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ON OPERATOR-VALUED FOURIER MULTIPLIER THEOREMS

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Abstract. The classical Fourier multiplier theorems of Marcinkiewicz and Mikhlin are extended to vector-valued functions and operator-valued multiplier functions on $\mathbb{Z}^d$ or $\mathbb{R}^d$ which satisfy certain R-boundedness conditions.

1. Introduction and motivation

Let X and Y be real (or complex) Banach spaces and $\mathscr{B}(X,Y)$ the Banach space of bounded linear operators $T : X \to Y$, endowed with the usual operator norm. By $\mathscr{S}(\mathbb{R}^d;X)$ we denote the Schwartz space of rapidly decreasing functions from $\mathbb{R}^d$ to X and by $^\wedge$, $^\vee$ we denote the Fourier transform and the inverse Fourier transform. For $1 \le p < \infty$ let $L^p(\mathbb{T}^d;X)$ and $L^p(\mathbb{R}^d;X)$ be the usual Bochner spaces of p-integrable X-valued functions on the d-dimensional circle $\mathbb{T}^d$ and $\mathbb{R}^d$ respectively.

In the first part of this article we are interested in obtaining Fourier multiplier theorems on $L^p(\mathbb{T}^d;X)$ in the following sense.

For $z \in \mathbb{T}^d$ set $\mathfrak{e}_\mathfrak{x}(z) = z^\mathfrak{x}$, $\mathfrak{x} \in \mathbb{Z}^d$. We say that a function $M : \mathbb{Z}^d \to \mathscr{B}(X,Y)$ is a **Fourier multiplier** on $L^p(\mathbb{T}^d;X)$ if the operator

$$f = \sum_{\mathfrak{x}\in\mathbb{Z}^d} \hat{f}(\mathfrak{x}) \otimes \mathfrak{e}_\mathfrak{x} \;\longmapsto\; \mathscr{K}_M f = \sum_{\mathfrak{x}\in\mathbb{Z}^d} M(\mathfrak{x})\hat{f}(\mathfrak{x}) \otimes \mathfrak{e}_\mathfrak{x}\,, \tag{1.1}$$

first defined for f with a finitely valued Fourier transformation $\hat{f}$, extends uniquely to a bounded operator from $L^p(\mathbb{T}^d;X)$ to $L^p(\mathbb{T}^d;Y)$. We denote the set of such multipliers by $\mathscr{M}_p(\mathbb{Z}^d;X,Y)$.

For $d = 1$ the assumption of the Marcinkiewicz theorem requires that for the dyadic decomposition $I_n = \{\mathfrak{x} \in \mathbb{Z} : 2^{n-1} < |\mathfrak{x}| \le 2^n\}$ we have

$$\operatorname{var}(M_{I_n}) \;\le\; C \tag{1.2}$$

for all $n \in \mathbb{N}$. For multipliers $M(\mathfrak{x}) = m(\mathfrak{x})I_X$ with a scalar function m it was shown in [Bou2] that (1.2) implies $M \in \mathscr{M}_p(\mathbb{Z};X)$ if and only if X is a UMD-space. A UMD-space can be characterized by the fact that the special multiplier $m_0(\mathfrak{x}) = \operatorname{sign}(\mathfrak{x})$ belongs to $\mathscr{M}_p(\mathbb{Z};X)$ (see [Bou1], [Bu]). Indeed, Bourgain shows how "to built up" general scalar multipliers with (1.2) from modifications of m_0. It is well known, that all subspaces and quotient spaces of $L^q(\Omega)$-spaces with $1 < q < \infty$ are UMD-spaces.

For operator-valued multipliers $M(\mathfrak{x}) \in \mathscr{B}(X)$ the variation (1.2), taken with respect to the operator norm, always implies that $M \in \mathscr{M}_p(\mathbb{Z};X)$ if and only if

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X is a Hilbert space (Schwartz showed that (1.2) is sufficient in Hilbert spaces [BL], and Pisier observed the converse). So besides the UMD property for X and Y one needs additional assumptions on the multiplier function M. Recently it was shown in [We], in the context of the Mikhlin-multiplier theorem for operator-valued multipliers, that this additional condition can be expressed in terms of R-boundedness:

