ON ONE-FACTORIZATIONS OF COMPLETE GRAPHS

Dedicated to the memory ot Hanna Neumann

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(Received 21 March 1972)

Communicated by G. Szekeres

1. Introduction

We use standard graph notation and definitions, as in [1]: in particular K_n is the complete graph on n vertices and $K_{n,n}$ is the regular complete bigraph of order $2n$.

Given a graph G, a factor of G is a spanning subgraph of G and a factorization is a sequence of edge-disjoint factors whose union is G. A one-factor is a factor which is a regular graph of degree 1; a one-factorization is a factorization whose factors are all one-factors. It is well-known that K_{2n} and $K_{n,n}$ always have one-factorizations. If K_{2n} has vertex-set $\{1,2,\cdots,2n\}$ then [1, p. 85] $\mathscr{G}_{2n} = \{G_1, G_2, \cdots, G_{2n-1}\}$ is a one-factorization where

$$G_i = \{(2n,i)\} \cup \{(i-j, i+j): j = 1,2,\cdots,n-1\}, \tag{1}$$

$i-j$ and $i+j$ being taken as integers modulo $2n-1$ in the range $\{1,2,\cdots,2n-1\}$. If the vertices of $K_{n,n}$ are written as $1_1, 2_1, \cdots, n_1, 1_2, 2_2, \cdots, n_2$ where the induced subgraph of $1_\alpha, 2_\alpha, \cdots, n_\alpha$ is null then $\mathscr{X}_n = \{X_1, X_2, \cdots, X_n\}$ is a one-factorization if

$$X_i = \{(j_1, (j-i+1)_2): j = 1,2,\cdots,n\}, \tag{2}$$

$j-i+1$ being taken as integers modulo n in the range $\{1,2,\cdots,n\}$.

Two factorizations $\mathscr{F}$ and $\mathscr{F}'$ of G are isomorphic if there is a permutation of the vertices of G which sends each member of $\mathscr{F}$ into a member of $\mathscr{F}'$. It is easy to see that, up to isomorphism, K_2, K_4 and K_6 have unique one-factorizations. There are six non-isomorphic one-factorizations of K_8. We shall prove

THEOREM 1. *When $n \geqq 4$, there are two non-isomorphic one-factorizations of K_{2n}.*

Given any positive integers i, k and n, we shall write d_{ik} for the greatest common divisor $(i-k, 2n-1)$ of $i-k$ and $2n-1$, and

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$$v_{ik} = (2n-1)/d_{ik};$$

$\equiv$ will denote congruence modulo $2n-1$.

2. Divisions

Suppose $F_{i_1}, F_{i_2}, \cdots, F_{i_t}$ are members of a factorization $\mathscr{F}$ of a graph G. We say that they form a *t-division* if $F_{i_1} \cup F_{i_2} \cup \cdots \cup F_{i_t}$ is a disconnected graph, and refer to the vertex-sets of the components of the union as the components of the division. If $F_{i_1}, F_{i_2}, \cdots, F_{i_t}$ are a t-division then $F_{i_\alpha}, F_{i_\beta}$ will necessarily be a 2-division if $\alpha \neq \beta$, and each component of the t-division will be a union of the components of the 2-division.

If $\mathscr{F}$ is a one-factorization of G then $F_{i_1} \cup F_{i_2} \cup \cdots \cup F_{i_t}$ is regular of degree t. Therefore each component of a t-division contains more than t vertices. In particular if G is of order $2n$ then an $(n-1)$-division has two components of order n; no n-division can occur. (In fact no $(n-1)$-division can occur when n is odd, as the components have one-factors and consequently must be of even order.)

