J. Math. Kyoto Univ. (JMKYAZ)
13-3 (1973) 595-603

On one-dimensional stochastic differential equations with non-sticky boundary conditions

By

Shojiro MANABE and Tokuzo SHIGA

(Communicated by Professor Yoshizawa, January 6, 1973)

1. Consider the following one-dimensional stochastic differential equation

$$(1) \qquad dX_t = a(X_t)dB_t.$$

It is well-known that if $a(x)$ is Hölder continuous of order α ($\geqq 1/2$) the pathwise uniqueness holds for (1). ([4]). But, for example, if $a(x) = |x|^{\alpha}$, $(0<\alpha<1/2)$, (1) has infinitely many solutions, ([1], [3]).

In this paper we shall discuss the uniqueness problem for (1) by imposing a non-sticky boundary condition:

$$(2) \qquad \begin{cases} X_t - X_0 = \int_0^t a(X_s)dB_s, \\ \int_0^t \chi_{\{0\}}(X_s)ds = 0, \end{cases}$$

where $\chi_{\{0\}}(x)=0$ for $x \neq 0$, $\chi_{\{0\}}(0)=1$. In the case that $a(x)$ is an odd function, we shall prove the pathwise uniqueness of the absolute value of solution of (2) under some regularity conditions. Moreover we shall. show the pathwise uniqueness of the following Skorokhod equation with the reflecting barrier boundary condition, applying the same method.

$$(3)\qquad \begin{cases} X_t - X_0 = \int_0^t a(X_s)dB_s + \varphi_t, \\ \int_0^t \chi_{\{0\}}(X_s)ds = 0, \\ X_t \geqq 0, \text{ for } \forall t \geqq 0, \end{cases}$$

where $\{\varphi_t\}$ is an increasing process such that $\int_0^t \chi_{\{0\}}(X_s)d\varphi_s = \varphi_t$.

Let $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$ be a probability space with an increasing family of σ-fields $\mathcal{F}_t$ which are sub σ-fields of $\mathcal{F}$ and $\{B_t\}$ be a $\mathcal{F}_t$-Brownian motion; i.e. $\{B_t\}$ is a continuous $\mathcal{F}_t$-martingale such that $E[(B_t - B_s)^2 | \mathcal{F}_s] = t - s$ for $t > s \geqq 0$.

Theorem 1. *Let $a(x)$ be a bounded continuous function on R^1 which satisfies the following conditions,*

(i) *$a(x)$ is an odd function and continuously differentiable on $R_{\{0\}}$,*

(ii) $\int_0^\delta 1/a(x)^2 dx < +\infty$ *for some* $\delta > 0$,

(iii) $\lim_{x \downarrow 0} xa'(x)/a(x)$ *exists and* $\lim_{x \downarrow 0} xa'(x)/a(x) \neq 1/2$.

Under these conditions, if two $\mathcal{F}_t$-adapted processes $\{X_t^1\}$, $\{X_t^2\}$ satisfy the equation (2) for the same Brownian motion $\{B_t\}$, then with probability one, $|X_t^1| = |X_t^2|$ for all $t \geqq 0$.

Here we give some remarks.

Remark 1. Under the conditions of Theorem 1, there is a diffusion process which satisfies the equation (2). However it is obvious that the pathwise uniqueness does not hold for the equation (2), because if $X_0 = 0$ and $\{X_t\}$ is a solution of (2) then $\{-X_t\}$ is, also, a solution of (2).

Remark 2. If $a(x) = |x|^\alpha s(x)$, (where $0 < \alpha < 1/2$, and $s(x)$ is a slowly varying odd function at the origin) or $a(x)$ is a linear combination of such functions, $a(x)$ satisfies the above conditions.

Remark 3. It is not difficult to check the following facts by elementary calculations.

(a) Under the condition (ii) if $\lim_{x\downarrow 0} xa'(x)/a(x)$ exists, then $\lim_{x\downarrow 0} xa'(x)/a(x)\leqq 1/2$.