A subset $\mathcal{T} \subset \mathscr{B}(X,Y)$ is called R-bounded if there is a constant C such that for all $T_0, T_1, \ldots, T_n \in \mathcal{T}$, $x_0, x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$

$$\int_0^1 \| \sum_{k=0}^n \varepsilon_k(t) T_k x_k \|_Y \, dt \;\le\; C \int_0^1 \| \sum_{k=0}^n \varepsilon_k(t) x_k \|_X \, dt, \tag{1.3}$$

where (ε_k) is the sequence of Rademacher functions on $[0,1]$. This concept was already used in [Bou2] and [BG] in connection with multiplier theorems, and more recently a detailed study was given in [CPSW]. If $X = Y = L^q(\Omega)$ for some $1 \le q < \infty$, then (1.3) is equivalent to the square function estimate

$$\left\| \left(\sum_{k=0}^n |T_k x_k|^2 \right)^{1/2} \right\|_{L^q} \;\le\; \tilde{C} \left\| \left(\sum_{k=0}^n |x_k|^2 \right)^{1/2} \right\|_{L^q} \tag{1.4}$$

known from harmonic analysis.

In this paper we use R-boundedness to give an appropriate form of the Marcinkiewicz condition for operator-valued multipliers. In place of (1.2) we assume that for some absolutely convex R-bounded set $\mathcal{T}$ we have

$$\sum_{k \in I_n} \|M(k+1) - M(k)\|_{\mathcal{T}} \;\le\; C \tag{1.5}$$

for all $n \in \mathbb{N}$, where $\| \cdot \|_{\mathcal{T}}$ denotes the Minkowski functional of $\mathcal{T}$. If X and Y are UMD-spaces, M satisfies (1.5) and

$$\{M(\pm 2^{k-1})\, k \in \mathbb{N}\} \text{ is } R\text{-bounded}, \tag{1.6}$$

then we show in section 3 that $M \in \mathscr{M}_p(\mathbb{Z}; X, Y)$. We also give d-dimensional versions of this result. Our proof follows the techniques of [Zi], who proved d-dimensional generalizations of Bourgain's result for scalar multipliers. In Remark 1.1 below we point out that (1.6) is necessary for M to be in $\mathscr{M}_p(\mathbb{Z}; X, Y)$. If M belongs to $\mathscr{M}_p(\mathbb{Z}; X, Y)$ and (1.2) holds, it follows that $\{M(\mathfrak{x})\, \mathfrak{x} \in \mathbb{Z}\}$ is R-bounded, and this indicates that R-boundedness arises naturally in the context of multiplier theorems. The second part of the paper will treat the continuous case. In analogy to the discrete setting we say that a function $M : \mathbb{R}^d \setminus \{0\} \to \mathscr{B}(X,Y)$ is a Fourier multiplier, i.e. $M \in \mathscr{M}_p(\mathbb{R}^d; X, Y)$, if the operator

$$f \;\longmapsto\; \mathscr{K}_M f = (M(\cdot)\hat{f}(\cdot))^\vee\,, \tag{1.7}$$

first defined for $f \in \mathscr{S}(\mathbb{R}^d; X)$, extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$.

For multiplier theorems of the Mikhlin type, one considers the sets

$$\{|x|^{|\gamma|}(D^\gamma M)(x) \,:\, x \in \mathbb{R}^d \setminus \{0\},\, \gamma \le (1, \ldots, 1)\} \tag{1.8}$$

or

$$\{x^\gamma (D^\gamma M)(x) \,:\, x \in \mathbb{R}^d \setminus \{0\},\, \gamma \le (1, \ldots, 1)\}. \tag{1.9}$$

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Again the norm boundedness of these sets is only sufficient in Hilbert spaces. If X and Y are UMD-spaces and (1.8) is R-bounded, we show then in Theorem 4.4 that $M \in \mathscr{M}_p(\mathbb{R}^d; X, Y)$ holds. For the finer condition (1.9) we need besides the R-boundedness of (1.9) and the UMD property an additional assumption on X and Y, which is the property (α), introduced by Pisier (see [Pi2]). In particular every q-concave Banach-lattice with $q < \infty$, or more generally, every Banach space with local unconditional structure and finite cotype has property (α) (cf. [Pi2], [DJT], Theorem 14.1). We reduce these theorems to the discrete case in section 3, again following the method in [Zi].

