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# ON OPERATOR IDEALS DETERMINED BY SEQUENCES

MANUEL GONZÁLEZ AND ANTONIO MARTINÓN

We associate with an operator ideal $\mathcal{A}$ (in the sense of Pietsch) a class of bounded sequences $s_{\mathcal{A}}$ by using the $\mathcal{A}$-variation of Astala. If $\mathcal{A}$ and $\mathcal{B}$ are operator ideals, and we define $(\mathcal{A}, \mathcal{B})$ as the class of operators which map a sequence of $s_{\mathcal{A}}$ into a sequence of $s_{\mathcal{B}}$, we obtain the following:

**THEOREM.** *If $T_n: X \to Y$ is a sequence of operators and for every sequence $(x_n) \subset X$ in $s_{\mathcal{A}}$ there exists $p$ such that $(T_p x_n)$ belongs to $s_{\mathcal{B}}$, then $T_m \in (\mathcal{A}, \mathcal{B})$ for some $m$.*

The compact operators, weakly compact operators and some other operator ideals can be represented as $(\mathcal{A}, \mathcal{B})$. Hence several results of Tacon and other authors are a consequence of this theorem.

## 1. INTRODUCTION

Tacon [14], by using techniques of non-standard analysis, showed that if $T: X \to X$ is a (linear and continuous) operator on the Banach space $X$, then $T$ is power compact (that is, $T^m$ is compact for some $m \in \mathbb{N}$) if and only if for every bounded sequence $(x_n) \subset X$ there exists $p \in \mathbb{N}$ such that $(T^p x_n)$ is a relatively compact sequence. This result is obtained as corollary of the following theorem: if $T_n: X \to Y$ is a sequence of operators mapping a Banach space $X$ into a Banach space $Y$, and for every bounded sequence $(x_n) \subset X$ there exists $p \in \mathbb{N}$ such that $(T_p x_n)$ is relatively compact, then $T_m$ is compact for some $m \in \mathbb{N}$. He proved similar results for weakly compact operators.

Using standard techniques Barría [3] proved the above results in the case $X = Y$ a Hilbert space; Brown and Foias [5] in the case $X = Y$ a Banach space, and Buoni, Klein, Scott and Wadhwa [6, 7] proved the results of Tacon, and analogous results for the classes of completely continuous, weakly completely continuous and Rosenthal operators.

In this paper, using the $\mathcal{A}$-variation associated to an operator ideal $\mathcal{A}$, due to Astala [1], we introduce a class of operator ideals and, using analogous techniques to those in [5], we prove a theorem from which we derive the results of Tacon for all the previously considered classes [14, 3, 5, 6, 7], and for some other classes of operators,

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for example, the dual of the completely continuous operators, the $V$ operators, the $V^*$ operators, the Dunford-Pettis operators, the Grothendieck operators and certain class of operators $\mathcal{SC}^S$, related to the strictly cosingular operators $\mathcal{SC}$. We show also that the equality of the classes $\mathcal{SC}$ and $\mathcal{SC}^S$ is equivalent to the positive solution of an old problem in Banach space theory: does every infinite dimensional Banach space have a infinite dimensional separable quotient?

In the following $X$, $Y$, $Z$ will be Banach spaces; $X^*$ the dual space of $X$; $\mathcal{L}(X, Y)$ the class of all operators between $X$ and $Y$; $T^* \in \mathcal{L}(Y^*, X^*)$ the conjugate operator of $T \in \mathcal{L}(X, Y)$; $B_X$ the closed unit ball of $X$; $\ell_\infty(X)$ the space of all bounded sequences $(x_n)$ in $X$ attached with the norm $\|(x_n)\| := \sup\{\|x_n\| : n \in \mathbb{N}\}$. The range of the sequence $(x_n)$ will be denoted by $\{x_n\}$.

## 2. The main results

Let $\mathcal{A}$ be an operator ideal in the sense of Pietsch [13]. $\mathcal{A}(X, Y)$ will be the class of all operators of $\mathcal{L}(X, Y)$ belonging to $\mathcal{A}$. The operator ideal $\mathcal{A}$ is called closed if $\mathcal{A}(X, Y)$ is closed in $\mathcal{L}(X, Y)$, for every $X$, $Y$. $\mathcal{A}$ is called surjective if for any surjective operator $Q \in \mathcal{L}(Z, X)$, an operator $T \in \mathcal{L}(X, Y)$ belongs to $\mathcal{A}$ whenever $TQ \in \mathcal{A}(Z, Y)$; equivalently, if $S \in \mathcal{A}(Z, Y)$, $T \in \mathcal{L}(X, Y)$ and $TB_X \subset SB_Z$, then $T \in \mathcal{A}(X, Y)$. We use the notation $\mathcal{A}^\wedge$ for the smallest surjective closed operator ideal containing $\mathcal{A}$.