(b) $\lim_{x\downarrow 0} xa'(x)/a(x)$ exists and $\lim_{x\downarrow 0} xa'(x)/a(x)<1/2$ if and only if $\lim_{x\downarrow 0} a'(x)\int_0^x 1/a(y)\,dy$ exists and $\lim_{x\downarrow 0} a'(x)\int_0^x 1/a(y)\,dy<1$.

(c) The conditions (ii) and (iii) imply $\lim_{x\downarrow 0} 1/a(x)\int_0^x 1/a(y)\,dy=0$.

(d) By the conditions (i) and (ii) there exists $\delta>0$ such that $a(x)\neq 0$ for all $x\in(0, \delta)$.

2. In order to prove Theorem 1 it is sufficient to show the uniqueness of (2) up to the first exist time from a neighborhood of the origin, because $a(x)$ is smooth except the origin. Therefore without loss of generality, we may assume that $a(x)>0$ for all $x>0$ and $a'(x)\int_0^x 1/a(y)\,dy$ is bounded on R^1.

Lemma 1. *Under the assumption of Theorem* 1 *the equation* (2) *is transformed into*

$$Y_t-Y_0=\int_0^t\sqrt{|Y_s|}\,dB_s+\int_0^t b(Y_s)\,ds \tag{4}$$

by putting $g(x)=\frac{1}{4}\left(\int_0^x 1/a(y)\,dy\right)^2$ *and* $Y_t=g(X_t)$, *where* $b(x)$ *is a bounded continuous function and* $b(0)>0$.

Proof. First, note that $g(x)$ is an even function and maps $[0, \infty)$ one-to-one onto itself. We show that the Ito formula is applicable to $g(x)$. $g'(x)=\frac{1}{2a(x)}\int_0^x 1/a(y)dy$ is continuous, (by Remark 3 (*c*)). $g''(x)=\frac{1}{2a(x)^2}\left[1-a'(x)\int_0^x 1/a(y)dy\right]$ is locally integrable, and continuous except at the origin. Therefore we can choose a sequence $\{g_n\}\subset C^2(R^1)$ such that $\{g_n(x)\}$ and $\{g_n'(x)\}$ converge uniformly to $g(x)$ and $g'(x)$, respectively, and moreover $\{g_n''(x)\}$ converges to $g''(x)$ for all $x\in R^1/\{0\}$ and $\{g_n''(x)a(x)\}$ converges boundedly to $g''(x)a(x)^2 \chi_{R^1/\{0\}}(x)$ for all $x\in R^1$. Applying the Ito formula for g_n, we have

$$g_n(X_t)-g_n(X_0)=\int_0^t g_n'(X_s)a(X_s)dB_s+\frac{1}{2}\int_0^t g_n'(X_s)a(X_s)^2 ds.$$

As $n\to\infty$, we get

$$g(X_t)-g(X_0)=\int_0^t g'(X_s)a(X_s)dB_s+\frac{1}{2}\int_0^t g''(X_s)a(X_s)^2\chi_{R^1/\{0\}}(X_s)ds.$$

So, using non-sticky condition

$$g(X_t)-g(X_0)=\int_0^t g'(X_s)a(X_s)\,dB_s+\frac{1}{2}\int_0^t g''(X_s)a(X_s)^2 ds.$$

Denote the inverse function of g by h, and put $b(x)=\frac{1}{4}[1-a'(h(x))\int_0^{h(x)} 1/a(z)\,dz]$. Then $b(0)>0$ by Remark 3, (b). Noting that $a(-x)=-a(x)$, $g'(-x)=-g'(x)$, $g'(x)a(x)=\sqrt{g(x)}$, $g''(h(x))a^2(h(x))=b(x)$ and $X_t=h(Y_t)\ \mathrm{sgn}\ (X_t)$, we obtain

$$Y_t-Y_0=\int_0^t \sqrt{|Y_s|}\ dB_s+\int_0^t b(Y_s)ds.$$

This lemma implies that in order to prove Theorem 1 it is sufficient to show the pathwise uniqueness of the solution of (4). Therefore we shall prove it in more general form as fol lows;

