PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 2, February 1994

ON OPEN THREE-MANIFOLDS OF QUASI-POSITIVE RICCI CURVATURE

SHUN-HUI ZHU

(Communicated by Christopher Croke)

ABSTRACT. It is proved that an open three-manifold of Ricci curvature nonnegative and positive at one point is diffeomorphic to the three-dimensional Euclidean space.

The Ricci curvature of a Riemannian manifold is said to be quasi-positive if it is nonnegative everywhere and strictly positive in any direction at (at least) one point. The purpose of this paper is to prove the following

Theorem. *A complete open three-manifold of quasi-positive Ricci curvature is diffeomorphic to* R^3.

We remark that the same statement as in the theorem for dimension four and above is not true, as shown by the examples of Sha and Yang [SY].

To put the theorem in perspective, we briefly review some earlier results. The question about the structure of quasi-positively curved manifolds was first raised by Cheeger and Gromoll [CG1]. The original question was about sectional curvature. Recall that Gromoll and Meyer [GM] proved that any complete open manifold of positive sectional curvature is diffeomorphic to R^n. Later, Cheeger and Gromoll [CG1] studied the structure of complete manifolds of nonnegative sectional curvature, obtaining the celebated Soul Theorem. In the same paper, they made the famous conjecture that any complete open manifold of quasi-positive sectional curvature is diffeomorphic to R^n. Many authors have studied this conjecture, and, to my knowledge, it is still open in general [Wa].

Similar problems about Ricci curvature were first studied by Schoen and Yau. They proved [ScY] that any open three-manifold of positive Ricci curvature is diffeomorphic to R^3. In the same paper, they conjectured that an open three-manifold of nonnegative Ricci curvature either is diffeomorphic to R^3 or its universal covering is isometric to $\Sigma \times R^1$ where Σ is a surface of nonnegative curvature.This conjecture was partially proved by Anderson and Rodriguez [AR] and Shi [Sh], with an additional requirement that the absolute value of sectional

Received by the editors July 8, 1991 and, in revised form, June 8, 1992.
1991 *Mathematics Subject Classification.* Primary 53C20.
Key words and phrases. Ricci curvature, three-manifolds.
Research supported by NSF grant 8610730.

©1994 American Mathematical Society
0002-9939/94 $1.00 + $.25 per page

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

curvature be bounded. Of course, our theorem is an immediate consequence of the conjecture of Schoen and Yau, which remains open.

We will carry out the proof of our theorem along the similar line as in [ScY]. The main change we have to make is to incorporate at appropriate places and in appropriate forms the following theorem of Aubin.

Theorem (Aubin). *Let* (M^n, g) *be a Riemannian manifold of nonnegative Ricci curvature and* $p_0 \in M$ *with* $\mathrm{Ric}(P_0) > 0$. *For any curve* γ *of finite length in* M *with* $\gamma(0) = P_0$, *we can deform the metric on* M *so that it has nonnegative Ricci curvature everywhere and positive Ricci curvature in a neighbourhood of* γ. *Furthermore, at points where* $\mathrm{Ric}(g) > 0$, *the Ricci curvature remains positive in the new metric.*

The above form of the theorem is not explicitly stated in [Au] but follows from the proof in [Au, pp. 398–399].

Remark. It follows from the above theorem that a compact manifold of quasi-positive Ricci curvature admits metric of strictly positive Ricci curvature. But the same does not follow for open manifolds.

Lemma. *Let* (M^3, g) *be an open three-manifold of quasi-positive Ricci curvature. Then* M *is contractible.*

Proof. To fix a notation, we fix a point $P_0 \in M$ with $\mathrm{Ric}(P_0) > 0$. Our strategy is to prove that $\pi_1(M) = \pi_2(M) = 0$. Then, since M is open and of dimension three, $H_k(M, Z) = 0$ for all $k \geq 3$. By the Hurewicz Theorem, we have $\pi_k(M) = 0$ for all $k \geq 1$. Hence, M is contractible by the Whitehead Theorem.