Definition 2.1: [1]. Let $D \subset X$ be bounded. Then the $\mathcal{A}$-variation of $D$ is defined by

$$h_\mathcal{A}(D) := \inf\{\varepsilon > 0 : \exists Z, \exists K \in \mathcal{A}(Z, X), D \subset KB_Z + \varepsilon B_X\}.$$

It is verified [1] that $h_{\mathcal{A}^\wedge} = h_\mathcal{A}$ and moreover

$$h_\mathcal{A}(D) = 0 \Leftrightarrow \exists Z, \exists K \in \mathcal{A}^\wedge(Z, X), D \subset KB_Z.$$

If we denote $\overline{aco}D$ the closed absolutely convex hull of $D$, then $h_\mathcal{A}(\overline{aco}\, D) = h_\mathcal{A}(D)$.

Using $h_\mathcal{A}$ we can define the following subspaces of $\ell_\infty(X)$.

Definition 2.2: [10, 11]. $s_\mathcal{A}(X) := \{(x_n) \in \ell_\infty(X) : h_\mathcal{A}(\{x_n\}) = 0\}$.

$s_\mathcal{A}(X)$ is a closed subspace of $\ell_\infty(X)$; moreover, for every $T \in \mathcal{L}(X, Y)$ we have that $(x_n) \in s_\mathcal{A}(X) \Rightarrow (Tx_n) \in s_\mathcal{A}(Y)$ [10, 11].

Note that the class $\mathcal{L}$ of all operators is a closed surjective operator ideal: $\mathcal{L} = \mathcal{L}^\wedge$. Moreover, $h_\mathcal{L} = 0$ and $s_\mathcal{L} = \ell_\infty$.

Definition 2.3: Let $\mathcal{A}$, $\mathcal{B}$ be operator ideals. We define the class $(\mathcal{A}, \mathcal{B})$ in the following way:

$$(\mathcal{A}, \mathcal{B})(X, Y) := \{T \in \mathcal{L}(X, Y) : (x_n) \in s_\mathcal{A}(X) \Rightarrow (Tx_n) \in s_\mathcal{B}(Y)\}.$$

It is immediate to show that $(\mathcal{A}, \mathcal{B})$ is an operator ideal.

**Lemma 2.4.** *Let $\mathcal{A}$, $\mathcal{B}$ be operator ideals and $T \in \mathcal{L}(X, Y)$. Then*

$$C := \{(x_n) \in s_{\mathcal{A}}(X) : (Tx_n) \notin s_{\mathcal{B}}(Y)\}$$

*is either empty, when $T \in (\mathcal{A}, \mathcal{B})$, or open dense in $s_{\mathcal{A}}(X)$, when $T \notin (\mathcal{A}, \mathcal{B})$.*

Proof: If $C$ is not empty, then its complement in $s_{\mathcal{A}}(X)$ is a proper closed subspace of $s_{\mathcal{A}}(X)$. Consequently, $C$ is open dense in $s_{\mathcal{A}}(X)$. ∎

**Theorem 2.5.** *Let $\mathcal{A}$, $\mathcal{B}$ be operator ideals. For a sequence of operators $(T_n) \subset \mathcal{L}(X, Y)$, the following assertions are equivalent:*

(1) *For every $(x_n) \in s_{\mathcal{A}}(X)$, there is $p \in \mathbb{N}$ such that $(T_p x_n) \in s_{\mathcal{B}}(Y)$.*
(2) *There exists $m \in \mathbb{N}$ such that $T_m \in (\mathcal{A}, \mathcal{B})$.*

Proof: Suppose (1). If $(x_n) \in s_{\mathcal{A}}(X)$, then there exists $p \in \mathbb{N}$ such that $(T_p x_n) \in s_{\mathcal{B}}(Y)$, hence

$$(x_n) \notin \{(x_n) \in s_{\mathcal{A}}(X) : (T_p x_n) \notin s_{\mathcal{B}}(Y)\}.$$

Since $(x_n)$ is arbitrary, it follows that

$$\bigcap_{p \in \mathbb{N}} \{(x_n) \in s_{\mathcal{A}}(X) : (T_p x_n) \notin s_{\mathcal{B}}(Y)\} = \emptyset.$$

If neither of the operators $T_p$ belongs to $(\mathcal{A}, \mathcal{B})$, by the lemma we would have a sequence of open dense subsets with empty intersection in the Banach space $s_{\mathcal{A}}(X)$, in contradiction with the Baire Category Theorem; hence there exists $m \in \mathbb{N}$ such that $T_m \in (\mathcal{A}, \mathcal{B})$. The other implication is obvious. ∎

Remark 2.6. The above theorem holds, with the same proof, for a sequence $(T_n)$ of operators such that $T_n \in \mathcal{L}(X, Y_n)$.