We also give a new criterion for the R-boundedness of a function $x \in I \to M(x)$. For one-dimensional intervals, R-boundedness follows if M is of bounded variation (see Theorem 2.7). For d-dimensional intervals we give an integrability criterion (see Theorem 4.1).

The next remark illustrates how the notion of R-boundedness is necessary if one considers operator-valued Fourier multipliers on vector-valued L^p-spaces.

1.1. *Remark.* Let us assume that we have a multiplier $M \in \mathscr{M}_p(\mathbb{Z}; X, Y)$. For trigonometric polynomials $f = \sum_{k=1}^n x_k \otimes \mathfrak{e}_{2^k}$ we therefore obtain

$$\|\sum_{k=1}^n M(2^k) x_k \otimes \mathfrak{e}_{2^k}\|_{L^p(\mathbb{T};Y)} \ \le\ C_1 \|\sum_{k=1}^n x_k \otimes \mathfrak{e}_{2^k}\|_{L^p(\mathbb{T};X)} . \tag{1.10}$$

Now, it is known (see [Pi1]) that there is a universal constant $C > 0$ such that for any Banach space E and any finite sequence $y_1, \dots, y_n$ in E:

$$\frac{1}{C}\|\sum_{k=1}^n \varepsilon_k y_k\|_{L^p([0,1];E)} \ \le\ \|\sum_{k=1}^n y_k \otimes \mathfrak{e}_{2^k}\|_{L^p(\mathbb{T};E)} \ \le\ C\|\sum_{k=1}^n \varepsilon_k y_k\|_{L^p([0,1];E)},$$

where $(\varepsilon_k)_{k\ge 1}$ denotes the sequence of Rademacher functions on $[0,1]$. These inequalities in connection with (1.10) lead to

$$\|\sum_{k=1}^n \varepsilon_k M(2^k) x_k\|_{L^p([0,1];Y)} \ \le\ C_2 \|\sum_{k=1}^n \varepsilon_k x_k\|_{L^p([0,1];X)} .$$

This means that the collection $\{M(2^k) : k \in \mathbb{N}\} \subset \mathscr{B}(X, Y)$ is R-bounded. By an application of Proposition 1.3 in [Bl] we get that $\mathcal{T} = \{M(r) : r \in \mathbb{Z}\}$ is R-bounded.

2. R-boundedness

In this section we list some important results about R-bounded collections $\mathcal{T}$ of bounded linear operators. Let $(\Omega, \mathfrak{A}, P)$ be a probability space and $(\varepsilon_k)_{k=0}^\infty$ a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $(\Omega, \mathfrak{A}, P)$. With $L^p(\Omega; X)$ we denote the Bochner space of p-integrable X-valued functions on $(\Omega, \mathfrak{A}, P)$.

2.1. **Definition.** A collection $\mathcal{T} \subset \mathscr{B}(X, Y)$ is said to be **R-bounded** if there exist a constant $C > 0$ such that for all $T_0, T_1, \dots, T_n \in \mathcal{T}$, $x_0, x_1, \dots, x_n \in X$ and all $n \in \mathbb{N}$

$$\|\sum_{k=0}^n \varepsilon_k T_k x_k\|_{L^1(\Omega;Y)} \ \le\ C \|\sum_{k=0}^n \varepsilon_k x_k\|_{L^1(\Omega;X)}. \tag{2.1}$$

The smallest constant C, for which (2.1) holds is denoted by $R(\mathcal{T})$.

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The notion of R-boundedness was already implicitly used in [Bou1], [Bou2] and was introduced in the paper [BG]. Detailed studies about collections of R-bounded operators can be found in [CPSW] and [We]. In [We] the reader will find as well a new characterization of maximal L^p-regularity of abstract differential equations using this notation.

Note. We want to emphasize that the definition of collections of R-bounded operators does not depend on the probability space $(\Omega, \mathfrak{A}, P)$ and the sequence of random variables (ε_k).

2.2. *Remark.* Here are some rather known facts about R-boundedness:

(a) Using Kahane's inequality (see [LT]) we can replace (2.1) by $(1 \leq p < \infty)$

$$\| \sum_{k=0}^{n} \varepsilon_k T_k x_k \|_{L^p(\Omega;Y)} \ \leq \ C_p \, \| \sum_{k=0}^{n} \varepsilon_k x_k \|_{L^p(\Omega;X)}. \tag{2.2}$$

(b) It is easy to see that R-bounded collections $\mathcal{T} \subset \mathscr{B}(X,Y)$ are necessarily bounded in $\mathscr{B}(X,Y)$. If X and Y are both Hilbert spaces, (2.2) shows that the converse also holds.

