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On octonary codes and their covering radii

Manoj K. Raut   Manish K. Gupta

Dhirubhai Ambani Institute of Information and Communication Technology
Gandhinagar
Gujarat, 382007
India
manoj_raut@daiict.ac.in   mankg@computer.org

Abstract

This paper introduces new reduction and torsion codes for an octonary code and determines their basic properties. These could be useful for the classification of self-orthogonal and self dual codes over $\mathbb{Z}_8$. We also focus our attention on the covering radius problem of octonary codes. In particular, we determine lower and upper bounds of the covering radius of several classes of repetition codes, simplex codes of type α and type β and their duals, MacDonald codes, and Reed-Muller codes over $\mathbb{Z}_8$.

1 Introduction

Codes over finite rings is a well studied area investigated by many researchers in the last two decades [2, 3, 5–7, 9, 14, 16–18, 21–23, 27–29, 31–33]. In particular, octonary codes have received attention by many researchers [2, 5, 9, 19, 21, 23, 31]. For any octonary linear code, we introduce binary and quaternary (over $\mathbb{Z}_4$) reduction and torsion codes and study their basic properties with respect to self-orthogonality and self-duality. One of the important properties of error correcting codes is that of determining the covering radius. The covering radius of binary linear codes has been studied in [10, 11]. It is shown in [4, 11] that the problem of computing covering radii of codes is both NP-hard and Co-NP hard. In fact, this problem is strictly harder than any NP-complete problem, unless NP=co-NP. The covering radius of codes over $\mathbb{Z}_4$ has been investigated with respect to Lee and Euclidean distances [1]. Several upper and lower bounds on the covering radius of codes has been studied in [1]. More recently, covering radius of codes over $\mathbb{Z}_{2^s}$ has been defined in [26] and upper and lower bounds on the covering radius of several classes of codes over $\mathbb{Z}_4$ have been obtained [26]. We extend some of these results to octonary codes in this paper.

A *linear code* $\mathbb{C}$, of length n, over $\mathbb{Z}_8$ is an additive subgroup of $\mathbb{Z}_8^n$. An element of $\mathcal{C}$ is called a *codeword of* $\mathcal{C}$ and a *generator matrix* of $\mathcal{C}$ is a matrix whose rows generate $\mathcal{C}$. The *Hamming weight* $w_H(\mathbf{x})$ of a vector $\mathbf{x}$ in $\mathbb{Z}_8^n$ is the number of non-zero components of $\mathbf{x}$. The *Homogeneous weight* $w_{HW}(\mathbf{x})$ [13] of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_8^n$ is given by $\sum_{i=1}^n w_{HW}(x_i)$, where

$$w_{HW}(x_i) = \begin{cases} 2, & x_i \neq 4 \\ 4, & x_i = 4. \end{cases}$$

The *Lee weight* $w_L(\mathbf{x})$ of a vector $\mathbf{x} \in \mathbb{Z}_8^n$ is $\sum_{i=1}^n \min\{x_i, 8 - x_i\}$. The *Euclidean weight* $w_E(\mathbf{x})$ of a vector $\mathbf{x} \in \mathbb{Z}_8^n$ is $\sum_{i=1}^n \min\{x_i^2, (8 - x_i)^2\}$.

The Hamming, Homogeneous, Lee and Euclidean distances $d_H(\mathbf{x}, \mathbf{y})$, $d_{HW}(\mathbf{x}, \mathbf{y})$, $d_L(\mathbf{x}, \mathbf{y})$, and $d_E(\mathbf{x}, \mathbf{y})$ between two vectors $\mathbf{x}$ and $\mathbf{y}$ are $w_H(\mathbf{x} - \mathbf{y})$, $w_{HW}(\mathbf{x} - \mathbf{y})$, $w_L(\mathbf{x} - \mathbf{y})$ and $w_E(\mathbf{x} - \mathbf{y})$, respectively. The minimum Hamming, Homogeneous, Lee and Euclidean weights, d_H, d_{HW}, $d_L$and d_E of $\mathcal{C}$ are the smallest Hamming, Homogeneous, Lee and Euclidean weights among all non-zero codewords of $\mathcal{C}$ respectively. One can define an isometry between $\mathbb{Z}_8^n \to \mathbb{Z}_2^{4n}$ as a coordinate-wise extension of the function from $\mathbb{Z}_8$ to $\mathbb{Z}_2^4$ defined by $0 \to (0,0,0,0), 1 \to (0,1,0,1), 2 \to (0,0,1,1), 3 \to (0,1,1,0), 4 \to (1,1,1,1), 5 \to (1,0,1,0), 6 \to (1,1,0,0), 7 \to (1,0,0,1)$ [8]. Such an isometry ϕ is called as the generalized *Gray map*. The image $\phi(\mathbf{C})$, of a linear code $\mathbf{C}$ over $\mathbb{Z}_8$ of length n by the generalized Gray map, is a binary code of length $4n$ [27].

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two vectors in $\mathbb{Z}_8^n$. Then the inner product of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\mathbf{x} \cdot \mathbf{y} = (x_1 y_1 + x_2 y_2 + \ldots + x_n y_n)(\bmod 8)$. The *dual code* $\mathcal{C}^\perp$ of $\mathcal{C}$ is defined as $\{\mathbf{x} \in \mathbb{Z}_8^n \mid \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in \mathcal{C}\}$, where $\mathbf{x} \cdot \mathbf{y}$ is the inner product of $\mathbf{x}$ and $\mathbf{y}$. $\mathcal{C}$ is *self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and $\mathcal{C}$ is *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$.

Two codes are said to be *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. Codes differing by only a permutation of coordinates are called *permutation-equivalent*. Let $\mathcal{C} \subseteq \mathbb{Z}_8^n$. If $\mathcal{C}$ has M codewords and minimum Homogeneous and Euclidean distances d_{HW} and d_E respectively then $\mathcal{C}$ is called an (n, M, d_{HW}, d_E) code. For more details about the octonary codes the reader is referred to any of the papers from [19, 23].

This paper is organized as follows. In Section 2, we define new torsion and reduction codes for an octonary code and obtain their basic properties. In Section 3 we present some results of the covering radius of octonary codes. Section 4, we discuss about the covering radius of the Octonary repetition codes. Octonary simplex codes of type α and β is discussed in Section 5. In Section 6, we consider MacDonald codes $\mathbb{Z}_8$. Finally, Section 7 considers Reed-Muller codes and Section 8 considers Octacode. Last section concludes the paper.

2 Reduction and Torsion Codes

The standard form of generator matrix G of the linear code $\mathcal{C}$ over $\mathbb{Z}_8$ [19] is of the form

$$G = \begin{pmatrix} I_{k_0} & A_{0,1} & A_{0,2} & A_{0,3} \\ 0 & 2I_{k_1} & 2A_{1,2} & 2A_{1,3} \\ 0 & 0 & 4I_{k_2} & 4A_{2,3} \end{pmatrix}, \tag{1}$$

where the matrices $A_{i,j}$ are binary matrices for $i > 0$. A code with a generator matrix in this form is of type $\{k_0, k_1, k_2\}$ and has $8^{k_0}4^{k_1}2^{k_2}$ vectors.

