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ON Φ-SYMMETRIC KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study ϕ-symmetric Kenmotsu manifolds.

1. Introduction

The notion of local symmetry of Riemannian manifolds have been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [10] introduced the notion of locally ϕ-symmetry on Sasakian manifolds. Generalizing the notion of locally ϕ-symmetry, one of the authors, De, [4] introduced the notion of ϕ-recurrent Sasakian manifolds. In the context of contact Geometry the notion of ϕ-symmetry is introduced and studied by Boeckx, Buecken and Vanhecke [3] with several examples. On the other hand Kenmotsu [6] defined a type of contact metric manifold which is called nowadays Kenmotsu manifold. It may be mentioned that a Kenmotsu manifold is not a Sasakian manifold. Also a Kenmotsu manifold is not compact because of $div\xi = 2n$. In [6], Kenmotsu showed that locally a Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and Kahler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant. The present paper is organized as follows:
Section 2 is devoted to preliminaries. In section 3 we prove that a ϕ-symmetric Kenmotsu manifold is an Einstein manifold. In the next section it is proved that a three-dimensional Kenmotsu manifold is locally ϕ-symmetric if and only if the scalar curvature is constant. Finally we give some examples of ϕ-symmetric and locally ϕ-symmetric Kenmotsu manifolds.

2. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold, where ϕ is a (1,1) tensor field, η is a 1-form and g is the Riemannian metric. It is well known that

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

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$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X, Y on M [1]. If, moreover,

$$(\nabla_X \phi)Y = -\eta(Y)\phi(X) - g(X, \phi Y)\xi, \quad X,\ Y \in \chi(M), \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g, then (M, ϕ, ξ, η, g) is called an almost Kenmotsu manifold [6].

Kenmotsu manifolds have been studied by many authors such as De and Pathak [4], Jun, De and Pathak [5], Binh, Tamassy, De and Tarafdar [2], Özgür and De [9], Özgür [7], [8] and many others. In Kenmotsu manifolds the following relations hold [6]:

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{2.8}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.10}$$

$$S(X, \xi) = -2n\eta(X), \tag{2.11}$$

$$(\nabla_Z R)(X, Y)\xi = g(Z, X)Y - g(Z, Y)X - R(X, Y)Z, \tag{2.12}$$

for every vector fields X, Y, Z, where R is the Riemannian curvature tensor and S is the Ricci tensor.

Definition 2.1. A Kenmotsu manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0, \tag{2.13}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced for Sasakian manifold by Takahashi[10].

Definition 2.2. A Kenmotsu manifold is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0, \tag{2.14}$$

for arbitrary vector fields X, Y, Z, W.

3. ϕ-symmetric Kenmotsu manifolds

Let us consider a ϕ-symmetric Kenmotsu manifold. Then by virtue of (2.2) and (2.14) we have

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = 0, \tag{3.1}$$

from which it follows that

$$-g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)g(\xi,U) = 0. \tag{3.2}$$

Let $\{e_i\}$, $i = 1, 2, ..., (2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.2) and taking summation over i, $1 \le i \le 2n+1$, we get

$$-(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = 0. \tag{3.3}$$

The second term of (3.3) by putting $Z = \xi$ takes the form

$$\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi), \tag{3.4}$$

which is denoted by E. In this case E vanishes. Namely we have

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) &= g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) \\ &- g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi) \end{aligned} \tag{3.5}$$

at $p \in M$. Since $\{e_i\}$ is an orthonormal basis, $\nabla_X e_i = 0$ at P. Using (2.9) we have

$$\begin{aligned} g(R(e_i,\nabla_W Y)\xi,\xi) &= g(\eta(e_i)\nabla_W Y - \eta(\nabla_W Y)e_i,\xi) \\ &= \eta(e_i)g(\nabla_W Y,\xi) - \eta(\nabla_W Y)g(e_i,\xi) \\ &= g(e_i,\xi)g(\nabla_W Y,\xi)g(e_i,\xi) \\ &= 0. \end{aligned} \tag{3.6}$$

Using (3.6) in (3.5) we obtain

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi). \tag{3.7}$$

Since $g(R(e_i,Y)\xi,\xi) = -g(R(\xi,\xi)Y,e_i) = 0$ we have

$$g(\nabla_W R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W \xi) = 0. \tag{3.8}$$

By using (3.8) in (3.7) we get

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla_W \xi) - g(R(e_i,Y)\nabla_W \xi,\xi).$$