(c) If X and Y are q-concave Banach lattices $(1 \leq q < \infty)$ (for the Definition see [LT] 1.d.3.(iii)) the definitions (2.1) and (2.2) are equivalent to (see [LT] Theorem 1.d.6.(i))

$$\left\| \left(\sum_{k=0}^{n} |T_k x_k|^2 \right)^{1/2} \right\|_Y \ \leq \ C \left\| \left(\sum_{k=0}^{n} |x_k|^2 \right)^{1/2} \right\|_X. \tag{2.3}$$

(d) If $\mathcal{T} := \{a_k I_X \,:\, k \in \mathbb{N}_0\} \subset \mathscr{B}(X)$, then $R(\mathcal{T}) \leq 2\|a\|_\infty$.

In the following we want to present four practical methods to enlarge an R-bounded collection $\mathcal{T}$. By $\operatorname{aco}(\mathcal{T})$ we denote the real or complex absolute convex hull of a collection $\mathcal{T} \subset \mathscr{B}(X,Y)$. With this in mind, we can formulate the first statement.

2.3. **Lemma.** *Let $\mathcal{T} \subset \mathscr{B}(X,Y)$ be an R-bounded collection with R-bound $R(\mathcal{T})$. Then the absolute convex hull* $\operatorname{aco}(\mathcal{T})$ *as well as the strong closure of $\mathcal{T}$ are R-bounded with R-bounds not larger than $2R(\mathcal{T})$.*

The statement is based on ideas introduced in [Bou2]. For the proof we refer to [CPSW]. The next two lemmas can also be found in [CPSW].

2.4. **Lemma.** *Let $\mathcal{S} \subset \mathscr{B}(X_2,X_3), \mathcal{T} \subset \mathscr{B}(X_1,X_2)$ be two collections which are R-bounded. Then the collection*

$$\mathcal{S}\mathcal{T} = \{ST \,:\, S \in \mathcal{S}, T \in \mathcal{T}\}$$

is R-bounded with an R-bound not greater than $R(\mathcal{S})R(\mathcal{T})$.

Let E be a Banach space, $X = L^p(\Lambda; E)$ for some σ-finite measure space $(\Lambda, \mathfrak{B}, \mu)$ and $1 \leq p < \infty$. For $\varphi \in L^\infty(\Lambda)$ we denote by M_φ the pointwise multiplication operator on X.

2.5. **Lemma.** *Let $X = L^p(\Lambda; E)$ and $\mathcal{T} \subset \mathscr{B}(X)$. If $\mathcal{T}$ is R-bounded, then the collection*

$$\{M_\varphi T M_\psi \,:\, \varphi, \psi \in L^\infty(\Lambda) \text{ with } \|\varphi\|_\infty, \|\psi\|_\infty \leq 1, T \in \mathcal{T}\}$$

is R-bounded as well.

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The following extension result is also useful and is taken from [We], Proposition 2.11.

2.6. **Lemma.** *For $T \in \mathscr{B}(X,Y)$ define the operator $(\tilde{T}f)(\lambda) := T(f(\lambda))$, $f \in L^p(\Lambda; X)$, $\lambda \in \Lambda$, $1 \le p < \infty$. Then, if $\mathcal{T} \subset \mathscr{B}(X,Y)$ is R-bounded, the collection $\tilde{\mathcal{T}} = \{\tilde{T} : T \in \mathcal{T}\} \subset \mathscr{B}(L^p(\Lambda; X), L^p(\Lambda; Y))$ is also R-bounded.*

In the next theorem we give a sufficient condition on the regularity of operator-valued functions which ensures R-boundednes for their range collection. This result generalizes in particular Proposition 2.5 in [We]. Other statements of this type can be found in Corollary 3.5 and Theorem 4.1.

2.7. **Theorem.** *If X, Y are arbitrary Banach spaces and the function $M : I \to \mathscr{B}(X,Y)$ on an interval $I = [a,b) \subset \mathbb{R}$ is of bounded variation, then the collection*

$$\mathcal{M} \;:=\; \{M(x) \,:\, x \in I\}$$

is R-bounded with $R(\mathcal{M}) \le C(\|M(a)\| + \operatorname{var}(M))$.