LEMMA 1. *If G_i and G_k are any two factors in $\mathscr{G}_{2n}$ then $G_i \cup G_k$ consists of a cycle of length $v_{ik}+1$ and $\frac{1}{2}(d_{ik}-1)$ cycles of length $2v_{ik}$.*

PROOF. Since $G_i \cup G_k$ is a regular graph of degree 2, it is a union of disjoint cycles. If one such cycle is

$$\gamma_0, \gamma_1, \cdots, \gamma_t,$$

where $\gamma_0 = \gamma_t$, it is necessarily true that $\{\gamma_0, \gamma_1\}, \{\gamma_2, \gamma_3\}, \cdots, \{\gamma_{2x}, \gamma_{2x+1}\}, \cdots$ are all in the same one-factor. The edge $\{\gamma_{t-1}, \gamma_0\}$ cannot be in this one-factor, because γ_0 cannot have degree 2 in a one-factor. So all the cycles are of even length, and the edges are alternately in G_i and G_k.

Suppose the cycle containing vertex $2n$ is of length $2m$; write it as

$$\alpha_0, \alpha_1, \cdots, \alpha_{2m-1}, \alpha_{2m} \tag{3}$$

where $\alpha_0 = \alpha_{2m} = 2n$. Without loss of generality we can assume $\alpha_1 = i$ and $\alpha_{2m-1} = k$. Since (3) is a cycle, $\alpha_{2x-1} \neq k$ when $0 < x < m$. The edge $\{\alpha_{2x}, \alpha_{2x+1}\}$ belongs to G_i, and from (1) the typical edge of G_i (other than $\{2n, i\}$) has form $\{j, 2i-j\}$, so

$$\alpha_{2x+1} \equiv 2i - \alpha_{2x}, \tag{4}$$

and similarly

$$\alpha_{2x} \equiv 2k - \alpha_{2x-1}, \tag{5}$$

provided α_{2x} is not $2n$ and α_{2x-1} is not i or k. So

$$\alpha_{2x+1} \equiv 2(i-k) + \alpha_{2x-1}$$

$$\equiv 2x(i-k)+i,$$

provided $1 \leqq x \leqq m-1$. In particular

(6) $$\alpha_{2x+1} = k \text{ if and only if } (2x+1)(i-k) \equiv 0,$$

provided that $\alpha_t \neq i, k$ or $2n$ for $1 < t < 2x+1$. Since $x = m-1$ is to be the smallest positive solution of $\alpha_{2x+1} = k$, and $2x+1 = v_{ik}$ is the smallest positive solution of $(2x+1)(i-k) \equiv 0$, we have $2m = v_{ik}+1$, and the cycle (3) is of length $v_{ik}+1$.

Now consider any z not in the cycle (3). Suppose that the cycle containing z in $G_i \cup G_k$ if of length $2l$; call it

(7) $$\beta_0, \beta_1, \cdots, \beta_{2l},$$

where $z = \beta_0 = \beta_{2l}$. Without loss of generality we may assume $\{\beta_0, \beta_1\} \in G_i$ and $\{\beta_{2l-1}, \beta_{2l}\} \in G_k$. Analogously to (4) and (5) we obtain

$$\beta_{2x+1} \equiv 2i - \beta_{2x},$$

$$\beta_{2x} \equiv 2k - \beta_{2x-1},$$

and consequently

$$\beta_{2x+1} \equiv 2(x-y)(i-k) + \beta_{2y+1}.$$

Since none of i, k or $2n$ can occur in this cycle, we need place no restriction on this equation, provided the subscripts $2x+1$ and $2y+1$ are reduced modulo $2k$, so

(8) $$\beta_{2x+1} = \beta_{2y+1} \text{ if and only if } 2(x-y)(i-k) \equiv 0.$$

By definition $\beta_{2x+1} = \beta_{2y+1}$ if and only if $2l$ divides $(2x+1)-(2y+1)$, that is, if and only if l divides $x-y$. But $2(x-y)(i-k) \equiv 0$ if and only if v_{ik} divides $2(x-y)$, that is, if and only if v_{ik} divides $x-y$ (since v_{ik} is odd). So $l = v_{ik}$, and the cycle (8) has length $2v_{ik}$.