The matrix (1) can also be written in the following form over $\mathbb{Z}_8$:

$$G = \begin{pmatrix} I_{k_0} & B_{0,1} + 2B_{0,1}^1 + 4B_{0,1}^2 & B_{0,2} + 2B_{0,2}^1 + 4B_{0,2}^2 & B_{0,3} + 2B_{0,3}^1 + 4B_{0,3}^2 \\ 0 & 2I_{k_1} & 2A_{1,2} & 2A_{1,3} \\ 0 & 0 & 4I_{k_2} & 4A_{2,3} \end{pmatrix}, \tag{2}$$

where $B_{0,i}, B_{0,i}^1$, and $B_{0,i}^2$ are binary matrices for $i > 0$ and the matrices $A_{i,j}$ are binary matrices for $i > 0$. For a quaternary linear code one can define reduction code and torsion code [14, 27]. These codes have been generalized for a linear code $\mathcal{C}$ over $\mathbb{Z}_8$ in the form of four binary torsion/reduction codes in [19]. For $0 \le i \le 3$,

$$Tor_i(\mathcal{C}) = \{v \pmod 2 \mid 2^i v \in \mathcal{C}\}.$$

The generator matrices of $Tor_0(\mathcal{C}), Tor_1(\mathcal{C}), Tor_2(\mathcal{C})$ are the following three binary matrices:

$$G_{Tor_0} = \begin{pmatrix} I_{k_0} & B_{0,1} + 2B_{0,1}^1 + 4B_{0,1}^2 & B_{0,2} + 2B_{0,2}^1 + 4B_{0,2}^2 & B_{0,3} + 2B_{0,3}^1 + 4B_{0,3}^2 \end{pmatrix}$$

$$G_{Tor_1} = \begin{pmatrix} I_{k_0} & B_{0,1} + 2B_{0,1}^1 + 4B_{0,1}^2 & B_{0,2} + 2B_{0,2}^1 + 4B_{0,2}^2 & B_{0,3} + 2B_{0,3}^1 + 4B_{0,3}^2 \\ 0 & I_{k_1} & A_{1,2} & A_{1,3} \end{pmatrix}$$

$$G_{Tor_2} = \begin{pmatrix} I_{k_0} & B_{0,1} + 2B_{0,1}^1 + 4B_{0,1}^2 & B_{0,2} + 2B_{0,2}^1 + 4B_{0,2}^2 & B_{0,3} + 2B_{0,3}^1 + 4B_{0,3}^2 \\ 0 & I_{k_1} & A_{1,2} & A_{1,3} \\ 0 & 0 & I_{k_2} & A_{2,3} \end{pmatrix},$$

where

$$\mid \mathcal{C} \mid = \mid Tor_0(\mathcal{C}) \mid\mid Tor_1(\mathcal{C}) \mid\mid Tor_2(\mathcal{C}) \mid = 2^{3k_0+2k_1+k_2}$$

The reduction and torsion code of quaternary linear code can also be generalized for linear codes over $\mathbb{Z}_8$ in another interesting way. We define two binary (over $\mathbb{Z}_2$) torsion codes and two quaternary ($\mathbb{Z}_4$) torsion codes for a given linear code over $\mathbb{Z}_8$ as follows:

$$\begin{aligned} \mathcal{C}^{(1)} &= \{c \pmod 2 \mid c \in \mathcal{C}\}, \\ \mathcal{C}^{(2)} &= \{c \pmod 4 \mid c \in \mathcal{C}\}, \\ \mathcal{C}^{(3)} &= \{c \mid 2c \in \mathcal{C}\}, \\ \mathcal{C}^{(4)} &= \{c \mid 4c \in \mathcal{C}\}. \end{aligned}$$

The generator matrices $G^{(1)}, G^{(2)}, G^{(3)}, G^{(4)}$ of $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \mathcal{C}^{(3)}, \mathcal{C}^{(4)}$ are obtained from equation (2) as follows:

$$G^{(1)} = \begin{pmatrix} I_{k_0} & B_{0,1} & B_{0,2} & B_{0,3} \end{pmatrix},$$

$$G^{(2)} = \begin{pmatrix} I_{k_0} & B_{0,1} + 2B_{0,1}^1 & B_{0,2} + 2B_{0,2}^1 & B_{0,3} + 2B_{0,3}^1 \\ 0 & 2I_{k_1} & 2A_{1,2} & 2A_{1,3} \end{pmatrix}, \tag{3}$$

$$G^{(3)} = \begin{pmatrix} I_{k_0} & B_{0,1} + 2B^1_{0,1} & B_{0,2} + 2B^1_{0,2} & B_{0,3} + 2B^1_{0,3} \\ 0 & I_{k_1} & A_{1,2} & A_{1,3} \\ 0 & 0 & 2I_{k_2} & 2A_{2,3} \end{pmatrix}, \quad (4)$$

$$G^{(4)} = \begin{pmatrix} I_{k_0} & B_{0,1} & B_{0,2} & B_{0,3} \\ 0 & I_{k_1} & A_{1,2} & A_{1,3} \\ 0 & 0 & I_{k_2} & A_{2,3} \end{pmatrix}.$$

Note that the number of elements of $\mathcal{C}$ is $8^{k_0}4^{k_1}2^{k_2} = 2^{3k_0+2k_1+k_2}$. The number of elements of $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \mathcal{C}^{(3)}, \mathcal{C}^{(4)}$ are $2^{k_0}, 4^{k_0}2^{k_1}, 4^{k_0+k_1}2^{k_2}$ and $2^{k_0+k_1+k_2}$ respectively. Thus for a linear code $\mathcal{C}$ over $\mathbb{Z}_8$ we have the following relationship.

Proposition 1

$$\mid \mathcal{C} \mid = \mid \mathcal{C}^{(1)} \mid \times \mid \mathcal{C}^{(3)} \mid = \mid \mathcal{C}^{(2)} \mid \times \mid \mathcal{C}^{(4)} \mid .$$

Remark 1 *Note that the above proposition admits a more natural proof by considering the kernel and image of the reduction map and applying the first isomorphism theorem.*

Note that if $k_1 = k_2 = 0$ then $\mathcal{C}^{(2)} = \mathcal{C}^{(3)}$. It is easy to observe the following.

Proposition 2

$$\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(4)} \textit{ and } \mathcal{C}^{(2)} \subseteq \mathcal{C}^{(3)}.$$

The next result is a simple generalization of self-orthogonality characterization from [24]. For $\mathbf{c} \in \mathcal{C}$ and $0 \leq i \leq 7$, let $w_i(\mathbf{c})$ denotes the composition of symbol i in the codeword $\mathbf{c}$.

Proposition 3 *A linear code $\mathcal{C}$ over $\mathbb{Z}_8$ is self-orthogonal if and only if each generator matrix of $\mathcal{C}$ has all its rows $\omega_1 + \omega_3 + \omega_5 + \omega_7 + 4\omega_2 + 4\omega_6 = 0 \pmod 8$ and every pair of rows of the generator matrix is orthogonal.*

Proof. The proof is straightforward. □

Now we determine few relationships among various reduction and torsion codes if code $\mathcal{C}$ is self-orthogonal or self-dual.