Proof. Assume that M has the form

$$M(t) = M(a) + \sum_{j=1}^{m} \chi_{A_j}(t) M_j$$

with $A_j \subset I$ and $M_j \in \mathscr{B}(X,Y)$. Then, using Lemma 2.4 in [We], we obtain

$$R(\{M(x) \,:\, x \in I\}) \;\le\; \|M(a)\| + \sum_{j=1}^{m} \|M_j\| \,. \tag{2.4}$$

We will now show that for general M there is sequence of the form

$$M_k(t) = M(a) + \sum_j \chi_{A_{k,j}}(t) M_{k,j} \tag{2.5}$$

with

$$\sum_j \|M_{k,j}\| \;\le\; C \operatorname{var}(M) \quad \forall\, k \in \mathbb{N}$$

and

$$\|M_k(t) - M(t)\| \;\to\; 0 \tag{2.6}$$

for $k \to \infty$ and all $t \in I$. With this approximation property for M the claim follows by an application of Lemma 2.3.

Without loss of generality we assume that M is continuous from the left. If we put $\Delta(t) := \lim_{s \searrow t} M(s) - M(t)$, the bounded variation property on M states that for each $n \in \mathbb{N}$ the set

$$\mathscr{T}_n := \{t \in I \,:\, \frac{1}{n+1} \le \|\Delta(t)\| < \frac{1}{n}\}$$

has to be finite, i.e. $\mathscr{T}_n = \{t_{n,1}, \ldots, t_{n,k(n)}\}$. If we put

$$N(t) \;:=\; \sum_{n=1}^{\infty} \sum_{x \in \mathscr{T}_n} \Delta(x) \chi_{(x,b)}(t),$$

one can show that N is continuous from the left and $\operatorname{var}(N) \le \operatorname{var}(M)$ holds. Moreover the function $L := M - N$ is also of bounded variation with $\operatorname{var}(L) \le$

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$2\,\mathrm{var}(M)$, continuous and therefore uniformly continuous on $[a,b)$. Obviously the definition of N allows us to construct a sequence (N_k) of functions of the form

$$N_k = M(a) + \sum_{j=1}^{m_k} \chi_{A_{k,j}}(t) N_{k,j}$$

with $\|N_k(t) - N(t)\| \to 0$ for $k \to \infty$ and all $t \in [a,b)$ and $\mathrm{var}(N_k) \leq \mathrm{var}(N)$. To approximate L we choose for a given $\varepsilon > 0$ a $\delta > 0$ such that $|s-t| < \delta$ implies $\|L(s) - L(t)\| \leq \varepsilon$ and that for a partition (t_j) in (a,b) with $\sup_j |t_j - t_{j-1}| \leq \delta$ we have

$$\left| \sum_j \|L(t'_j) - L(t'_{j-1})\| - \mathrm{var}(L) \right| \;\leq\; \varepsilon$$

for all refinements (t'_j) of (t_j). Now the function

$$\tilde{L}(t) \;:=\; M(a) + \sum_j [L(t_j) - L(t_{j-1})]\chi_{[t_j,b)}(t)$$

satisfies $\mathrm{var}(\tilde{L}) \leq \mathrm{var}(L) + \varepsilon$ and $\|L(t) - \tilde{L}(t)\| \leq \varepsilon$. In this way we can find a sequence (L_k) of the form

$$L_k = M(a) + \sum_{j=1}^{m_k} \chi_{A_{k,j}}(t) M_j$$

satisfying $\|L_k(t) - L(t)\| \to 0$ for $k \to \infty$, $t \in [a,b)$ and $\mathrm{var}(L_k) \leq 2\,\mathrm{var}(L)$. Now the sequence $M_k := N_k + L_k$ has the required properties (2.5) and (2.6). □

Since outside the Hilbert space setting bounded sets of operators are usually not R-bounded anymore, we have to replace the operator norm in various estimates and definitions by the following norms "measuring" R-boundedness:

2.8. **Notation.** For a bounded collection $\mathcal{T} \subset \mathscr{B}(X,Y)$ we denote the Minkowski functional of $\mathrm{aco}\,(\mathcal{T})$ by

$$\|\cdot\|_{\mathcal{T}} : \begin{cases} \mathscr{B}(X,Y) & \longrightarrow \quad [0,\infty], \\ T & \longmapsto \quad \|T\|_{\mathcal{T}} := \inf\{t > 0 \,:\, T \in t \cdot \mathrm{aco}(\mathcal{T})\}. \end{cases} \tag{2.7}$$

Here are some obvious facts that will be used constantly in the next two sections.