**Corollary 2.7.** *Let $\mathcal{A}$, $\mathcal{B}$ be operator ideals, and $T \in \mathcal{L}(X, X)$. There exists $m \in \mathbb{N}$ such that $T^m \in (\mathcal{A}, \mathcal{B})$ if and only if for every $(x_n) \in s_{\mathcal{A}}(X)$, there is $p \in \mathbb{N}$ such that $(T^p x_n) \in s_{\mathcal{B}}(X)$.*

Given two operator ideals $\mathcal{A}$, $\mathcal{B}$, the left quotient $\mathcal{A}^{-1} \circ \mathcal{B}$ is an operator ideal defined in the following way [**13**, 3.2.1]

$$(\mathcal{A}^{-1} \circ \mathcal{B})(X, Y) := \{T \in \mathcal{L}(X, Y) : \forall Z, \forall S \in \mathcal{A}(Y, Z), ST \in \mathcal{B}\}.$$

It is easy to verify that if $\mathcal{B}$ is closed, then $\mathcal{A}^{-1} \circ \mathcal{B}$ is closed; if $\mathcal{B}$ is surjective, then $\mathcal{A}^{-1} \circ \mathcal{B}$ is surjective.

Now we show that the operator ideals of the type $\mathcal{A}^{-1} \circ (\mathcal{B}, \mathcal{C})$ can be written in the form $(\mathcal{B}, \mathcal{A}^{-1} \circ \mathcal{C})$. Consequently, Theorem 2.5 can be applied to these operator ideals.

**Lemma 2.8.** *Let $\mathcal{C}$ be a closed surjective operator ideal, $(x_n) \in \ell_\infty(X)$, and $K\colon \ell_1 \to X$ defined by $Ke_n := x_n$, where $(e_n)$ is the canonical basis of $\ell_1$. Then*

$$(x_n) \in s_{\mathcal{C}}(X) \text{ if and only if } K \in \mathcal{C}.$$

Proof: Clearly $K$ is a linear and continuous operator.

If $K \in \mathcal{C}$, from $\{x_n\} = \{Ke_n\} \subset KB_{\ell_1}$ and $K \in \mathcal{C} = \mathcal{C}^\wedge$ we obtain $(x_n) \in s_{\mathcal{C}}$.

Conversely, if $(x_n) \in s_{\mathcal{C}}(X)$ we have $h_{\mathcal{C}}(\{x_n\}) = 0$, and consequently $h_{\mathcal{C}}(\overline{aco}\{x_n\}) = 0$. Now, since

$$KB_{\ell_1} = K\overline{aco}\{e_n\} \subset \overline{aco}\{Ke_n\} = \overline{aco}\{x_n\},$$

we conclude that $K \in \mathcal{C}$. ∎

**Lemma 2.9.** *Let $\mathcal{A}$ be an operator ideal and $\mathcal{C}$ a closed surjective operator ideal. If we denote by $\mathcal{A}^{-1} \circ s_{\mathcal{C}}$ the class defined in following way*

$$(\mathcal{A}^{-1} \circ s_{\mathcal{C}})(X) := \{(x_n) \in \ell_\infty(X) : \forall Y, \forall A \in \mathcal{A}(X, Y), (Ax_n) \in s_{\mathcal{C}}(Y)\},$$

*then we have that $s_{\mathcal{A}^{-1}\circ\mathcal{C}} = \mathcal{A}^{-1} \circ s_{\mathcal{C}}$.*

Proof: Suppose $(x_n) \in (\mathcal{A}^{-1} \circ s_{\mathcal{C}})(X)$. We consider the operator $K\colon \ell_1 \to X$, defined by $Ke_n := x_n$. For every operator $A \in \mathcal{A}(X, Y)$ we have that $(Ax_n) \in s_{\mathcal{C}}(Y)$. Using Lemma 2.8 we find that the operator $AK$ belongs to $\mathcal{C}(\ell_1, Y)$, and consequently $K \in (\mathcal{A}^{-1} \circ \mathcal{C})(\ell_1, X)$. Hence $(x_n) \in s_{\mathcal{A}^{-1}\circ\mathcal{C}}$.