Theorem 2. *Consider the following equation*

$$X_t-X_0=\int_0^t a(X_s)dB_s+\int_0^t b(X_s)ds. \tag{5}$$

Suppose that $a(x)$, $b(x)$ are bounded continuous functions which satisfy the following conditions;

(i) *there exists a positive increasing function* $\rho(x)$ *defined on* $(0, \infty)$ *such that* $|a(x)-a(y)|\leqq\rho(|x-y|)$ *for* $\forall x$, $\forall y\in R^1$, *and*

$$\int_0^\delta 1/\rho(x)^2dx=+\infty \quad \text{for some } \delta>0,$$

(ii) $a(0)=0$, $a(x)\neq 0$ *for* $\forall x\neq 0$ *and* $b(0)\neq 0$.

Then the pathwise uniquenss holds for (5).

The following lemma is essential for the proof of Theorem 2.

Lemma 2. *Under the assumption of Theorem* 2 *any solution* $\{X_t\}$ *of the equation* (5) *satisfies*

$$\int_0^t \chi_{\{0\}}(X_s)ds=0, \quad \forall t\geqq 0, \quad \textit{with probability one.}$$

Proof. Let $r_0=1>r_1>r_2\ldots>r_n\rightarrow 0$ be defined by $\int_{r_n}^{r_{n-1}} 1/\rho(x)^2 dx=n$ and choose a continuous $g_n(x)$ defined on $[0, \infty)$ such that $g_n(x)=$between 0 and $\frac{2}{n}\rho^{-2}(x)$ for every x of (r_n, r_{n-1}) and vanishes in the other part, and satisfies $\int_{r_n}^{r_{n-1}} g_n(x)\, dx=1$. Let $f_n(x)$ be defined by $f_n(x)=x-\int_0^x \mathrm{dy} \int_0^y g_n(z)\, dz$ and extend on $(-\infty, +\infty)$ symmetrically. Then $f_n(x)$ is twice continuously differentiable and $\{f_n(x)\}$, $\{f_n'(x)\}$ converge boundedly to 0, $\chi_{\{0\}}(x)$, respectively. By the Ito formula

$$f_n(X_t)-f_n(X_0)=\int_0^t f_n'(X_s)a(X_s)dB_s+\frac{1}{2}\int_0^t f_n''(X_s)a(X_s)^2ds + \int_0^t f_n'(X_s)b(X_s)ds.$$

So we get

$$E[f_n(X_t)-f_n(X_0)]=\frac{1}{2}E[\int_0^t f_n''(X_s)a(X_s)^2ds] + E[\int_0^t f_n'(X_s)b(X_s)ds=I_1+I_2.$$

Then we have

$$|I_1| \leqq \frac{t}{2} \max_{r_n \leqq |x| \leqq r_{n-1}} [g_n(x)\rho^2(x)] \leqq \frac{t}{n} \to 0, \text{ as } n \to \infty.$$

For I_2, by bounded convergence theorem

$$I_2 \longrightarrow b(0) E[\int_0^t \chi_{\{0\}}(X_s) ds].$$

But it is obvious that $E[f_n(X_t) - f_n(X_0)] \to 0$, as $n \to \infty$. Hence, noting $b(0) \neq 0$, $\int_0^t \chi_{\{0\}}(X_s)\, ds = 0$ follows with probability one.