Using (2.6) we obtain

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) &= -g(R(e_i,Y)\xi,W) + \eta(W)g(R(e_i,Y)\xi,\xi) \\ &\quad +g(R(e_i,Y)W,\xi) - \eta(W)g(R(e_i,Y)\xi,\xi) \\ &\quad = 0, \end{aligned}$$

i.e.,

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = 0. \tag{3.9}$$

Using (3.9) from (3.3) we get

$$(\nabla_W S)(Y,\xi) = 0. \tag{3.10}$$

We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W(S(Y,\xi)) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi).$$

Using (2.6), (2.7), and (2.11) we get

$$(\nabla_W S)(Y,\xi) = -2ng(W,Y) - S(Y,W). \tag{3.11}$$

Using (3.11) in (3.10) we obtain

$$S(Y,W) = -2ng(W,Y). \tag{3.12}$$

This leads to the following:

Theorem 3.1. *A ϕ-symmetric Kenmotsu manifold is an Einstein manifold.*

4. Three-dimensional locally ϕ-symmetric Kenmotsu manifolds

It is known [4] that in a three dimensional Kenmotsu manifold the curvature tensor has the following form

$$\begin{aligned} R(X,Y)Z = \tfrac{r+4}{2}[g(Y,Z)X - g(X,Z)Y] \\ -\tfrac{r+6}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.1}$$

Taking covariant differentiation of (4.1) we have

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &- \frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &- \frac{r+6}{2}[g(Y,Z)(\nabla_W\eta)(X)\xi + g(Y,Z)\eta(X)\nabla_W\xi \\ &- g(X,Z)(\nabla_W\eta)(Y)\xi - g(X,Z)\eta(Y)\nabla_W\xi + (\nabla_W\eta)(Y)\eta(Z)X \\ &+ \eta(Y)(\nabla_W\eta)(Z)X - (\nabla_W\eta)(X)\eta(Z)Y - \eta(X)(\nabla_W\eta)(Z)Y]. \end{aligned} \tag{4.2}$$

Now applying ϕ^2 to both sides of (4.2) we obtain

$$\begin{aligned} \phi^2(\nabla_W R)(X,Y)Z &= -\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi \\ &+ g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\ &+ \frac{r+6}{2}[(\nabla_W\eta)(Y)\eta(Z)X + \eta(Y)(\nabla_W\eta)(Z)X \\ &- (\nabla_W\eta)(X)\eta(Z)Y - (\nabla_W\eta)(Z)\eta(X)Y \\ &- (\nabla_W\eta)(Y)\eta(Z)\eta(X)\xi + \eta(Z)(\nabla_W\eta)(X)\eta(Y)\xi]. \end{aligned} \tag{4.3}$$

Now taking X, Y, Z orthogonal to ξ and using (2.14), we finally get

$$\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] = 0. \tag{4.4}$$

Thus we can state the following:

Theorem 4.1. *A three-dimensional Kenmotsu manifold is locally ϕ-symmetric if and only if the scalar curvature is constant.*

5. **Examples**

In this section we give some examples of ϕ-symmetric Kenmotsu manifolds.

Example 5.1. It is known that [6] a conformally flat Kenmotsu manifold of dimension greater than three is a space of constant curvature -1.

Hence the conformally flat Kenmotsu manifold of dimension greater than three is ϕ-symmetric.

Example 5.2. Kenmotsu [6] proved that if a Kenmotsu manifold is a space of constant ϕ-holomorphic sectional curvature, then the manifold is a space of constant curvature.

Therefore a Kenmotsu manifold of constant ϕ-holomorphic sectional curvature is ϕ-symmetric.

Example 5.3. We consider the three-dimensional manifolod $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3 The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = -z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = e_1, \qquad [e_2, e_3] = e_2.$$

The Rimennian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula.

Koszul's formula yields

$$\begin{array}{lll} \nabla_{e_1} e_3 = e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -e_3, \\ \nabla_{e_2} e_3 = e_2, & \nabla_{e_2} e_2 = e_3, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

From the above it follows that the manifold satisfies $\nabla_X \xi = X - \eta(X)\xi$, for $\xi = e_3$. Hence the manifold is Kenmotsu Manifold. With the help of the above results we can verify the following results.

$$\begin{array}{lll} R(e_1, e_2)e_2 = -e_1, & R(e_1, e_3)e_3 = -e_1, & R(e_2, e_1)e_1 = -e_2, \\ R(e_2, e_3)e_3 = -e_2, & R(e_3, e_1)e_1 = -e_3, & R(e_3, e_2)e_2 = -e_3. \end{array}$$

From the above expressions of the curvature tensor we obtain that the manifold under consideration is locally ϕ-symmetric. Also it follows that the scalar curvature r of the manifold is equal to -6.

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