Conversely, suppose $(x_n) \in s_{\mathcal{A}^{-1}\circ\mathcal{C}}(X)$. For some Banach space $Z$ and operator $K \in (\mathcal{A}^{-1} \circ \mathcal{C})(Z, X)$ we have that $\{x_n\} \subset KB_Z$. If $A \in \mathcal{A}(X, Y)$, then $\{Ax_n\} \subset AKB_Z$ and $AK \in \mathcal{C}$. Hence $(Ax_n) \in s_{\mathcal{C}}$, for every $A \in \mathcal{A}$, and consequently $(x_n) \in (\mathcal{A}^{-1} \circ s_{\mathcal{C}})(X)$. ∎

**Proposition 2.10.** *Let $\mathcal{A}$, $\mathcal{B}$ be operator ideals and $\mathcal{C}$ a closed surjective operator ideal. Then $\mathcal{A}^{-1} \circ (\mathcal{B}, \mathcal{C}) = (\mathcal{B}, \mathcal{A}^{-1} \circ \mathcal{C})$.*

Proof: The following chain of equivalences holds:

$$\begin{aligned}
&T \in [\mathcal{A}^{-1} \circ (\mathcal{B}, \mathcal{C})](X, Y)\\
&\Leftrightarrow \forall Z, \forall A \in \mathcal{A}(Y, Z),\ AT \in (\mathcal{B}, \mathcal{C})(X, Z)\\
&\Leftrightarrow \forall Z, \forall A \in \mathcal{A}(Y, Z),\ (x_n) \in s_{\mathcal{B}}(X) \Rightarrow (ATx_n) \in s_{\mathcal{C}}(Z)\\
&\Leftrightarrow (x_n) \in s_{\mathcal{B}}(X) \Rightarrow (Tx_n) \in (\mathcal{A}^{-1} \circ s_{\mathcal{C}})(Z) = s_{\mathcal{A}^{-1}\circ\mathcal{C}}(Z)\\
&\Leftrightarrow T \in (\mathcal{B}, \mathcal{A}^{-1} \circ \mathcal{C})(X, Y)
\end{aligned}$$

∎

## 3. Examples

We show several classical operator ideals which can be represented as $(\mathcal{A}, \mathcal{B})$ for suitable operator ideals $\mathcal{A}$ and $\mathcal{B}$.

**3.1 The compact operators $\mathcal{C}o$.** We use the following well-known characterisation of $\mathcal{C}o$:

$$T \in \mathcal{C}o \Leftrightarrow ((x_n) \text{ bounded } \Rightarrow \{Tx_n\} \text{ relatively compact})$$

$\mathcal{C}o$ is a closed surjective operator ideal: $\mathcal{C}o = \mathcal{C}o^{\wedge}$ [**13**, 1.4.2, 4.2.5, 4.7.12].

The $\mathcal{C}o$-variation $h_{\mathcal{C}o}$ agree with the Hausdorff measure of noncompactness [**1**, **2**]. Hence $s_{\mathcal{C}o}$ is the class of all bounded sequences with relatively compact range; that is, every subsequence has a convergent subsequence. Hence

$$T \in \mathcal{C}o \Leftrightarrow ((x_n) \in s_{\mathcal{L}}(X) \Rightarrow (Tx_n) \in s_{\mathcal{C}o})$$

and $\mathcal{C}o = (\mathcal{L}, \mathcal{C}o)$.

**3.2 The weakly compact operators $\mathcal{WC}o$.**

$T \in \mathcal{WC}o \Leftrightarrow ((x_n)$ bounded $\Rightarrow \{Tx_n\}$ weakly relatively compact) [**13**, 1.5.2]. $\mathcal{WC}o$ is a closed surjective operator ideal: $\mathcal{WC}o = \mathcal{WC}o^{\wedge}$ [**13**, 1.4.2, 4.2.5, 4.7.12].

The $\mathcal{WC}o$-variation $h_{\mathcal{WC}o}$ agrees with the De Blasi measure of weak noncompactness [**1**, **8**]. Hence $s_{\mathcal{WC}o}$ is the class of all bounded sequences with relatively weakly compact range; that is, every subsequence has a weakly convergent subsequence. Hence

$$T \in \mathcal{WC}o \Leftrightarrow ((x_n) \in s_{\mathcal{L}} \Rightarrow \{Tx_n\} \in s_{\mathcal{WC}o})$$

and $\mathcal{WC}o = (\mathcal{L}, \mathcal{WC}o)$.

