Below we give some known examples of $*$-doubles which have finite Gröbner bases.

5. We present an example of O^*-algebra which is not C^*-representable. Consider the $*$-algebra:

$$Q_{4,\alpha} = \mathbb{C}\big\langle q_1, \ldots, q_4, q_1^*, \ldots, q_4^* \mid q_j^2 = q_j, q_j^{*2} = q_j^*, \text{ for all } 1 \le j \le 4, \sum_{j=1}^{4} q_j = \alpha, \sum_{j=1}^{4} q_j^* = \overline{\alpha}\big\rangle.$$

which is the *-double of the algebra

$$B_{n,\alpha} = \mathbb{C}\langle q_1, \ldots, q_4 \mid q_j^2 = q_j, \ \sum_j q_j = \alpha\rangle$$

This algebra has the following Gröbner basis:

$S = \{q_1q_1 - q_1, q_2q_2 - q_2, -q_3q_2 - 2q_1 - 2q_2 - 2q_3 + \alpha + 2\alpha q_1 + 2\alpha q_2 + 2\alpha q_3 - \alpha^2 - q_1q_2 - q_1q_3 - q_2q_1 - q_2q_3 - q_3q_1, q_3q_3 - q_3, -q_3q_1q_2 - 3\alpha + 5\alpha^2 - 2\alpha^3 + q_2(6 - 10\alpha + 4\alpha^2) + q_3(6 - 10\alpha + 4\alpha^2) + q_1(8 - 13\alpha + 5\alpha^2) + (3 - 2\alpha)q_1q_2 + (6 - 4\alpha)q_1q_3 + (6 - 4\alpha)q_2q_1 + (6 - 4\alpha)q_2q_3 + (3 - 2\alpha)q_3q_1 + q_1q_2q_1 + q_1q_2q_3 + q_1q_3q_1 + q_2q_1q_3 + q_2q_3q_1)\}$. More detailed treatment of this algebra can be found in [12, 1]. Note that when $\alpha = 0$ the $*$-algebra $Q_{4,0} = B_{4,0} * B_{4,0}$ has only zero representation in bounded operators (see [1]). Thus for this $*$-algebra only representations in unbounded operators could exist.

6. That the generators in the previous example are idempotents is not important for O^*-representability, we can consider the following example:

$$T_{3,\alpha} = \mathbb{C}\big\langle q_1, q_2, q_3, q_1^*, q_2^*, q_3^* \mid q_j^3 = q_j, q_j^{*3} = q_j^* \text{ for } 1 \le j \le 3, \sum_j q_j = \alpha, \sum_j q_j^* = \overline{\alpha}\big\rangle.$$

It is the $*$-double of the algebra $\mathbb{C}\langle q_1, q_2, q_3 \mid q_j^3 = q_j, \ \sum_j q_j = \alpha\rangle$. We will find its Gröbner basis. We have the following set of relations $\{q_1^3 - q_1, q_2^3 - q_2, q_3^3 - q_3, q_1 + q_2 + q_3 - \alpha\}$. From these relations it follows that this algebra is generated by q_1 and q_2. Thus we can consider the following equivalent set of relations: $\{q_1^3 - q_1, q_2^3 - q_2, (\alpha - q_1 - q_2)^3 - (\alpha - q_1 - q_2)\}$. Introduce the following order on the generators $q_2 > q_1$. All relations are already normalized, i.e. all leading coefficients are equal to 1. The greatest words in these relations are q_1^3, q_2^3 and $q_1^2q_2$. Thus we have no reductions. The first and the third relations form two compositions. From one side they intersect by the word q_1. And the result of this composition is $(q_1^3 - q_1)q_1q_2 - q_1^2((\alpha - q_1 - q_2)^3 - (\alpha - q_1 - q_2))$. On the other hand they intersect by the word q_1^2. The result of this composition is $(q_1^3 - q_1)q_2 - q_1((\alpha - q_1 - q_2)^3 - (\alpha - q_1 - q_2))$. Another composition is formed by the third and the second relations. Their greatest words intersect by the word q_2. Result of this composition is $((\alpha - q_1 - q_2)^3 - (\alpha - q_1 - q_2))q_2^2 - q_1^2(q_2^3 - q_2)$. Hence we have three new relations. After performing reductions we will have the following set of relations:

$S = \{q_1^3 - q_1, -q_2^2q_1 + 3\alpha q_1^2 + 3\alpha q_2^2 + \alpha^3 + q_1(-1 - 3\alpha^2) + q_2(-1 - 3\alpha^2) + 3\alpha q_1q_2 - q_1q_2^2 - q_1^2q_2 + 3\alpha q_2q_1 - q_2q_1^2 - q_1q_2q_1 - q_2q_1q_2, q_2^3 - q_2, -q_2q_1q_2q_1^2 + -\alpha^3 + 9\alpha^5 - q_1^2(-3\alpha - 37\alpha^3) - q_2^2(3\alpha - 27\alpha^3) - q_2(-1 + 6\alpha^2 + 27\alpha^4) - q_1(18\alpha^2 + 30\alpha^4) - (-12\alpha - 45\alpha^3)q_1q_2 - 27\alpha^2q_1q_2^2 - (1 + 30\alpha^2)q_1^2q_2 + 9\alpha q_1^2q_2^2 - (6\alpha - 18\alpha^3)q_2q_1 - (1 + 3\alpha^2)q_2q_1^2 - (-2 + 15\alpha^2)q_1q_2q_1 + 3\alpha q_1q_2q_1^2 + 3\alpha q_1^2q_2q_1 - q_1^2q_2q_1^2 - (-1 + 9\alpha^2)q_2q_1q_2 + 6\alpha q_1q_2q_1q_2 - q_1^2q_2q_1q_2 - 3\alpha q_2q_1q_2q_1 + q_1q_2q_1q_2q_1\}$

Some of these relations do form compositions but all of them reduce to zero. Hence it is a Gröbner basis. Thus $T_{3,\alpha}$ is O^*-representable for every complex parameter α.

5 APPENDIX: Noncommutative Gröbner bases.

For the convenience of the reader we review some relevant facts from noncommutative Gröbner bases theory (see [16, 2]) with some straightforward reformulations.

The reader should keep in mind that a Gröbner basis is just a special set of defining relations of a given algebra and thus is a subset of a free algebra. The main advantage of having a Gröbner basis for an algebra is that one can algorithmically solve the equality problem, i.e. one can decide for a given two noncommutative polynomial in the algebra generators if they represent the same element of the algebra or not.

The Gröbner basis always exists whatever system of generator one chooses but the procedure to find a Gröbner basis does not always terminate.

Below we will present only those aspects of the Gröbner bases theory which are necessary for this paper. Let W_n denote the free semigroup with generators $x_1, \ldots, x_n$. For a word $w = x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}$ (where $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$, and $\alpha_1, \ldots, \alpha_k \in \mathbb{N} \cup \{0\}$) the length of w, denoted by $|w|$, is defined to be $\alpha_1 + \ldots + \alpha_k$. Let $F_n =$

$\mathbb{C}\langle x_1, \ldots, x_n\rangle$ denote the free associative algebra with generators $x_1, \ldots, x_n$. We will sometimes omit subscript n. Fix the linear order on W_n such that $x_1 > x_2 > \ldots > x_n$, the words of the same length ordered lexicographically and the words of greater length are considered greater. Any $f \in F_n$ is a linear combination $\sum_{i=1}^{k} \alpha_k w_i$ of distinct words w_1, w_2, …, w_k with complex coefficients $\alpha_i \neq 0$ for all $i \in \{1, \ldots, k\}$). Let $\hat{f}$ denote the greatest of these words, say w_j. The coefficient α_j we denote by $\mathrm{lc}(f)$ and call *leading coefficient.* Then denote $\hat{f} - (\alpha_j)^{-1} f$ by $\bar{f}$. The degree of $f \in F_n$, denoted by $\deg(f)$, is defined to be $|\hat{f}|$. The elements of the free algebra F can be identified with functions $f : W \to \mathbb{C}$ with finite support via the map $f \to \sum_{w \in W} f(w)w$. For a word $z \in W$ and an element $f \in F$ we will write $z \prec f$ if $f(z) \neq 0$.