Proposition 4 *If $\mathcal{C}$ is a self-orthogonal code over $\mathbb{Z}_8$ then $\mathcal{C}^{(1)}, \mathcal{C}^{(4)}$ are self-orthogonal codes over $\mathbb{Z}_2$ and $\mathcal{C}^{(2)}, \mathcal{C}^{(3)}$ are self-orthogonal codes over $\mathbb{Z}_4$.*

Proof. The self-orthogonality of $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}$ follows from [19]. It remains to see the self- orthogonality of $\mathcal{C}^{(3)}$, and $\mathcal{C}^{(4)}$. Let $v \in \mathcal{C}^{(3)}$. By definition of $\mathcal{C}^{(3)}$ we have $2v \in \mathcal{C}$. As $\mathcal{C}$ is self orthogonal, $< 2v, u >= 0 \pmod 4$ for all $u \in \mathcal{C}$. So $2\sum v_i u_i \equiv 0 \pmod 8$ for all $u \in \mathcal{C}$. Then $\sum v_i u_i \equiv 0 \pmod 4$ for all $u \in \mathcal{C}$. This implies $< v, u >= 0$ for all $u \in \mathcal{C}^{(3)}$ as $\mathcal{C}^{(3)}$ is a code over $\mathbb{Z}_4$. So $v \in {\mathcal{C}^{(3)}}^{\perp}$. Hence $\mathcal{C}^{(3)}$ is self orthogonal. The self-orthogonality of $\mathcal{C}^{(4)}$ can be proved similarly. □

Proposition 5 *If $\mathcal{C}$ is self orthogonal code over $\mathbb{Z}_8$ then $\mathcal{C}^{(4)} \subseteq {\mathcal{C}^{(1)}}^{\perp}$ and $\mathcal{C}^{(3)} \subseteq {\mathcal{C}^{(2)}}^{\perp}$.*

Proof. Let $v \in \mathcal{C}^{(4)}$. By definition of $\mathcal{C}^{(4)}$, we have $4v \in \mathcal{C}$. As $\mathcal{C}$ is self-orthogonal, $< 4v, u >= 0 \pmod 8$ for all $u \in \mathcal{C}$. So $4\sum v_i u_i = 0 \pmod 8$ for all $u \in \mathcal{C}$. $\sum v_i u_i = 0 \pmod 2$ for all $u \in \mathcal{C}$. $\sum (v_i \pmod 2)(u_i \pmod 2) = 0 \pmod 2$ for all $u \pmod 2 \in \mathcal{C}$. As $\mathcal{C}^{(4)}$ is a code over $\mathbb{Z}_2$, $v_i \pmod 2 = v_i$ and as $\mathcal{C}^{(1)}$ is a code over $\mathbb{Z}_2$, $\sum v_i(u_i \pmod 2) = 0 \pmod 2$ for all $u \in \mathcal{C}^{(1)}$. So $< v, u >= 0 \pmod 2$ for all $u \in \mathcal{C}^{(1)}$. This implies $v \in {\mathcal{C}^{(1)}}^{\perp}$. The second inclusion $\mathcal{C}^{(3)} \subseteq {\mathcal{C}^{(2)}}^{\perp}$ can be proved similarly. $\square$

Proposition 6 *If $\mathcal{C}$ is self-dual over $\mathbb{Z}_8$ then $\mathcal{C}^{(4)} = {\mathcal{C}^{(1)}}^{\perp}$ and $\mathcal{C}^{(3)} = {\mathcal{C}^{(2)}}^{\perp}$.*

Proof. We know that $\mathcal{C}^{(4)} \subseteq {\mathcal{C}^{(1)}}^{\perp}$ from Proposition 5. It remains to show ${\mathcal{C}^{(1)}}^{\perp} \subseteq \mathcal{C}^{(4)}$. Let $v \in {\mathcal{C}^{(1)}}^{\perp}$. So $< v, w >\equiv 0 \pmod 2$ for all $w \in \mathcal{C}^{(1)}$. $\sum v_i w_i \equiv 0 \pmod 2$ for all $w \in \mathcal{C}^{(1)}$. $4\sum v_i w_i \equiv 0 \pmod 8$ for all $w \in \mathcal{C}$. $\sum 4v_i w_i \equiv 0 \pmod 8$ for all $w \in \mathcal{C}$. $< 4v, w >= 0 \pmod 8$ for all $w \in \mathcal{C}$. $4v \in \mathcal{C}^{\perp} = \mathcal{C}$. This implies $v \in \mathcal{C}^{(4)}$. Hence proved. The proof of second result is similar. $\square$

We know that $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(3)}$ are codes over $\mathbb{Z}_4$. Thus it is natural to consider the torsion and reduction code of $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(3)}$. We get the following:

$$\begin{aligned}
\mathcal{C}^{(21)} &= \{c \pmod 2 \mid c \in \mathcal{C}^{(2)}\}, \\
\mathcal{C}^{(22)} &= \{c \mid 2c \in \mathcal{C}^{(2)}\}, \\
\mathcal{C}^{(31)} &= \{c \pmod 2 \mid c \in \mathcal{C}^{(3)}\}, \\
\mathcal{C}^{(32)} &= \{c \mid 2c \in \mathcal{C}^{(3)}\}.
\end{aligned}$$

The generator matrices $G^{(21)}, G^{(22)}, G^{(31)}, G^{(32)}$ of $\mathcal{C}^{(21)}, \mathcal{C}^{(22)}, \mathcal{C}^{(31)}, \mathcal{C}^{(32)}$ are obtained from (3) and (4) as follows:

$$G^{(21)} = \begin{pmatrix} I_{k_0} & B_{0,1} & B_{0,2} & B_{0,3} \end{pmatrix} = G^{(1)},$$

$$G^{(22)} = \begin{pmatrix} I_{k_0} & B_{0,1} & B_{0,2} & B_{0,3} \\ 0 & I_{k_1} & A_{1,2} & A_{1,3} \end{pmatrix},$$

$$G^{(31)} = \begin{pmatrix} I_{k_0} & B_{0,1} & B_{0,2} & B_{0,3} \\ 0 & I_{k_1} & A_{1,2} & A_{1,3} \end{pmatrix} = G^{(22)},$$

$$G^{(32)} = \begin{pmatrix} I_{k_0} & B_{0,1} & B_{0,2} & B_{0,3} \\ 0 & I_{k_1} & A_{1,2} & A_{1,3} \\ 0 & 0 & I_{k_2} & A_{2,3} \end{pmatrix}.$$

All the codes $\mathcal{C}^{(21)}, \mathcal{C}^{(22)}, \mathcal{C}^{(31)}, \mathcal{C}^{(32)}$ are codes over $\mathbb{Z}_2$. It is easy to see the following results from their generator matrices:

Proposition 7 *If $\mathcal{C}$ is a code over $\mathbb{Z}_8$ then $\mathcal{C}^{(21)} = \mathcal{C}^{(1)}$, $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(22)}$, $\mathcal{C}^{(31)} = \mathcal{C}^{(22)}$, $\mathcal{C}^{(31)} \subseteq \mathcal{C}^{(32)}$, and $\mathcal{C}^{(32)} = \mathcal{C}^{(4)}$.*

It is also natural to obtain the following from [14].

Proposition 8 *If $\mathcal{C}$ is self-orthogonal over $\mathbb{Z}_8$ then*

1. $\mathcal{C}^{(21)} \subseteq \mathcal{C}^{(22)} \subseteq \mathcal{C}^{(21)^{\perp}}$.
2. $\mathcal{C}^{(31)} \subseteq \mathcal{C}^{(32)} \subseteq \mathcal{C}^{(31)^{\perp}}$.

Thus we have an interesting family of codes from a linear octonary codes having beautiful inclusions.

3 Covering Radius of Octonary Codes

In this section, first we collect some known facts of the covering radius of codes over $\mathbb{Z}_8$ with respect to Homogeneous and Euclidean distances [26] and then derive some of its properties. Let d be either a Homogeneous distance or Euclidean distance. Then the covering radius of code $\mathcal{C}$ over $\mathbb{Z}_8$ with respect to distance d is given by

$$r_d(\mathcal{C}) = \max_{\mathbf{u}\in\mathbb{Z}_8^n} \left\{ \min_{\mathbf{c}\in\mathcal{C}} d(\mathbf{u}, \mathbf{c}) \right\}.$$

We can easily see [26] that $r_d(\mathcal{C})$ is the minimum value r_d such that

$$\mathbb{Z}_8^n = \cup_{\mathbf{c}\in\mathcal{C}} S_{r_d}(\mathbf{c}),$$

where

$$S_{r_d}(\mathbf{u}) = \{\mathbf{v} \in \mathbb{Z}_8^n \mid d(\mathbf{u}, \mathbf{v}) \leq r_d\}$$

for any element $\mathbf{u} \in \mathbb{Z}_8^n$.