**3.3 The completely continuous operators $\mathcal{CC}$.**

$T \in \mathcal{CC} \Leftrightarrow ((x_n)$ weakly convergent $\Rightarrow (Tx_n)$ convergent) [**13**, 1.6.1]. Consequently we obtain that

$$T \in \mathcal{CC} \Leftrightarrow ((x_n) \in s_{\mathcal{WC}o} \Rightarrow (Tx_n) \in s_{\mathcal{C}o})$$

and $\mathcal{CC} = (\mathcal{WC}o, \mathcal{C}o)$.

We note that $\mathcal{CC}$ is a closed operator ideal [**13**, 1.6.2, 4.2.5], but it is not surjective: $\mathcal{CC}^{\wedge} = \mathcal{L}$ [**13**, 4.7.13]. Hence $h_{\mathcal{CC}} = h_{\mathcal{L}} = 0$ and $s_{\mathcal{CC}} = s_{\mathcal{L}} = \ell_\infty$.

The completely continuous operators $\mathcal{CC}$ can also be characterised in the following way [**13**, 1.6.3]:

$$T \in \mathcal{CC} \Leftrightarrow ((x_n) \text{ weakly Cauchy } \Rightarrow (Tx_n) \text{ convergent}).$$

**3.4 The Rosenthal operators $\mathcal{R}o$.**

$T \in \mathcal{R}o \Leftrightarrow ((x_n)$ bounded $\Rightarrow (Tx_n)$ has a weakly Cauchy subsequence) [**13**, 3.2.4]. We have that $\mathcal{R}o = \mathcal{CC}^{-1} \circ \mathcal{C}o$ [**13**, 3.2.4], hence $\mathcal{R}o$ is a closed surjective operator ideal: $\mathcal{R}o = \mathcal{R}o^\wedge$. Using Proposition 2.10 we obtain

$$\mathcal{R}o = \mathcal{CC}^{-1} \circ \mathcal{C}o = \mathcal{CC}^{-1} \circ (\mathcal{L}, \mathcal{C}o) = (\mathcal{L}, \mathcal{CC}^{-1} \circ \mathcal{C}o) = (\mathcal{L}, \mathcal{R}o).$$

By Lemma 2.9, it follows that

$$s_{\mathcal{R}o}(X) := \{(x_n) \in \ell_\infty(X) : \forall Y, \forall A \in \mathcal{CC}(X, Y), \{Ax_n\} \text{ relatively compact}\}.$$

Then $s_{\mathcal{R}o}$ is the class of all bounded sequences such that every subsequence has a weakly Cauchy subsequence. In fact, if $(x_n) \in s_{\mathcal{R}o}(X)$ and $(y_n)$ is a subsequence of $(x_n)$, for every $f \in X^* \subset \mathcal{CC}$, there exists a subsequence $(z_n)$ of $(y_n)$ such that $(fz_n)$ is convergent, hence $(z_n)$ is weakly Cauchy; conversely, if $(x_n)$ is a sequence such that every subsequence $(y_n)$ has a weakly Cauchy subsequence $(z_n)$ and if $A \in \mathcal{CC}(X, Y)$, then $(Az_n)$ is convergent, hence $\{Ax_n\}$ is relatively compact.

From the second characterisation of $\mathcal{CC}$ we obtain

$$T \in \mathcal{CC} \Leftrightarrow ((x_n) \in s_{\mathcal{R}o} \Rightarrow (Tx_n) \in s_{\mathcal{C}o}),$$

hence $\mathcal{CC} = (\mathcal{R}o, \mathcal{C}o)$.

**Proposition 3.5.** *If $\mathcal{CC}^* := \{T : T^* \in \mathcal{CC}\}$, then $\mathcal{WC}o^{-1} \circ \mathcal{C}o = \mathcal{CC}^*$.*

Proof: Suppose $T \in \mathcal{CC}^*(X, Y)$ and $A \in \mathcal{WC}o(Y, Z)$. If $(x_n) \in \ell_\infty(Z^*)$, then $(A^*x_n) \in s_{\mathcal{WC}o}(Y^*)$, hence $(T^*A^*x_n) \in s_{\mathcal{C}o}(X^*)$. Consequently $T^*A^* \in (\mathcal{L}, \mathcal{C}o) = \mathcal{C}o$, hence $AT \in \mathcal{C}o$ and $T \in \mathcal{WC}o^{-1} \circ \mathcal{C}o$.