Let us first prove $\pi_2(M) = 0$. If $\pi_2(M) \neq 0$, then $\pi_2(\widetilde{M}) \neq 0$, where $\widetilde{M}$ is the universal covering space of M. The Sphere Theorem in three-dimensional topology [He] says that there exists an embedded S^2 in $\widetilde{M}$ which is not homotopically trivial. If $\widetilde{M} \backslash S^2$ were connected, we could take a loop in $\widetilde{M}$ intersecting S^2 at exactly one point. This loop could not be null-homotopic. This would contradict $\pi_1(\widetilde{M}) = 0$. Thus S^2 divides $\widetilde{M}$ into two connected components. By Van Kampen's theorem, each component is simply connected. If one of these were compact, then since S^2 is a trivial element in H_2 of the compact set, by the Hurewicz Theorem it is trivial in π_2. This is a contradiction. Therefore, S^2 divides $\widetilde{M}$ into two noncompact components. This implies the existence of a line, namely, a geodesic which is minimizing between any two of its points. Now the Cheeger-Gromoll Splitting Theorem [CG2] implies that $\widetilde{M}$ is a product of a line and a compact surface, $M = R^1 \times \Sigma$. Thus, for any point $x \in M$, $\mathrm{Ric}(\frac{\partial}{\partial t}) = 0$ where $\frac{\partial}{\partial t}$ generates the R^1 factor in $M = R^1 \times \Sigma$. This contradicts the assumption that $\mathrm{Ric}(P_0) > 0$. Hence $\pi_2(M) = 0$.

We next prove that $\pi_1(M) = 0$. Since $\dim(M) = 3$ and M is open, $H_k(M, Z) = 0$ for $k \geq 3$. By the Hurewicz Theorem, all higher homotopy groups of M vanish. Therefore, M is a $K(\pi, 1)$ space, and $H^i(\pi_1(M)) = H^i(M) = 0$ for $i \geq 3$. Since infinitely many cohomology groups of a finite cyclic group are nonzero, $\pi_1(M)$ is torsionfree.

Suppose $\pi_1(M) \neq 0$. By passing to a covering space, we can assume $\pi_1(M, P_0) = Z$. Let γ be a loop at P_0, generating the fundamental group. Let M_i be an exhaustion of M, i.e., $M_i \subset M_{i+1}$, $M = \bigcup M_i$, with ∂M_i a

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

disjoint union of smooth closed surfaces. Without loss of generality, we can assume $\gamma \subset M_i$ for any $i > 0$.

By the theorem of Aubin, we can deform the metric to g' so that g' is positively Ricci curved along γ, i.e., for any $x \in \gamma$, $\vec{v} \in T_x M$, we have $\mathrm{Ric}(\vec{v}) > 0$. By the Poincaré duality, for each i, we can find a compact orientable surface Σ_i, so that $\partial \Sigma_i \subset \partial M_i$, with $\Sigma \cap \gamma \neq \varnothing$. We then perturb the metric near ∂M_i so that it has positive mean curvature in the new metric. Thus there exists a minimal surface, minimizing among all surfaces in M_i which are homologous to Σ_i with the same boundary $\partial \Sigma_i$. We denote the minimal surface also by Σ_i. Note that $\Sigma_i \cap \gamma \neq \varnothing$.

Since Σ_i minimizes area in homology, it follows that the area of Σ_i inside any compact domain Ω of M has as uniform bound independent of i. The compactness and regularity theorems for minimal surfaces then imply that a subsequence of $\Sigma_i \cap \Omega$ converges to a properly embedded minimal surface Σ (with respect to g') with $\Sigma \cap \gamma \neq \varnothing$. Since Σ_i is area minimizing Σ is stable on each compact subset. Now by a theorem of Fischer-Colbrie and Schoen [F-CS], for any $x \in \Sigma$, we have $\mathrm{Ric}(\vec{n}(x)) = 0$ where $\vec{n}(x)$ is the unit vector normal to Σ at x. In particular, $\mathrm{Ric}(\vec{n}(x_0)) = 0$ for $x_0 \in \Sigma \cap \gamma$. This contradicts the fact that g' has positive Ricci curvature along γ. Therefore, $\pi_1(M, P_0) = 1$. Hence M is contractible. Q.E.D.

Proof of the theorem. By a theorem of Stallings [St], a contractible three-manifold is diffeomorphic to R^3 if and only of it is irreducible and simply connected at infinity.

We first show that M is simply connected at infinity. If not, we would have a compact connected set $K \subset M$, and a sequence of Jordan curves γ_i tending uniformly to infinity such that any disc D_i spanning γ_i has the property $D_i \cap K \neq \varnothing$. Without loss of generality, we can assume $B_{P_0}(\frac{1}{2}) \subset K$ where $B_P(r)$ is the metric ball of radius r around P. Since K is compact and connected, we can apply the above-quoted theorem of Aubin finitely many times to obtain a metric g' in M, such that $\mathrm{Ric}(g') \geq 0$ everywhere in M and $\mathrm{Ric}(g') > 0$ at every point in K. By the same procedure as in the proof of the lemma, we get a sequence of area minimizing discs D_i with $\partial D_i = \gamma_i$ and $D_i \cap K \neq \varnothing$. Note that, for any $x_i \in D_i \cap K$,