The coset of $\mathcal{C}$ is the translate $\mathbf{u} + \mathcal{C} = \{\mathbf{u} + \mathbf{c} \mid \mathbf{c} \in \mathcal{C}\}$ where $\mathbf{u} \in \mathbb{Z}_8^n$. A vector of least weight in a coset is called a *coset leader*. The following proposition is well known [26].

Proposition 9 *The covering radius of $\mathcal{C}$ with respect to the general distance d is the largest minimum weight among all cosets.*

Proposition 10 *For any octonary code over $\mathbb{Z}_8$,*

$$\begin{array}{rcccl} \frac{1}{2}r_{HW}(\mathcal{C}) & \leq & r_E(\mathcal{C}) & \leq & 5r_{HW}(\mathcal{C}), \\ r_L(\mathcal{C}) & \leq & r_E(\mathcal{C}), & & \\ r_{HW}(\mathcal{C}) & \leq & 2r_L(\mathcal{C}). & & \end{array}$$

Proof. We observe that $\frac{1}{2}d_{HW}(\mathbf{x}, \mathbf{y}) \leq d_E(\mathbf{x}, \mathbf{y}) \leq 5d_{HW}(\mathbf{x}, \mathbf{y})$, so the first inequality follows. As $d_L(\mathbf{x}, \mathbf{y}) \leq d_E(\mathbf{x}, \mathbf{y})$, the second inequality follows. Further the third inequality follows since we have $d_{HW}(\mathbf{x}, \mathbf{y}) \leq 2d_L(\mathbf{x}, \mathbf{y})$. □

The following proposition is also well known [26].

Proposition 11 *Let $\mathcal{C}$ be a code over $\mathbb{Z}_8$ and $\phi(\mathcal{C})$ the generalized Gray map image of $\mathcal{C}$. Then $r_{HW}(\mathcal{C}) = r_H(\phi(\mathcal{C}))$.*

The following two results are the two upper bounds of the covering radius of codes over $\mathbb{Z}_8$ with respect to Homogeneous weight.

Proposition 12 (Sphere-Covering Bound) *For any code $\mathcal{C}$ of length n over $\mathbb{Z}_8$,*

$$\begin{aligned} \frac{2^{4n}}{|\mathcal{C}|} &\leq \sum_{i=0}^{r_{HW}(\mathcal{C})} \binom{4n}{i}, \\ \frac{2^{4n}}{|\mathcal{C}|} &\leq \sum_{i=0}^{r_E(\mathcal{C})} V_i, \\ \textit{where } \sum_{i=0}^{16n} V_i x^i &= (1 + 2x + 2x^4 + 2x^9 + x^{16})^n. \end{aligned}$$

Proof. The proof of both inequality over $\mathbb{Z}_8$ is similar to the proof over $\mathbb{Z}_4$ given in [1] (see also [26]) and hence omitted. □

Let $\mathcal{C}$ be a code over $\mathbb{Z}_8$ and let $s(\mathcal{C}^{\perp}) = |\left\{i \mid A_i(\mathcal{C}^{\perp}) \neq 0, i \neq 0\right\}|$, where $A_i(\mathcal{C}^{\perp})$ is the number of codewords of homogenous weight i in $\mathcal{C}^{\perp}$.

Theorem 1 (Delsarte Bound) *Let $\mathcal{C}$ be a code over $\mathbb{Z}_8$ then $r_{HW}(\mathcal{C}) \leq s(\mathcal{C}^{\perp})$ and $r_E(\mathcal{C}) \leq 5s(\mathcal{C}^{\perp})$.*

Proof. The first result is obtained in [26]. The second result follows from [1] and Proposition 10. □

The following result of Mattson [10] is useful for computing covering radii of codes over rings [26].

Proposition 13 (Mattson) *If $\mathcal{C}_0$ and $\mathcal{C}_1$ are codes over $\mathbb{Z}_8$ generated by matrices G_0 and G_1 respectively and if $\mathcal{C}$ is the code generated by*

$$G = \left(\begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array} \right),$$

then $r_d(\mathcal{C}) \leq r_d(\mathcal{C}_0) + r_d(\mathcal{C}_1)$ and the covering radius of $\mathcal{D}$ (concatenation of $\mathcal{C}_0$ and $\mathcal{C}_1$) satisfies the following

$$r_d(\mathcal{D}) \geq r_d(\mathcal{C}_0) + r_d(\mathcal{C}_1),$$

for all distances d over $\mathbb{Z}_8$.

Now we determine a bound on the covering radius of octonary code and its corresponding reduction and torsion codes. The following result is a generalization of Theorem 4.4 of [1].

Theorem 2 *For a code over $\mathbb{Z}_8$, let d_1, d_2, d_3, d_4 denote the minimum Hamming distances of linear codes $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \mathcal{C}^{(3)}, \mathcal{C}^{(4)}$ respectively. If $d_1 \geq 8, d_2 \geq 18, d_3 \geq \frac{25}{4}, d_4 \geq \frac{25}{16}$ then*

$$\begin{aligned} r_E(C) &\geq 9 \; min \left\{ \lfloor \tfrac{d_1}{8} \rfloor, \lfloor \tfrac{d_2}{18} \rfloor, 4\lfloor \tfrac{d_3}{25} \rfloor, 16\lfloor \tfrac{d_4}{25} \rfloor \right\}, \\ r_{HW}(C) &\geq 2 \; min \left\{ \lfloor \tfrac{d_1}{8} \rfloor, \lfloor \tfrac{d_2}{18} \rfloor, 4\lfloor \tfrac{d_3}{25} \rfloor, 16\lfloor \tfrac{d_4}{25} \rfloor \right\}. \end{aligned}$$

Proof. Let $t = \min\left\{\lfloor\frac{d_1}{8}\rfloor, \lfloor\frac{d_2}{18}\rfloor, 4\lfloor\frac{d_3}{25}\rfloor, 16\lfloor\frac{d_4}{25}\rfloor\right\}$. Hence $t > 0$. Let $\mathbf{x} = (00\ldots0\underbrace{44\ldots4}_{t})$. Let $\mathcal{C}$ be a code over $\mathbb{Z}_8$. Let $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathcal{C}$ such that $c_i = 0$ or 4. Hence $\frac{\mathbf{c}}{4} \in \mathcal{C}^{(4)}$. So $wt(\mathcal{C}^{(4)}) \geq d_4$ as the minimum Hamming distance of $\mathcal{C}^{(4)}$ is d_4. Thus $wt(\frac{\mathbf{c}}{4}) \geq d_4 \geq t$. Let $\mathbf{c} = (00\ldots0\underbrace{44\ldots4}_{\geq d_4})$. Hence

$$\begin{array}{rcccl} d_E(\mathbf{c},\mathbf{x}) & = & 16(d_4 - t) & \geq & 9t, \\ d_{HW}(\mathbf{c},\mathbf{x}) & = & 4(d_4 - t) & \geq & 2t. \end{array}$$

Similarly for $\mathbf{c} \in \mathcal{C}$ such that $\frac{\mathbf{c}}{2} \in \mathcal{C}^{(3)}$ we get

$$\begin{array}{rcccl} d_E(\mathbf{c},\mathbf{x}) & \geq & 4d_3 - 16t & \geq & 9t, \\ d_{HW}(\mathbf{c},\mathbf{x}) & \geq & 2d_3 - 4t & \geq & 2t. \end{array}$$

For $\mathbf{c} \in \mathcal{C}$ such that $\mathbf{c} \pmod 4 \in \mathcal{C}^{(2)}$ we have

$$\begin{array}{rcccl} d_E(\mathbf{c},\mathbf{x}) & \geq & d_2 - 9t & \geq & 9t, \\ d_{HW}(\mathbf{c},\mathbf{x}) & = & 2d_2 & \geq & 2t. \end{array}$$