Definition 9. *We will say that two elements $f, g \in F_n$ form a composition $w \in W$ if there are words $x, z \in W$ and nonempty word $y \in W$ such that $\hat{f} = xy$, $\hat{g} = yz$ and $w = xyz$. Denote the result of the composition $\beta f z - \alpha x g$ by $(f, g)_w$, where α and β are the leading coefficients of f and g respectively.*

If f and g are as in the preceding definition then $f = \alpha xy + \alpha \bar{f}$ and $g = \beta yz + \beta \bar{g}$ and $(f, g)_w = \alpha\beta(\bar{f} z - x\bar{g})$. We will also say that f and g intersect by y. Remark that there may exist many such y for a given f and g, and the property "intersect" is not symmetrical. It is also obvious that $(f, g)_w < w$. Notice that two elements f and g may form compositions in many ways and f may form composition with itself.

The following definition is due to Bokut [2].

Definition 10. *A subset $S \subseteq F_n$ is called closed under compositions if for any two elements f, $g \in S$ the following properties holds.*

1. *If $f \neq g$ then the word $\hat{f}$ is not a subword in $\hat{g}$.*
2. *If f and g form a composition w then there are words a_j, $b_j \in W_n$, elements $f_j \in S$ and complex α_j such that $(f, g)_w = \sum_{j=1}^{m} \alpha_j a_j f_j b_j$ and $a_j f_j b_j < w$, for $j = 1, \ldots, m$.*

Definition 11. *A set $S \subseteq F$ is called a Gröbner basis of an ideal $\mathcal{I} \subseteq F$ if for any $f \in \mathcal{I}$ there is $s \in S$ such that $\hat{s}$ is a subword in $\hat{f}$. A Gröbner basis S of $\mathcal{I}$ is called minimal if no proper subset of S is a Gröbner basis of $\mathcal{I}$.*

If S is closed under compositions then S is a minimal Gröbner basis for the ideal $\mathcal{I}$ generated by S (see [2]). Henceforth we will consider only minimal Gröbner bases. Thus we will say that S is a Gröbner basis of an associative algebra $A = F/\mathcal{I}$ if S is closed under composition and generates $\mathcal{I}$ as an ideal of F. Let GB be a Gröbner basis for A and let $\hat{GB} = \{\hat{s} | s \in GB\}$. Denote by $BW(GB)$ the subset of those words in W_n that contain no word from $\hat{GB}$ as a subword. It is a well known fact that $BW(GB)$ is a linear basis for A. Henceforth we will write simply BW since we will always deal with a fixed Gröbner basis.

If $S \subseteq F$ is closed under compositions and $\mathcal{I}$ is an ideal generated by S then each element $f+\mathcal{I}$ of the factor algebra $F/\mathcal{I}$ is the unique linear combination of basis vectors $\{w+\mathcal{I}\}_{w\in BW}$

$$f+\mathcal{I} = \sum_{i=1}^{n} c_i(w_i+\mathcal{I}).$$

We can define an operator $\mathrm{R}_S : F \to F$ by the following rule $\mathrm{R}_S(f) = \sum_{i=1}^{n} c_i w_i$. The element $\mathrm{R}_S(f)$ can be considered as a canonical form of the element f in the factor algebra $F/\mathcal{I}$. Computing canonical forms we can algorithmically decide if two elements are equal in $F/\mathcal{I}$.

For example for a finite dimensional Lie algebra $\mathcal{L}$ with linear basis $\{e_i\}_{i\in M}$ and structure constants C_{ij}^k ($[e_i, e_j] = \sum_k C_{ij}^k e_k$) the set of relations $e_ie_j - e_je_i - \sum_k C_{ij}^k e_k$ with $i > j$ constitute a Gröbner basis for the universal enveloping associative algebra $U(\mathcal{L})$ and the canonical form is given by the PBW theorem.

Clearly R_S is a retraction on a subspace K in F spanned by BW. We can consider a new operation on the space K: $f \diamond g = \mathrm{R}_S(fg)$ for $f,\ g \in K$. Then $(K, +, \diamond)$ becomes an algebra which is isomorphic to $F/\mathcal{I}$.