We have shown that $G_i \cup G_k$ has one cycle of length $v_{ik}+1$ and all other cycles of length $2v_{ik}$. Since G has $2n$ vertices, the number of cycles of length $2v_{ik}$ must be

$$\frac{2n - v_{ik} - 1}{2v_{ik}},$$

that is $\frac{1}{2}(d_{ik}-1)$.

Theorem 2. *When $n > 2$, $\mathscr{G}_{2n}$ cannot contain an $(n-1)$-division.*

Proof. An $(n-1)$-division would have two components of order n. Suppose $n > 2$, so that $n-1 \geqq 2$, and let G_i and G_k be two different factors in an $(n-1)$-division. The 2-division $\{G_i, G_k\}$ has one component of size $v_{ik}+1$ and $(d_{ij}-1)$ components of size $2v_{ik}$. So one of the components of the $(n-1)$-division must be a union of disjoint sets of size $2v_{ik}$. So v_{ik} divides n; since v_{ik} also divides $2n-1$

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we have $v_{ik} = 1$ and $d_{ik} = 2n-1$, which is impossible since $1 \leqq i,\ k \leqq 2n-1$ and $i \neq k$.

THEOREM 3. *If* $n \neq 5$, *no 2-division of* $\mathscr{G}_{2n}$ *has a component of order* $2n-4$.

PROOF. Consider the 2-division $\{G_i, G_k\}$ whose components have sizes $2v_{ik}$ and $v_{ik}+1$. Since v_{ik} divides the odd number $2n-1$ and as observed in the above proof $v_{ik} > 1$, $v_{ik} \geqq 3$. If $v_{ik}+1 = 2n-4$ we have $v_{ik} = 2n-5 \mid 2n-1$, $v_{ik} \mid 4$, which is a contradiction. If $2v_{ik} = 2n-4$ then $v_{ik} \mid (2n-4, 2n-1)$, so $v_{ik} \mid 3$; so $v_{ik} = 3$ and $n = 5$.

3. Proof of theorem 1

We shall exhibit:

(A) a one-factorization $\mathscr{H}_{2n}$ of K_{2n} which contains an $(n-1)$-division, for every even n;

(B) a one-factorization $\mathscr{L}_{2n}$ of K_{2n} which contains a 2-division with a component of order $2n-4$, for every odd n greater than 5;

(C) two non-isomorphic one-factorizations of K_{10}.

Theorem 2 together with (A) proves Theorem 1 for even n, Theorem 3 together with (B) proves Theorem 1 for odd n greater than 5, and (C) completes the proof.

PART (A). In this case n is even, so K_n is one-factorable. Label the vertices of K_{2n} as $1_1, 2_1, \cdots, n_1, 1_2, 2_2, \cdots, n_2$, and let $F_{\alpha,1}, F_{\alpha,2}, \cdots, F_{\alpha,n-1}$ be the factors in some one-factorization of the K_n with vertices $1_\alpha, 2_\alpha, \cdots, n_\alpha$.

Then write

$$\begin{aligned} H_i &= F_{1,i} \cup F_{2,i} \qquad & i &= 1, 2, \cdots, n-1 \\ H_i &= X_{i-n+1} \qquad & i &= n, n+1, \cdots, 2n-1 \end{aligned}$$

where X_i are as defined in (2). Write $\mathscr{H}_{2n} = \{H_1, H_2, \cdots, H_{2n-1}\}$. Then clearly $\mathscr{H}_{2n}$ is a one-factorization of K_{2n} and contains an $(n-1)$-division

$$\{H_1, H_2, \cdots, H_{n-1}\}.$$

PART (B). When n is odd, write $n = 2m+1$, and write the vertices of K_{4m+2} as $1_1, 2_1, \cdots, (2m+1)_1, 1_2, 2_2, \cdots, (2m+1)_2$. Write $G_{\alpha,1}, G_{\alpha,2}, \cdots, G_{\alpha,2m}$ for the factors in the one-factorization $\mathscr{G}_{2m+2}$ of the K_{m+2} with vertices $1_\alpha, 2_\alpha, \cdots, (2m+2)_\alpha$, as defined in (1), for $\alpha = 1, 2$; write $G^*_{\alpha,i}$ for $G_{\alpha,i}$ with $(i_\alpha, (2m+2)_\alpha)$ deleted; and write