Finally for $\mathbf{c} \in \mathcal{C}$ such that $\mathbf{c} \pmod 2 \in \mathcal{C}^{(1)}$ we have

$$\begin{array}{rcccl} d_E(\mathbf{c},\mathbf{x}) & = & d_1 + 8t & \geq & 9t, \\ d_{HW}(\mathbf{c},\mathbf{x}) & = & 2d_1 & \geq & 2t. \end{array}$$

Hence the result follows. □

4 Octonary Repetition Codes

A q-ary repetition code $\mathcal{C}$ over a finite field $\mathbb{F}_q = \{\alpha_0 = 0, \alpha_1 = 1, \alpha_2, \alpha_3, \ldots, \alpha_{q-2}\}$ is an [n,1,n]-code $\mathcal{C} = \{\overline{\alpha} \mid \alpha \in \mathbb{F}_q\}$, where $\overline{\alpha} = \{\alpha, \alpha, \ldots, \alpha\}$. The covering radius of $\mathcal{C}$ is $\lceil\frac{n(q-1)}{q}\rceil$ [20]. In [26], several classes of repetition codes over $\mathbb{Z}_4$ have been studied and their covering radius has been obtained. Now we generalize those results for codes over $\mathbb{Z}_8$. Consider the repetition codes over $\mathbb{Z}_8$. One can define seven basic repetition codes $\mathcal{C}_{\alpha_i}$, $(1 \leq i \leq n)$ of length n over $\mathbb{Z}_8$ generated by $G_{\alpha_1} = [\underbrace{11\ldots1}_{n}]$, $G_{\alpha_2} = [\underbrace{22\ldots2}_{n}]$, $G_{\alpha_3} = [\underbrace{33\ldots3}_{n}]$, $G_{\alpha_4} = [\underbrace{44\ldots4}_{n}]$, $G_{\alpha_5} = [\underbrace{55\ldots5}_{n}]$, $G_{\alpha_6} = [\underbrace{66\ldots6}_{n}]$, $G_{\alpha_7} = [\underbrace{77\ldots7}_{n}]$. So the repetition codes are $\mathcal{C}_{\alpha_1} = \mathcal{C}_{\alpha_3} = \mathcal{C}_{\alpha_5} = \mathcal{C}_{\alpha_7} = \{(00\ldots,0), (11\ldots1), (22\ldots2), (33\ldots3), (44\ldots4), (55\ldots5), (66\ldots6), (77\ldots7)\}$, $\mathcal{C}_{\alpha_2} = \mathcal{C}_{\alpha_6} = \{(00\ldots0), (22\ldots2), (44\ldots4), (66\ldots6)\}$ and $\mathcal{C}_{\alpha_4} = \{(00\ldots0), (44\ldots4)\}$. The following theorems determine the covering radius of $\mathcal{C}_{\alpha_i}$ for $1 \leq i \leq 7$.

Theorem 3 $r_E(\mathcal{C}_{\alpha_1}) = r_E(\mathcal{C}_{\alpha_3}) = r_E(\mathcal{C}_{\alpha_5}) = r_E(\mathcal{C}_{\alpha_7}) = \frac{11n}{2}$ *and* $r_{HW}(\mathcal{C}_{\alpha_1}) = r_{HW}(\mathcal{C}_{\alpha_3}) = r_{HW}(\mathcal{C}_{\alpha_5}) = r_{HW}(\mathcal{C}_{\alpha_7}) = 2n$.

Proof. We know that $r_E(\mathcal{C}_{\alpha_i}) = \max_{x\in\mathbb{Z}_8^n}\{d_E(x,\mathcal{C}_{\alpha_i})\}$. Let $\mathbf{x} \in \mathbb{Z}_8^n$. If $\mathbf{x}$ has composition
$(\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7)$ where $\sum_{i=0}^{7}\omega_i = n$, then

$$\begin{aligned}
d_E(\mathbf{x},\bar{0}) &= n-\omega_0+3\omega_2+8\omega_3+15\omega_4+8\omega_5+3\omega_6,\\
d_E(\mathbf{x},\bar{1}) &= n-\omega_1+3\omega_3+8\omega_4+15\omega_5+8\omega_6+3\omega_7,\\
d_E(\mathbf{x},\bar{2}) &= n-\omega_2+3\omega_0+3\omega_4+8\omega_5+15\omega_6+8\omega_7,\\
d_E(\mathbf{x},\bar{3}) &= n-\omega_3+8\omega_0+3\omega_1+3\omega_5+8\omega_6+15\omega_7,\\
d_E(\mathbf{x},\bar{4}) &= n-\omega_4+15\omega_0+8\omega_1+3\omega_2+3\omega_6+8\omega_7,\\
d_E(\mathbf{x},\bar{5}) &= n-\omega_5+8\omega_0+15\omega_1+8\omega_2+3\omega_3+3\omega_7,\\
d_E(\mathbf{x},\bar{6}) &= n-\omega_6+3\omega_0+8\omega_1+15\omega_2+8\omega_3+3\omega_4,\\
d_E(\mathbf{x},\bar{7}) &= n-\omega_7+3\omega_1+8\omega_2+15\omega_3+8\omega_4+3\omega_5.
\end{aligned}$$

Hence,

$$d_E(\mathbf{x},\mathcal{C}_{\alpha_1}) \leq \frac{8n+36(\omega_0+\omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6+\omega_7)}{8} = \frac{11n}{2}.$$

Thus $r_E(\mathcal{C}_{\alpha_1}) \leq \frac{11n}{2}$.

Let $\mathbf{x} = \underbrace{00\ldots0}_{t}\underbrace{11\ldots1}_{t}\underbrace{22\ldots2}_{t}\underbrace{33\ldots3}_{t}\underbrace{44\ldots4}_{t}\underbrace{55\ldots5}_{t}\underbrace{66\ldots6}_{t}\underbrace{77\ldots7}_{n-7t} \in \mathbb{Z}_8^n$, where $t = \lfloor\frac{n}{8}\rfloor$. Then $d_E(\mathbf{x},\bar{0}) = n+36t$, $d_E(\mathbf{x},\bar{1}) = 4n+12t$, $d_E(\mathbf{x},\bar{2}) = 9n-28t$, $d_E(\mathbf{x},\bar{3}) = 16n-84t$, $d_E(\mathbf{x},\bar{4}) = 9n-28t$, $d_E(\mathbf{x},\bar{5}) = 4n+12t$, $d_E(\mathbf{x},\bar{6}) = n+36t$, $d_E(\mathbf{x},\bar{7}) = 44t$. Thus

$$r_E(\mathcal{C}_{\alpha_1}) \geq \frac{44n+36t+12t-28t-84t-28t+12t+36t+44t}{8} = \frac{11n}{2}.$$

Thus $r_E(\mathcal{C}_{\alpha_1}) = r_E(\mathcal{C}_{\alpha_3}) = r_E(\mathcal{C}_{\alpha_5}) = r_E(\mathcal{C}_{\alpha_7}) = \frac{11n}{2}$. The gray map $\phi(\mathcal{C}_{\alpha_1})$ will be a binary repetition code of length $4n$. Thus $r_{HW}(\mathcal{C}_{\alpha_1}) = \lceil\frac{4n(2-1)}{2}\rceil = 2n = r_{HW}(\mathcal{C}_{\alpha_3}) = r_{HW}(\mathcal{C}_{\alpha_5}) = r_{HW}(\mathcal{C}_{\alpha_7})$. □

Theorem 4 *$r_E(\mathcal{C}_{\alpha_2}) = r_E(\mathcal{C}_{\alpha_6}) = 6n$ and $r_{HW}(\mathcal{C}_{\alpha_2}) = r_{HW}(\mathcal{C}_{\alpha_6}) = 2n$.*

Proof. The proof is similar to the proof of Theorem 3, hence omitted.