Conversely, suppose $T \in \mathcal{L}(X, Y)$ and $T \notin \mathcal{CC}^*$. There exists a sequence $(g_n) \subset Y^*$ such that the weak limit of $(g_n)$ is 0, but $(T^*g_n)$ does not converge to 0 in the norm topology. We define an operator

$$A : Y \to c_0, \quad Ay := (g_n(y)).$$

$A$ is weakly compact, because $A^*e_n^* = g_n$ and $(g_n)$ is weakly null [**9**, VII, Exercise 4(i)], where $(e_n^*)$ is the canonical basis of $c_0^*$. Consequently,

$$AT : X \to c_0, AT x = (g_n(Tx)),$$

but $g_n(Tx) = T^*g_n(x)$ and the sequence of general term

$$(AT)^* e_n^* = T^*A^*e_n^* = T^*g_n$$

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is not norm null in $X^*$, hence $AT \notin \mathcal{C}o$ [**9**, VII, Exercise 4(ii)]. ∎

**3.6 The dual of the completely continuous operators $\mathcal{CC}^*$.** From Propositions 2.10 and 3.5 we find that $\mathcal{CC}^*$ is a closed surjective operator ideal, $\mathcal{CC}^{*\wedge} = \mathcal{CC}^*$, and

$$\mathcal{CC}^* = \mathcal{WC}o^{-1} \circ (\mathcal{L}, \mathcal{C}o) = \left(\mathcal{L}, \mathcal{WC}o^{-1} \circ \mathcal{C}o\right) = (\mathcal{L}, \mathcal{CC}^*).$$

Moreover, from Lemma 2.9, we have that

$$s_{\mathcal{CC}^*}(X) = \{(x_n) \in \ell_\infty(X) : \forall Y, \forall A \in \mathcal{WC}o(X, Y), \{Ax_n\} \text{ relatively compact}\}.$$

**3.7 The $V$ operators $\mathcal{V}$.** Pelczynski introduced in [**12**] the class of Banach spaces with property $(V)$ as those spaces $X$ such that every unconditionally converging operator from $X$ into any Banach space is weakly compact. This property suggests that we consider the following class of operators $\mathcal{V}$:

$$T \in \mathcal{V}(X, Y) \Leftrightarrow [\forall Z, \forall S \in \mathcal{U}c(Y, Z), ST \in \mathcal{WC}o(X, Z)].$$

where $\mathcal{U}c$ is the operator ideal of all unconditionally convergent operators (that is: $\sum_1^\infty x_n$ weakly unconditionally Cauchy implies $\sum_1^\infty Tx_n$ unconditionally convergent). Clearly $X$ has property $(V)$ if and only if $I_X \in \mathcal{V}$.

We have $\mathcal{V} = \mathcal{U}c^{-1} \circ \mathcal{WC}o$. Consequently $\mathcal{V}$ is a closed surjective operator ideal and from Proposition 2.10 and Lemma 2.9 we obtain $\mathcal{V} = (\mathcal{L}, \mathcal{V})$ and

$$s_{\mathcal{V}}(X) := \{(x_n) \in \ell_\infty(X) : \forall Y, \forall A \in \mathcal{U}c(X, Y), \{Ax_n\} \text{ relatively weakly compact}\}.$$

**3.8 The $V^*$ operators $\mathcal{V}^*$.** Analogously as in the last example, we consider the following class of operators $\mathcal{V}^*$:

$$T \in \mathcal{V}^*(X, Y) \Leftrightarrow (\forall S \in \mathcal{L}(Y, \ell_1), ST \in \mathcal{C}o(X, \ell_1)).$$

$X$ has property $(V^*)$ ([**12**]; see also [4]) if and only if $I_X \in \mathcal{V}$.

If $op\{\ell_1\}$ is the operator ideal of all operators which factorise in $\ell_1$, then we have

$$\mathcal{V}^* = [op\{\ell_1\}]^{-1} \circ \mathcal{C}o$$

and consequently $\mathcal{V}^*$ is a closed surjective operator ideal: $\mathcal{V}^{*\wedge} = \mathcal{V}^*$.