$$L^*_i = G^*_{1,i} \cup G^*_{2,i} \cup \{(i_1, i_2)\}.$$

Now carry out the vertex-permutation defined by

$$\begin{aligned} (2m+2-i)_\alpha &\mapsto (2i)_\alpha \\ (i+1)_\alpha &\mapsto (2i+1)_\alpha \\ 1_\alpha &\mapsto 1_\alpha \end{aligned}$$

for $i = 1, 2, \cdots, m$ and $a = 1, 2$, writing L_i for the result of applying the permutation to L_i^*. Then $L_1, L_2, \cdots, L_{2m+1}$ are edge-disjoint one-factors of K_{4m+2}, and their union contains all the edges of the form (j_1, k_1) and (j_2, k_2) where $j \neq k$ and all the edges (j_1, j_2), but no edge of the form (j_1, k_2) with $j \neq k$. Now define

$$L_i = X_{i-2m}, \quad i = 2m+2,\ 2m+3, \cdots,\ 4m+1$$

where X_i are as defined in (2) with n replaced by $2m + 1$.

$$\begin{aligned} \mathscr{L}_{4m+2} &= \{L_1, L_2, \cdots, L_{4m+1}\} \text{ is a one-factorization of } K_{4m+2}. \text{ Now} \\ L_1 &= \{(1_1, 1_2), (2_1, 3_1), \cdots, ((2x)_1, (2x+1)_1), \cdots, ((2m)_1, (2m+1)_1), \\ &\qquad (2_2, 3_2), \cdots, ((2x)_2, (2x+1)_2), \cdots, ((2m)_2, (2m+1)_2)\}, \\ L_{2m+4} &= \{(1_1, (2m-1)_2), (2_1, (2m)_2), (3_1, (2m+1)_2), (4_1, 1_2), \cdots, ((2m+1)_1, \\ &\qquad (2m-2)_2)\}, \end{aligned}$$

and $L_1 \cup L_{2m+4}$ contains the cycle

$$1_1, 1_2, 4_1, 5_1, 2_2, 3_2, 6_1, 7_1, \cdots, (2m-1)_2, 1_1$$

of length $4m - 2$, that is $2n - 4$.

PART (C). Suitable 1-factorizations of K_{10} are G_{10}, which contains the 3-division $\{F_1, F_4, F_7\}$, and

$\{(1, 10),$	$(2, 3),$	$(4, 5),$	$(6, 7),$	$(8, 9)\},$
$\{(2, 10),$	$(1, 4),$	$(3, 9),$	$(5, 6),$	$(7, 8)\},$
$\{(3, 10),$	$(1, 8),$	$(2, 4),$	$(5, 7),$	$(6, 9)\},$
$\{(4, 10),$	$(1, 3),$	$(2, 6),$	$(5, 8),$	$(7, 9)\},$
$\{(5, 10),$	$(1, 9),$	$(2, 7),$	$(3, 8),$	$(4, 6)\},$
$\{(6, 10),$	$(1, 5),$	$(2, 9),$	$(3, 7),$	$(4, 8)\},$
$\{(7, 10),$	$(1, 2),$	$(3, 4),$	$(5, 9),$	$(6, 8)\},$
$\{(8, 10),$	$(1, 7),$	$(2, 5),$	$(3, 6),$	$(4, 9)\},$
$\{(9, 10),$	$(1, 6),$	$(2, 8),$	$(3, 5),$	$(4, 7)\},$

which contains no 3-division.

Reference

[1] F. Harary, *Graph Theory*, (Addison-Wesley, Reading, Mass., 1969).

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https://doi.org/10.1017/S1446788700014178 Published online by Cambridge University Press