Theorem 5 *$r_E(\mathcal{C}_{\alpha_4}) = 8n$ and $r_{HW}(\mathcal{C}_{\alpha_4}) = 2n$.*

Proof. The proof is similar to the proof of Theorem 3, hence omitted.

In order to determine the covering radius of Simplex code S_k^{α} over $\mathbb{Z}_8$, we have to define a block repetition code over $\mathbb{Z}_8$ and find its covering radius. Thus the covering radius of the block repetition code $BRep^{m_1+m_2+\ldots+m_7}$ with parameters

$$\begin{aligned}
n &= m_1+m_2+\ldots+m_7,\\
M &= 8,\\
d_{HW} &= \min\{2m_1+2m_2+2m_3+4m_4+2m_5+2m_6+2m_7,\\
&\qquad 2m_1+4m_2+2m_3+2m_5+4m_6+2m_7,\\
&\qquad 4m_1+4m_3+4m_5+4m_7\},\\
d_E &= \min\{m_1+4m_2+9m_3+16m_4+9m_5+4m_6+m_7,\\
&\qquad 4m_1+16m_2+4m_3+4m_5+16m_6+4m_7,\\
&\qquad 9m_1+4m_2+m_3+16m_4+m_5+4m_6+9m_7,\\
&\qquad 16m_1+16m_3+16m_5+16m_7\}.
\end{aligned}$$

and generator matrix $G = [\underbrace{11\ldots1}_{m_1}\underbrace{22\ldots2}_{m_2}\underbrace{33\ldots3}_{m_3}\underbrace{44\ldots4}_{m_4}\underbrace{55\ldots5}_{m_5}\underbrace{66\ldots6}_{m_6}\underbrace{77\ldots7}_{m_7}]$ is given in the following theorems.

Theorem 6 $r_E(BRep^{m_1+m_2+\ldots+m_7}) = \frac{11}{2}(m_1+m_3+m_5+m_7)+6(m_2+m_6)+8m_4.$

Proof. By proposition 13 and Theorem 3, 4, 5 we have $r_E(BRep^{m_1+m_2+\ldots+m_7}) \geq \frac{11}{2}(m_1+m_3+m_5+m_7)+6(m_2+m_6)+8m_4$.

On the other hand, let $\mathbf{x} = (\mathbf{x}_1 \mid \mathbf{x}_2 \mid \mathbf{x}_3 \mid \mathbf{x}_4 \mid \mathbf{x}_5 \mid \mathbf{x}_6 \mid \mathbf{x}_7) \in \mathbb{Z}_8^{m_1+m_2+\ldots+m_7}$ with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7$ have compositions $(p_0, p_1, \ldots, p_7)$, $(q_0, q_1, \ldots, q_7)$, $(r_0, r_1, \ldots, r_7)$, $(s_0, s_1, \ldots, s_7)$, $(t_0, t_1, \ldots, t_7)$, $(u_0, u_1, \ldots, u_7)$, $(w_0, w_1, \ldots, w_7)$ such that $p_0+p_1+\ldots+p_7 = m_1$, $q_0+q_1+\ldots+q_7 = m_2$, $r_0+r_1+\ldots+r_7 = m_3$, $s_0+s_1+\ldots+s_7 = m_4$, $t_0+t_1+\ldots+t_7 = m_5$, $u_0+u_1+\ldots+u_7 = m_6$, $w_0+w_1+\ldots+w_7 = m_7$.

$d_E(\mathbf{x}, \bar{0}) = m_1+m_2+m_3+m_4+m_5+m_6+m_7-p_0+3p_2+8p_3+15p_4+8p_5+3p_6-q_0+3q_2+8q_3+15q_4+8q_5+3q_6-r_0+3r_2+8r_3+15r_4+8r_5+3r_6-s_0+3s_2+8s_3+15s_4+8s_5+3s_6-t_0+3t_2+8t_3+15t_4+8t_5+3t_6-u_0+3u_2+8u_3+15u_4+8u_5+3u_6-w_0+3w_2+8w_3+15w_4+8w_5+3w_6$, where $\bar{0}$ is the first vector of $BRep^{m_1+m_2+\ldots+m_7}$.

$d_E(\mathbf{x}, \mathbf{c}_1) = m_1+m_2+m_3+m_4+m_5+m_6+m_7-p_1+3p_3+8p_4+15p_5+8p_6+3p_7-q_2+3q_0+3q_4+8q_5+15q_6+8q_7-r_3+8r_0+3r_1+3r_5+8r_6+15r_7-s_4+15s_0+8s_1+3s_2+3s_6+8s_7-t_5+8t_0+15t_1+8t_2+3t_3+3t_7-u_6+3u_0+8u_1+15u_2+8u_3+3u_4-w_7+3w_1+8w_2+15w_3+8w_4+3w_5$, where $\mathbf{c}_1 = (\underbrace{11\ldots1}_{m_1}\underbrace{22\ldots2}_{m_2}\underbrace{33\ldots3}_{m_3}\underbrace{44\ldots4}_{m_4}\underbrace{55\ldots5}_{m_5}\underbrace{66\ldots6}_{m_6}\underbrace{77\ldots7}_{m_7})$ is the second vector of $BRep^{m_1+m_2+\ldots+m_7}$.

$d_E(\mathbf{x}, \mathbf{c}_2) = m_1+m_2+m_3+m_4+m_5+m_6+m_7-p_2+3p_0+3p_4+8p_5+15p_6+8p_7-q_4+15q_0+8q_1+3q_2+3q_6+8q_7-r_6+3r_0+8r_1+15r_2+8r_3+3r_4-s_0+3s_2+8s_3+15s_4+8s_5+3s_6-t_2+3t_0+3t_4+8t_5+15t_6+8t_7-u_4+15u_0+8u_1+3u_2+3u_6+8u_7-w_6+3w_0+8w_1+15w_2+8w_3+3w_4$, where $\mathbf{c}_2 = (\underbrace{22\ldots2}_{m_1}\underbrace{44\ldots4}_{m_2}\underbrace{66\ldots6}_{m_3}\underbrace{00\ldots0}_{m_4}\underbrace{22\ldots2}_{m_5}\underbrace{44\ldots4}_{m_6}\underbrace{66\ldots6}_{m_7})$ is the third vector of $BRep^{m_1+m_2+\ldots+m_7}$.

$d_E(\mathbf{x}, \mathbf{c}_3) = m_1+m_2+m_3+m_4+m_5+m_6+m_7-p_3+8p_0+3p_1+3p_5+8p_6+15p_7-q_6+3q_0+8q_1+15q_2+8q_3+3q_4-r_1+3r_3+8r_4+15r_5+8r_6+3r_7-s_4+15s_0+8s_1+3s_2+3s_6+8s_7-t_7+3t_1+8t_2+15t_3+8t_4+3t_5-u_2+3u_0+3u_4+8u_5+15u_6+8u_7-w_5+8w_0+15w_1+8w_2+3w_3+3w_7$, where $\mathbf{c}_3 = (\underbrace{33\ldots3}_{m_1}\underbrace{66\ldots6}_{m_2}\underbrace{11\ldots1}_{m_3}\underbrace{44\ldots4}_{m_4}\underbrace{77\ldots7}_{m_5}\underbrace{22\ldots2}_{m_6}\underbrace{55\ldots5}_{m_7})$ is the fourth vector of $BRep^{m_1+m_2+\ldots+m_7}$.