Lemma 3. (*Yamada, T*[5]). *Consider the following two equations;*

(6) $$X_t^{(i)} - X_0^{(i)} = \int_0^t a(X_s^{(i)}) dBs + \int_0^t b^{(i)}(X_s^{(i)}) ds, \quad i = 1, 2,$$

where $a(x)$, $b^{(i)}(x)$ are bounded continuous functions. Suppose that
(i) *There exists a positive increasing function $\rho(x)$ on $(0, \infty)$ such that $|a(x) - a(y)| \leqq \rho(|x-y|)$ for any x, y of R^1 and*

$$\int_0^\delta 1/\rho(x)^2 dx = +\infty \quad \text{for some } \delta > 0,$$

(ii) *$b^{(1)}(x) < b^{(2)}(x)$ for any x of R^1.*

If $\{X_t^{(1)}\}$, $\{X_t^{(2)}\}$ are solutions of (6) *corresponding to $[a, b^{(1)}]$, $[a, b^{(2)}]$, respectively and $X_0^{(1)} \leqq X_0^{(2)}$, then $X_t^{(1)} \leqq X_t^{(2)}$ for $\forall t \geqq 0$ with probability one.*

Proof of Theorem 2. By virtue of Lemma 3 there exist the maximal solution $\{\bar{X}_t\}$ and the minimal solution $\{\underline{X}_t\}$ of the equation (5). It is easy to show that the maximal solution and the minimal solution have the strong Markov property because of their uniqueness. So $\{\bar{X}_t\}$ and $\{\underline{X}_t\}$ are diffusion processes with the same local generator $A = \frac{1}{2}$

$a^2(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}$ and have no stay*) at the origin by Lemma 2. However by Feller's general theory of one dimensional diffusion processes, there exists only one diffusion process which possesses A as its local generator and does not stay at the origin**). Therefore we have $P\{\bar{X}_t=X_t\}=1$ for $\forall t\geqq 0$. Hence the pathwise uniqueness follows from this.

Next, we study the Skorokhod equation with the reflecting barrier condition at the origin.

Theorem 3. *Suppose that $a(x)$ is bounded continuous function of $[0, \infty)$ such that*

(i) $a(0)=(0)$ *and continuously differentiable on* $(0, \infty,)$

(ii) $\int_0^\delta \frac{dx}{a(x)^2}<+\infty$ *for some* $\delta>0$,

(iii) $\lim_{x\downarrow 0}\frac{xa'(x)}{a(x)}$ *exists and* $\lim_{x\downarrow 0}\frac{xa'(x)}{a(x)}\neq\frac{1}{2}$.

Then the pathwise uniqueness holds for the equation (3).

Proof. If we extend $a(x)$ on $(-\infty, \infty)$ as an odd function, $a(x)$ satisfies the conditions of Theorem 1. Let $g(x)$, $g_n(x)$ be the same functions as in the proof of Lemma 1. Moreover we may assume $g_n'(0)=0$ because $\lim_{x\downarrow 0} g'(x)=0$ by Remark 3, (*c*). Applying the Ito formula for $g_n(x)$,

$$g_n(X_t)-g_n(X_0)=\int_0^t g_n'(X_s)a(X_s)dB_s+\frac{1}{2}\int_0^t g_n''(X_s)a(X_s)^2ds+\int_0^t g_n'(X_s)d\varphi_s.$$

Since $\int_0^t \chi_{\{0\}}(X_s)d\varphi_s=\varphi_t$ and $g_n'(0)=0$, the last term of the right hand side vanishes. So as $n\rightarrow\infty$,

---

*), **) By this, we mean that, with probability one, the set $\{t; X_t=0\}$ has the Lebesgue measure 0.

$$g(X_t)-g(X_0)=\int_0^t g'(X_s)a(X_s)dB_s+\frac{1}{2}\int_0^t g''(X_s)a(X_s)^2\chi_{R^1\diagup\{0\}}(X_s)ds.$$

By the same argument of Lemma 1 the equation (3) is transformed into the equation (4) in Lemma 1. Therefore, since $g(x)$ is a homeomorphism from $[0, \infty)$ onto itself, the pathwise uniqueness follows immediately from Theorem 2.