From Proposition 2.10, we obtain $\mathcal{V}^* = (\mathcal{L}, \mathcal{V}^*)$. Also,

$$s_{\mathcal{V}^*}(X) = \{(x_n) \in \ell_\infty(X) : \forall S \in \mathcal{L}(X, \ell_1), \{Sx_n\} \text{ relatively compact}\}.$$

In fact, if $(x_n) \in s_{\mathcal{V}^*}(X)$, then $h_{\mathcal{V}^*}(\{x_n\}) = 0$, and hence there exists $K \in \mathcal{V}^*(Z, X)$ such that $\{x_n\} \subset KB_Z$. For every $S \in \mathcal{L}(X, \ell_1)$ we obtain $\{Sx_n\} \subset SKB_Z$ with $SK \in \mathcal{C}o$; hence $SKB_Z$ is relatively compact, and consequently $\{Sx_n\}$ is relatively compact. Conversely, let $(x_n) \in \ell_\infty(X)$ such that for every $S \in \mathcal{L}(X, \ell_1)$ the sequence $(Sx_n)$ is relatively compact. We consider the operator

$$K : \ell_1 \to X, \quad Ke_n := x_n,$$

where $(e_n)$ is the canonical basis of $\ell_1$. From Lemma 2.8 we obtain $SK \in \mathcal{C}o(\ell_1, \ell_1)$; consequently $K \in \mathcal{V}^*$, and $(x_n) \in s_{\mathcal{V}^*}(X)$.

**3.9 The Dunford-Pettis operators $\mathcal{DP}$.** We consider the class $\mathcal{DP}$ defined in the following way [**13**, 3.2.5]:

$$T \in \mathcal{DP}(X, Y) \Leftrightarrow ((x_n) \subset X,\, w - \lim x_n = 0,\, (b_n) \subset Y^*,$$
$$w\text{-}\lim b_n = 0 \Rightarrow \lim b_n(Tx_n) = 0).$$

We note that $\mathcal{DP} = \mathcal{WC}o^{-1} \circ \mathcal{CC}$ [**13**, 3.2.5] and consequently $\mathcal{DP}$ is a closed operator, but $\mathcal{CC} \subset \mathcal{DP}$, hence $\mathcal{CC}^\wedge = \mathcal{DP}^\wedge = \mathcal{L}$, $h_{\mathcal{DP}} = h_{\mathcal{L}} = 0$ and $s_{\mathcal{DP}} = s_{\mathcal{L}} = \ell_\infty$. Moreover, by using Proposition 2.10, $\mathcal{DP} = (\mathcal{L}, \mathcal{DP})$.

**3.10 The Grothendieck operators $\mathcal{G}r$.**

$$T \in \mathcal{G}r \Leftrightarrow ((b_n) \subset Y^*,\, w^*\text{-}\lim b_n = 0 \Rightarrow w\text{-}\lim T^* b_n = 0) \quad [\mathbf{13}, 3.2.6].$$

If $\mathcal{S}$ is the operator ideal of all operators with separable range, then $\mathcal{G}r = \mathcal{S}^{-1} \circ \mathcal{WC}o$ [**13**, 3.2.6]. Hence $\mathcal{G}r$ is a closed surjective operator ideal: $\mathcal{G}r = \mathcal{G}r^\wedge$. Moreover, from Proposition 2.10 we obtain $\mathcal{G}r = (\mathcal{L}, \mathcal{G}r)$.

**3.11 The weakly completely continuous operators $\mathcal{WCC}$.** The class $\mathcal{WCC}$ is defined as follows:

$$T \in \mathcal{WCC} \Leftrightarrow ((x_n) \text{ weakly Cauchy } \Rightarrow (Tx_n) \text{ weakly convergent}).$$

$X$ is weakly sequentially complete if and only if $I_X \in \mathcal{WCC}$. It is immediate to prove that

$$T \in \mathcal{WCC} \Leftrightarrow ((x_n) \in s_{\mathcal{R}o} \Rightarrow (Tx_n) \in s_{\mathcal{WC}o});$$

hence $\mathcal{WCC} = (\mathcal{R}o, \mathcal{WC}o)$.

We note that $\mathcal{WCC}$ is an operator ideal, but it is not surjective. As $\mathcal{CC} \subset \mathcal{WCC}$ we have $\mathcal{CC}^\wedge = \mathcal{WCC}^\wedge = \mathcal{L}$. Hence $h_{\mathcal{CC}} = h_{\mathcal{L}} = 0$ and $s_{\mathcal{WCC}} = s_{\mathcal{L}} = \ell_\infty$.