$d_E(\mathbf{x}, \mathbf{c}_4) = m_1+m_2+m_3+m_4+m_5+m_6+m_7-p_4+15p_0+8p_1+3p_2+3p_6+8p_7-q_0+3q_2+8q_3+15q_4+8q_5+3q_6-r_4+15r_0+8r_1+3r_2+3r_6+8r_7-s_0+3s_2+8s_3+15s_4+8s_5+3s_6-t_4+15t_0+8t_1+3t_2+3t_6+8t_7-u_0+3u_2+8u_3+15u_4+8u_5+3u_6-w_4+15w_0+8w_1+3w_2+3w_6+8w_7$, where $\mathbf{c}_4 = (\underbrace{44\ldots4}_{m_1}\underbrace{00\ldots0}_{m_2}\underbrace{44\ldots4}_{m_3}\underbrace{00\ldots0}_{m_4}\underbrace{44\ldots4}_{m_5}\underbrace{00\ldots0}_{m_6}\underbrace{44\ldots4}_{m_7})$ is the fifth vector of $BRep^{m_1+m_2+\ldots+m_7}$.

$d_E(\mathbf{x}, \mathbf{c}_5) = m_1+m_2+m_3+m_4+m_5+m_6+m_7-p_5+8p_0+15p_1+8p_2+3p_3+3p_7-q_2+3q_0+3q_4+8q_5+15q_6+8q_7-r_7+3r_1+8r_2+15r_3+8r_4+3r_5-s_4+15s_0+8s_1+3s_2+3s_6+8s_7-t_1+3t_3+8t_4+15t_5+8t_6+3t_7-u_6+3u_0+8u_1+15u_2+8u_3+3u_4-w_3+8w_0+3w_1+$

$3w_5+8w_6+15w_7$, where $\mathbf{c}_5=(\underbrace{55\dots5}_{m_1}\underbrace{22\dots2}_{m_2}\underbrace{77\dots7}_{m_3}\underbrace{44\dots4}_{m_4}\underbrace{11\dots1}_{m_5}\underbrace{66\dots6}_{m_6}\underbrace{33\dots3}_{m_7})$ is the sixth vector of $BRep^{m_1+m_2+\dots+m_7}$.

$d_E(\mathbf{x},\mathbf{c}_6)=m_1+m_2+m_3+m_4+m_5+m_6+m_7-p_6+3p_0+8p_1+15p_2+8p_3+3p_4-q_4+15q_0+8q_1+3q_2+3q_6+8q_7-r_2+3r_0+3r_4+8r_5+15r_6+8r_7-s_0+3s_2+8s_3+15s_4+8s_5+3s_6-t_6+3t_0+8t_1+15t_2+8t_3+3t_4-u_4+15u_0+8u_1+3u_2+3u_6+8u_7-w_2+3w_0+3w_4+8w_5+15w_6+8w_7$, where $\mathbf{c}_6=(\underbrace{66\dots6}_{m_1}\underbrace{44\dots4}_{m_2}\underbrace{22\dots2}_{m_3}\underbrace{00\dots0}_{m_4}\underbrace{66\dots6}_{m_5}\underbrace{44\dots4}_{m_6}\underbrace{22\dots2}_{m_7})$ is the seventh vector of $BRep^{m_1+m_2+\dots+m_7}$.

$d_E(\mathbf{x},\mathbf{c}_7)=m_1+m_2+m_3+m_4+m_5+m_6+m_7-p_7+3p_1+8p_2+15p_3+8p_4+3p_5-q_6+3q_0+8q_1+15q_2+8q_3+3q_4-r_5+8r_0+15r_1+8r_2+3r_3+3r_7-s_4+15s_0+8s_1+3s_2+3s_6+8s_7-t_3+8t_0+3t_1+3t_5+8t_6+15t_7-u_2+3u_0+3u_4+8u_5+15u_6+8u_7-w_1+3w_3+8w_4+15w_5+8w_6+3w_7$, where $\mathbf{c}_7=(\underbrace{77\dots7}_{m_1}\underbrace{66\dots6}_{m_2}\underbrace{55\dots5}_{m_3}\underbrace{44\dots4}_{m_4}\underbrace{33\dots3}_{m_5}\underbrace{22\dots2}_{m_6}\underbrace{11\dots1}_{m_7})$ is the eighth vector of $BRep^{m_1+m_2+\dots+m_7}$. Thus

$$d(\mathbf{x},BRep^{m_1+m_2+\dots+m_7}) \;\leq\; \tfrac{11}{2}(m_1+m_3+m_5+m_7)+6(m_2+m_6)+8m_4.$$

Hence the equality.

Theorem 7 $\min\{2m_1+2m_2+2m_3+2m_4+2m_5+2m_6+2m_7, 2m_2+2m_3+2m_4+4m_5+2m_6+2m_7, 2m_1+2m_2+2m_4+2m_5+2m_6+4m_7, 4m_1+2m_2+2m_3+2m_4+2m_6+2m_7, 2m_1+2m_2+4m_3+2m_4+2m_5+2m_6\} \leq r_{HW}(BRep^{m_1+m_2+\dots+m_7}) \leq 11(m_1+m_3+m_5+m_7)+12(m_2+m_6)+16m_4$.

Proof. By choosing $\mathbf{x}=(\underbrace{11\dots\dots\dots1}_{m_1+m_2+\dots+m_7})\in\mathbb{Z}^{m_1+m_2+\dots+m_7}$ and computing the homogenous distance from each codeword we get $d_{HW}(\mathbf{x},BRep^{m_1+m_2+\dots+m_7}) = \min\{2m_1+2m_2+2m_3+2m_4+2m_5+2m_6+2m_7, 2m_2+2m_3+2m_4+4m_5+2m_6+2m_7, 2m_1+2m_2+2m_4+2m_5+2m_6+4m_7, 4m_1+2m_2+2m_3+2m_4+2m_6+2m_7, 2m_1+2m_2+4m_3+2m_4+2m_5+2m_6\}$. Hence the first inequality follows. The second inequality follows from Proposition 10 and Theorem 6. □

5 Octonary Simplex Codes of Type α and β

Simplex codes of type α and β have been studied in [25]. The linear code $\mathbf{S}_k^\alpha$ is a type α simplex code over $\mathbb{Z}_8$ with parameters $(n=8^k, M=8^k, d_{HW}=2^{3(k+1)-2})$ generated by

$$G_k^\alpha=\left[\begin{array}{c|c|c|c|c|c|c|c} 00\cdots0 & 11\cdots1 & 22\cdots2 & 33\cdots3 & 44\cdots4 & 55\cdots5 & 66\cdots6 & 77\cdots7\\ \hline G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha \end{array}\right] \quad (5)$$

with $G_1^\alpha=[01234567]$. The number of vectors in $\mathbf{S}_k^\alpha$ is 2^{3k}. The dual code of $\mathbf{S}_k^\alpha$ is denoted by $S_k^{\alpha\perp}$.

The linear code $\mathbf{S}_k^\beta$ is a type β simplex code over $\mathbb{Z}_8$ with parameters $(n = 2^{2(k-1)}(2^k-1), M = 8^k, d_{HW} = 2^{2k-1}(2^k-1))$ generated by

$$G_2^\beta = \left[\begin{array}{c|c|c|c|c} 11111111 & 0 & 2 & 4 & 6 \\ \hline 01234567 & 1 & 1 & 1 & 1 \end{array}\right]$$

and for $k > 2$

$$G_k^\beta = \left[\begin{array}{c|c|c|c|c} 11\cdots1 & 00\cdots0 & 22\cdots2 & 44\cdots4 & 66\cdots6 \\ \hline G_{k-1}^\alpha & G_{k-1}^\beta & G_{k-1}^\beta & G_{k-1}^\beta & G_{k-1}^\beta \end{array}\right],$$

where G_{k-1}^α is the generator matrix of S_{k-1}^α. The dual code of $\mathbf{S}_k^\beta$ is denoted by $S_k^{\beta\perp}$.