Remark 4. The absolute value of solution of the equation (2) is the unique solution of the equation (3). Now, we prove $Y_t=|X_t|$ is a solution of the equation (3) for any solution $\{X_t\}$ of the equation (2). Let $\{\varphi_t\}$ defined by

$$|X_t|-|X_0|-\int_0^t a(|X_s|)dB_s.$$

Then it is sufficient to show $\{\varphi_t\}$ is an increasing process such that $\int_0^t \chi_{\{0\}}(|X_s|)d\varphi_s=\varphi_t$. We can find a sequence $\{g_n(x)\}\subset C^2(R^1)$ such that

(i) $g_n(-x)=g_n(x)$ and $\{g_n(x)\}$ converges uniformly to $|x|$,

(ii) $|g_n'(x)|\leqq 1$,

(iii) $\mathrm{supp}(g_n'')\subset\left[-\frac{2}{n}, \frac{2}{n}\right]$ and $0\leqq g_n''(x)\leqq n$.

Applying the Ito formula for $g_n(x)$ and $\{X_t\}$, we have

$$g_n(X_t)-g_n(X_0)=\int_0^t g_n'(X_s)a(X_s)dB_s+\frac{1}{2}\int_0^t g_n''(X_s)a(X_s)^2ds.$$

Noting $g_n'(x)a(x)\to a(|x|)$, as $n\to\infty$, by the bounded convergence theorem, we have

$$E\left\{\left(\frac{1}{2}\int_0^t g_n''(X_s)a(X_s)^2ds-\varphi_t\right)^2\right\}\to 0, \text{ as } n\to\infty.$$

Therefore, φ_t is an increasing process and

$$|X_t|=|X_0|+\int_0^t a(|X_s|)dB_s+\varphi_t.$$

Next, applying the Ito formula for $g_n(x)$ and $|X_t|$, we have

$$g_n(|X_t|)-g_n(|X_0|)=\int_0^t g_n'(X_s)a(X_s)dB_s+\frac{1}{2}\int_0^t g_n''(|X_s|)a(|X_s|)^2ds$$
$$+\int_0^t g_n'(|X_s|)d\varphi_s.$$

Repeating the same argument, we see

$$E\left\{\left(\int_0^t g_n'(|X_s|)d\varphi_s\right)^2\right\}\to 0, \quad \text{as } n\to\infty.$$

Therefore, by the bounded convergence theorem,

$$E\left\{\left(\int_0^t \chi_{(0,\infty)}(|X_s|)d\varphi_s\right)^2\right\}=0.$$

Thus the process $\{\varphi_t\}$ satisfies $\int_0^t \chi_{\{0\}}(|X_s|)d\varphi_s=\varphi_t$.

Finally, we don't know whether the pathwise uniqueness hold in case that $a(x)$ is an even function. Another important problem is to formulate the uniqueness problem of (1) by imposing an appropriate random lateral condition different from the non-sticky boundary condition.

OSAKA UNIVERSITY
NARA WOMEN'S UNIVERSITY

Reference

[1] Girsanov, I. V.: *An example of non-uniqueness of the solution of the stochastic equation of K, Ito.* Theory of prob. and its appl. **7** (1962) 325–331 (English translation).

[2] Ito, K. and McKean, H. P. Jr: *Diffusion processes and their sample paths.* Springer, Berlin (1965).

[3] Tanaka, H. and Hasegawa, M.: *Stochastic differential equations.* Seminar on Probability Vol. **19** (1964), (in Japanese).

[4] Yamada, T. and Watanabe, S.: *On the uniqueness of solutions of stochastic differential equations.* J. Math. Kyoto Univ. **11** (1971) 155–167.

[5] Yamada, T.: *On a comparision theorem for solutions of stochastic differential equations and its applications.* to appear in J. Math. Kyoto Univ..