**3.12 The strictly cosingular operators $\mathcal{SC}$.** The class $\mathcal{SC}$ is defined as follows [**13**, 1.10.2]:

$T \in \mathcal{SC}(X, Y) \Leftrightarrow$ (for every quotient $Y/U$, $Q_U T$ surjection $\Rightarrow \dim(Y/U) < \infty$) where $Q_U: Y \to Y/U$ is the quotient map. $\mathcal{SC}$ is a closed surjective operator ideal: $\mathcal{SC} = \mathcal{SC}^\wedge$ [**13**, 1.10.4, 4.2.7, 4.7.14].

We shall denote by $\mathcal{SC}^s$ the operator ideal $(\mathcal{L}, \mathcal{SC})$. Clearly $\mathcal{SC}^s$ contains $\mathcal{SC}$, and we do not know if they coincide. However we can show that the equality is equivalent to an old open problem in Banach space theory.

**PROPOSITION 3.13.** *$\mathcal{SC} = \mathcal{SC}^s$ if and only if every infinite dimensional Banach space has an infinite dimensional separable quotient.*

PROOF: We recall that $\mathcal{SC}$ is the greatest surjective operator ideal $\mathcal{A}$ that verifies [**13**, 4.7.14]:

$$I_X \in \mathcal{A} \Rightarrow \dim(X) < \infty,$$

where $I_X$ is the identity operator on $X$. Consequently,

$$\mathcal{SC}^s = \mathcal{SC} \Leftrightarrow (I_X \in \mathcal{SC}^s \Rightarrow \dim(X) < \infty),$$

or equivalently

$$\ell_\infty(X) = s_{SC}(X) \Rightarrow \dim(X) < \infty.$$

From this, it is enough to prove that $I_X \in \mathcal{SC}^s$ if and only if $X$ has no infinite dimensional separable quotients.

First, supposing that $X$ has a separable quotient of infinite dimension $X/U$, we will show that $\ell_\infty(X) \neq s_{SC}(X)$.

Let $Q_U: X \to X/U$ denote the quotient map, and let $(y_n)$ be a sequence dense in $B_{X/U}$. We take $(x_n) \subset 2B_X$ such that $Q_U x_n = y_n$. If $T \in \mathcal{L}(Z, X)$ is an operator such that $\{x_n\} \subset TB_Z$, we have that $Q_U T$ is surjective [**15**, IV.5.4]; hence $T \in \mathcal{SC}$ and consequently $(x_n) \notin s_{SC}(X)$.

Now we prove the converse. If there is a sequence $(x_n) \in \ell_\infty(X) \setminus s_{SC}(X)$, then the operator $T: \ell_1 \to X$ defined by $Te_n := x_n$ is not strictly cosingular; hence there exists an infinite dimensional quotient $X/U$ such that $Q_U T$ is surjective. $X/U$ is a quotient of $\ell_1$, hence it is separable; and we conclude that $X$ has a separable quotient of infinite dimension. ∎

Finally we give sufficient conditions assuring that $\mathcal{SC}^s(X, Y) = \mathcal{SC}(X, Y)$. In these cases we can apply our theorem for strictly cosingular operators.

**PROPOSITION 3.14.**

(1) *If every infinite dimensional quotient of $Y$ has a infinite dimensional separable quotient, then $\mathcal{SC}^s(X, Y) = \mathcal{SC}(X, Y)$ for every $X$.*

(2) *If every infinite dimensional quotient of $X$ has a infinite dimensional separable quotient, then $\mathcal{SC}^s(X, Y) = \mathcal{SC}(X, Y)$ for every $Y$.*

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PROOF: Suppose $T \notin SC(X, Y)$; then there exists an infinite dimensional quotient of $Y$, $Y/U$, such that $Q_U T$ is surjective; moreover, $Y/U$ has a infinite dimensional separable quotient, $Y/V$ with $U \subset V$. We take a dense sequence $(y_n)$ in $B_{Y/V}$ and a bounded sequence $(x_n)$ in $X$ such that $Q_V T x_n = y_n$. As in the proof of the above proposition, we can prove that $(y_n) \notin s_{SC}(Y/V)$; then $(Tx_n) \notin s_{SC}(Y)$; hence $T \notin SC^s(X, Y)$.

(2) The proof is the same, noting that $Y/U$ is isomorphic to $X/T^{-1}U$, because $Q_U T$ is surjective. ∎

REMARK 3.15. As a consequence of the above descriptions we conclude that the results of Theorem 2.5 and Corollary 2.7 apply to the operator ideals considered in Examples 3.1–3.4 and 3.6–3.11.

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Departamento de Matemáticas
Universidad de Cantabria
39071 Santander
Spain

Departamento de Análisis Matemático
Universidad de La Laguna
38271 La Laguna (Tenerife)
Spain