Each element $s \in S$ in a Gröbner basis could be considered as a substitution rule $\hat{f} \to \bar{f}$ which tells us to replace each occurrence of the subword $\hat{f}$ with $\bar{f}$. The canonical form $\mathrm{R}_S(f)$ can be computed step by step by performing all possible substitutions described above. The order in which the substitutions performed is not essential, only a finite number of substitutions could occur. From this it follows that if $w \prec R_S(u)$ for some words w and u then $w < u$. For example, take algebra $A = \mathbb{C}\langle a,\ b | ba = qab\rangle$ for some complex q. Then considering $b > a$ we obtain that $S = \{ba - qab\}$ is a Gröbner basis for A. We have only one substitution rule $ba \to qab$. To obtain the canonical form of b^2a we compute $b(ba) \to q(ba)b \to q^2b^2a$. Thus $\mathrm{R}_S(b^2a) = q^2b^2a$. Much more complicated examples can be found in Section 4 of the present paper.

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References

[1] H. Bart, T. Erhardt, B. Sillbermann, Zero Sums of Idempotens in Banach Algebras, *Integr. Equat. Oper. Th.*, 19 (1994) 123–134.

[2] L. A. Bokut, Imbedding of rings. (Russian) *Uspekhi Mat. Nauk* 42 (1987), no. 4(256), 87–111.

[3] M.D. Choi, E.G. Effros, Injectivity and operator spaces. *J. Functional Analysis* 24 (1977), no. 2, 156–209.

[4] R.S. Doran, V.A. Belfi, Characterizations of C^*-algebras. The Gelfand-Naĭmark theorems. Monographs and Textbooks in Pure and Applied Mathematics, 101. Marcel Dekker, Inc., New York, 1986. xi+426 pp.

[5] J. Diximier, C^*-algebras (English translation), North Holland, 1977.

[6] P.E.T Jorgensen, L.M. Schmitt, R.F. Werner, 'Positive representations of general commutation relations allowing wick ordering', *Funct. Anal.* 134, No.1 (1995), 33-99.

[7] E. C. Lance, Finitely-presented C^*-algebras. 1997, Operator algebras and applications, Nato ASI Series, Ser. C: Math and Phys. Sci., Vol.495, Kluwer Acad. Publ., 1997, pp.255-266

[8] S. V. Popovych, Conditions for embedding a $*$-algebra into a C^*-algebra, *Methods of Funct. Analysis and topology.* 5, No.3 (1999) 40-48.

[9] S. V. Popovych, Monomial $*$-Algebras and Tapper's Conjecture *Methods of Funct. Analysis and topology.*, 8 No. 1 (2002) 70-75

[10] S. V. Popovych, Realization of finitely presented *-algebras by Hilbert space operators, *Functional analysis:Proc. of Ukr. Math. Congress - 2001*, (Kyiv:Inst. of Math. of NAS of Ukraine, 2002), pp. 245-253

[11] R.T. Powers, Selfadjoint algebras of unbounded operators. II. Trans. Amer. Math. Soc. 187 (1974), 261–293.

[12] V. I. Rabanovich, Yu. S. Samoilenko, When a sum of projections is a multiple of identity, *Funct. Anal. and Appl.*, 34 No.4 (2000) 91–93. (in Russian)

[13] P. Tapper, Embedding of *-algebras into C^*-algebras and C^*-ideals generated by words, *J. Operator Theory*, 41 (1999) 351-364.

[14] T. A., Sarymsakov, Sh. A. Ayupov, Dzh. Khadzhiev, V. I. Chilin, Uporyadochennye algebry. (Russian) [Ordered algebras] "Fan", Tashkent, 1983. 303 pp.

[15] K. Schmüdgen, Unbounded operator algebras and representation theory. Operator Theory: Advances and Applications, 37. Birkha"user Verlag, Basel, 1990. 380 pp.

[16] V. A. Ufnarovsky, I. Combinatorial and asimptotic methods in algebra, *Sovrem. Probl. Matem. Fundam. Naprav. 57 (1990) 5-177* (in Russian)