Theorem 8 *$r_{HW}(S_k^\alpha) \geq 2^{3k+1}$ and $r_E(S_k^\alpha) \leq 6(8^k-1)+2$.*

Proof. The proof can be obtained using the Proposition 13, Theorem 6, equation (5) and is similar to the $\mathbb{Z}_4$ case [26]. Hence it is omitted. □

Theorem 9 *$r_E(S_k^\beta) \leq \frac{3}{2}(8^k-1) - \frac{5}{3}(4^k-1) - \frac{39}{2} + r_E(S_2^\beta)$ and $r_{HW}(S_k^\beta) \leq 3(8^k - 1) - \frac{10}{3}(4^k-1) - 139 + r_{HW}(S_2^\beta)$.*

Proof. The first inequality is proved using Theorem 6 and is similar to the $\mathbb{Z}_4$ case [26]. The case of homogeneous weight is similar. □

Theorem 10 *$r_E(S_k^{\alpha\perp}) \leq 3$, $r_{HW}(S_k^{\alpha\perp}) = 1$ and $r_{HW}(S_k^{\beta\perp}) = 2$.*

Proof. By Lemmaa 4.2 and Theorem 4.3 of [25], $r_E(S_k^{\alpha\perp}) \leq 3$. By Theorem 4.3(3) of [25], $r_{HW}(S_k^{\alpha\perp}) \leq 1$. Sience $r_{HW}(S_k^{\alpha\perp}) \geq 1$, so $r_{HW}(S_k^{\alpha\perp}) = 1$. By Theorem 4.4 of [25] and by Theorem 1, $r_{HW}(S_k^{\beta\perp}) \leq 2$ and as $r_{HW}(S_k^{\beta\perp}) \geq 1$ thus $r_{HW}(S_k^{\beta\perp}) = 1$ or 2 but $r_{HW}(S_k^{\beta\perp}) \neq 1$ by Proposition 12. Hence the result follows. □

Theorem 11 *S_k^α and S_k^β are self orthogonal codes over $\mathbb{Z}_8$.*

Proof. The proof follows from Proposition 3.

6 Octonary MacDonald Codes of Type α and β

The q-ary MacDonald code $\mathbb{M}_{k,u}(q)$ over the finite field $\mathbb{F}_q$ is a unique $[\frac{q^k-q^u}{q-1}, k, q^{k-1} - q^{u-1}]$ code in which every nonzero codeword has weight either q^{k-1} or $q^{k-1} - q^{u-1}$ [15]. In [12], authors have defined the MacDonald codes over $\mathbb{Z}_4$ using the generator matrices of simplex codes. In a similar manner one can define MacDonald code over $\mathbb{Z}_{2^s}$. For $1 \leq u \leq k-1$, let $G^{\alpha}_{k,u}\left(G^{\beta}_{k,u}\right)$ be the matrix obtained from $G^{\alpha}_{k}\left(G^{\beta}_{k}\right)$ by deleting columns corresponding to the columns of $G^{\alpha}_{u}\left(G^{\beta}_{u}\right)$. i.e,

$$G^{\alpha}_{k,u} = \left[\begin{array}{cc} G^{\alpha}_{k} & \backslash \frac{\mathbf{0}}{G^{\alpha}_{u}} \end{array}\right],$$

and

$$G^{\beta}_{k,u} = \left[\begin{array}{cc} G^{\beta}_{k} & \backslash \frac{\mathbf{0}}{G^{\beta}_{u}} \end{array}\right],$$

where $[A\backslash\frac{\mathbf{0}}{B}]$ is the matrix obtained by deleting the matrix $\mathbf{0}$ and B from A where B is a $(k-u)\times 2^{su}$ matrix in (6) $\big($ resp. $(k-u)\times 2^{(s-1)(u-1)}(2^u-1)\big)$ matrix in (6). The code

$$\mathbb{M}^{\alpha}_{k,u} : [2^{sk}-2^{su}, sk]\left(\mathbb{M}^{\beta}_{k,u} : [2^{(s-1)(u-1)}(2^k-1) - 2^{(s-1)(u-1)}(2^u-1), sk]\right)$$

generated by the matrix $G^{\alpha}_{k,u}\left(G^{\beta}_{k,u}\right)$ is the punctured code of $S^{\alpha}_{k}\left(S^{\beta}_{k}\right)$ and is called a *MacDonald code* of type α (β).

The next theorem provides basic bounds on the covering radii of MacDonald codes over $\mathbb{Z}_8$.

Theorem 12

$$r_E(\mathbb{M}^{\alpha}_{k,u}) \quad \leq \quad 6(8^k - 8^r) + r_E(\mathbb{M}^{\alpha}_{r,u}) \; \textit{for} \; u < r \leq k.$$

Proof. Similar to $\mathbb{Z}_4$ case [26]. □

7 Octonary Reed-Muller Code

In this section, we give covering radius of octonary first order Reed Muller code [25]. Let $1 \leq i \leq m-2$. Let $\mathbf{v}_i$ be a vector of length 2^{m-2} consisting of successive blocks of 0's and 1's each of size $2^{(m-2)-i}$ and let $\mathbf{1} = (111\dots11) \in \mathbb{Z}_2^{2^{m-2}}$. Let G be a $(m-1)\times 2^{m-2}$ matrix given by (consisting of the rows as $\mathbf{1}$ and $4\mathbf{v}_i$ $(1 \leq i \leq m-2)$)

$$G = \left[\begin{array}{cccccccccc} 0 & 0 & \cdots & 0 & 0 & 4 & 4 & \cdots & 4 & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 4 & \cdots & 0 & 4 & 0 & 4 & \cdots & 0 & 4 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{array}\right].$$

The code generated by G is called the *first order Reed-Muller code over* $\mathbb{Z}_8$, denoted $\mathbb{R}^{1,m-2}$. It is a $(n = 2^{m-2}, M = 2^{m+1}, d_{HW} = 2^{m-1})$ type α linear code over $\mathbb{Z}_8$ [25]. The following proposition gives the covering radius of the first order binary Reed-Muller code for even m [30].

Proposition 14 *The covering radius of the binary first order Reed-Muller code* $RM(1,m)$ *for even* m *is given by*

$$r(RM(1,m)) = 2^{m-1} - 2^{\frac{m}{2}-1}.$$

From Propositions 11 and 14 we obtain the following result.

Theorem 13 $r_{HW}(\mathbb{R}^{1,m-2}) = 2^{m-1} - 2^{\frac{m}{2}-1}$*for even* m.

8 Octonary Octa Codes

The octa code over $\mathbb{Z}_8$ is generated by the following matrix.

$$G = \begin{bmatrix} 5 & 7 & 5 & 6 & 1 & 0 & 0 & 0 \\ 5 & 0 & 7 & 5 & 6 & 1 & 0 & 0 \\ 5 & 0 & 0 & 7 & 5 & 6 & 1 & 0 \\ 5 & 0 & 0 & 0 & 7 & 5 & 6 & 1 \end{bmatrix}.$$

From Proposition 12 we get the following result.

Theorem 14 *If $\mathcal{C}$ is the code generated by G then $r_{HW}(\mathcal{C}) \geq 6$.*

9 Conclusion

In this work, we have introduced new torsion and reduction codes for any linear octonary code and obtained a nice relationship between various reduction and torsion codes. Further, we have extended some of the results regarding covering radius of [26] to the octonary case. In particular, we have found exact values and the bounds of the covering radius of Repetition codes, Simplex codes of Type α and Type β and their duals, MacDonald codes, and first order Reed-Muller codes, Octacode over $\mathbb{Z}_8$. New reduction and torsion codes can be used to classify octonary linear codes.

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