$$\mathrm{dist}(x_i, \partial D_i) = \mathrm{dist}(x_i, \gamma_i) \to \infty.$$

On the other hand, by a theorem of Schoen [Sc, Theorem 2],

$$\mathrm{dist}(x_i, \partial D_i) \leq \exp(C/K_0), \tag{1}$$

where $K_0 = \inf_{B_{x_i}(1/2)} \mathrm{Ric} \geq \inf_K \mathrm{Ric} > 0$ and C is a constant depending only on the metric in $B_{x_0}(1)$. Thus C is uniformly bounded since K is compact. Hence, the right-hand side of (1) is independent of i. It follows that $\mathrm{dist}(x_i, \partial D_i)$ is uniformly bounded. This is a contradiction. Thus M is simply connected at infinity.

We now prove that M is irreducible; i.e., any embedded two sphere in M bounds a three ball. We will only briefly outline the proof. For details, we refer to the work of Anderson and Rodriguez [AR].

If, on the contrary, M contains a fake cell, we will look for an embedded minimal two-sphere containing the fake cell. This will lead to a contradiction

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

since the existence of a minimal two-sphere splits M as $S^2 \times R^1$ [MSY, Theorem 6], which is obviously irreducible. For the existence, we take an exhaustion of $M = \bigcup M_i$ with M_i containing the fake cell and try to obtain a minimal two-sphere S_i in M surrounding the fake cell. The desired minimal S^2 will then be the limit of S_i when i goes to infinity. For this to work, we need to overcome two difficulties. The first is that we need M_i to be homogeneously regular in order for S_i to exist [MSY]. This can be achieved by blowing up the metric on M_i near the boundary ∂M_i (this is possible essentially because M_i is compact). The second difficulty is to guarantee that S_i does not everywhere go to infinity when $i \to \infty$. For this, we use the level set of Busemann function as a barrier, since the level set can be approximated by a surface of nonnegative mean curvature. We will omit the details of this argument which are carried out in [AR] and [ScY]. Thus M is irreducible. This completes the proof of the theorem. Q.E.D.

References

[AG] U. Abresch and D. Gromoll, *On complete manifolds with nonnegative Ricci curvature*, J. Amer. Math. Soc. **3** (1990), 355–374.

[AR] M. Anderson and L. Rodriguez, *Minimal surfaces and* 3*-manifolds of nonnegative Ricci curvature*, Math. Ann. **284** (1989), 461–475.

[Au] T. Aubin, *Métriques riemanninennes et courbure*, J. Differential Geom. **4** (1970), 383–424.

[CG1] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443.

[CG2] ——, *The splitting theorem for manifold of nonnegative Ricci curvature*, J. Differential Geom. **6** (1971), 119–129.

[F-CS] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in* 3*-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), 199–211.

[GM] D. Gromoll and W. Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. (2) **90** (1969), 75–90.

[He] J. Hempel, 3*-manifolds*, Ann. of Math. Stud., no. 86, Princeton Univ. Press, Princeton, NJ, 1976.

[MSY] W. Meeks, L. Simon, and S.-T. Yau, *Embedded minimal surfaces, exotic spheres and manifolds with positive Ricci curvature*, Ann. of Math. (2) **116** (1981), 621–659.

[Sc] R. Schoen, *Estimates for stable minimal surfaces in three manifolds*, Ann. of Math. Stud., no. 103, Princeton Univ. Press, Princeton, NJ, 1985, pp. 111–126.

[ScY] R. Schoen and S.-T. Yau, *Complete three dimensional manifolds with positive Ricci curvature and scalar curvature*, Ann. of Math. Stud., no. 102, Princeton Univ. Press, Princeton, NJ, 1982, pp. 209–228.

[Sh] W. Shi, *Complete noncompact* 3*-manifolds with nonnegative Ricci curvature*, J. Differential Geom. **29** (1989), 353–360.

[St] J. Stallings, *Group theory and three dimensional manifolds*, Yale Univ. Press, New Haven, CT, 1971.

[SY] J. Sha and D. Yang, *Examples of manifolds of positive Ricci curvature*, J. Differential Geom. **29** (1989), 95–103.

[Wa] G. Walschap, *Open manifolds of nonnegative curvature*, preprint.

Department of Mathematics, University of California at Los Angeles, Los Angeles, California 90024

Current address: Mathematical Sciences Research Institute, Berkeley, California 94720

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use