2.9. *Remark.* a) If we set $\mathcal{T} := \{a_k I_X \,:\, k \in \mathbb{N}_0\} \subset \mathscr{B}(X)$, then we have

$$\mathrm{aco}(\mathcal{T}) = \{z I_X \,:\, |z| \leq \|a\|_\infty\}$$

and thus by Remark 2.2.(d) we get

$$R(\mathcal{T})\|\alpha I_X\|_{\mathcal{T}} \;\leq\; 2\|a\|_\infty \|\alpha I_X\|_{\mathcal{T}} \;\leq\; 2|\alpha|, \quad \alpha \in \mathbb{C}.$$

b) Let $\mathcal{T} \subset \mathscr{B}(X,Y)$ be an R-bounded collection. Then the following holds:

(i) $\|T_1 + T_2\|_{\mathcal{T}} \leq \|T_1\|_{\mathcal{T}} + \|T_2\|_{\mathcal{T}}$.

(ii) If a collection $\mathcal{M} = \{M_n \,:\, n \in \mathbb{N}\}$ has the property that

$$C := \sup\{\|M_n\|_{\mathcal{T}} \,:\, n \in \mathbb{N}\} \;<\; \infty\,,$$

then $\mathcal{M}$ is also R-bounded and the R-bound is not greater than $4CR(\mathcal{T})$.

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3. The discrete case

In this section we are interested in giving sufficient conditions on the function $M : \mathbb{Z}^d \to \mathscr{B}(X,Y)$ so that the operator, defined in (1.1), extends to a bounded operator. In particular we want to generalize the results given in [Bou2] and [Zi].

The next two sections will rather follow the examinations given in section 1 and 2 of [Zi]. For that reason we will use the same kind of notation which appears in that work.

3.1. **Definition.** Let $\alpha, \beta \in \mathbb{Z}^d$, $\alpha \leq \beta$ (coordinatewise) and $[\alpha;\beta] := \{\mathfrak{x} \in \mathbb{Z}^d : \alpha \leq \mathfrak{x} \leq \beta\}$. For a function $M : \mathbb{Z}^d \to \mathscr{B}(X,Y)$ we define the restriction of M to $G \subset \mathbb{Z}^d$ by

$$M_G(\mathfrak{x}) := \begin{cases} M(\mathfrak{x}) & : \quad \mathfrak{x} \in G, \\ 0 & : \quad \mathfrak{x} \notin G. \end{cases}$$

The difference operators Δ^{e_j} $(j = 1, \ldots, d)$ are defined for the unit vectors e_j of $\mathbb{Z}^d$ as

$$(\Delta^{e_j} M_{[\alpha;\beta]})(\mathfrak{x}) := \begin{cases} M_{[\alpha;\beta]}(\mathfrak{x}) - M_{[\alpha;\beta]}(\mathfrak{x} - e_j) & : \quad \mathfrak{x}_j \neq \alpha_j, \\ 0 & : \quad \mathfrak{x}_j = \alpha_j. \end{cases}$$

For arbitrary $\gamma = (\gamma_1, \ldots, \gamma_d) = \sum_{j=1}^d \gamma_j e_j$ with $\gamma_j \in \{0,1\}$ we set

$$\Delta^0 M_{[\alpha;\beta]} := M_{[\alpha;\beta]}, \quad \Delta^\gamma M_{[\alpha;\beta]} := \Delta^{\gamma_1 e_1} \circ \cdots \circ \Delta^{\gamma_d e_d} M_{[\alpha;\beta]}.$$

Next we generalize the defintion of a variation.

3.2. **Definition.** Let $M : \mathbb{Z}^d \to \mathscr{B}(X,Y)$ be an arbitrary function and $\mathcal{T} \subset \mathscr{B}(X,Y)$ a bounded collection. We define the $\mathcal{T}$-*variation* of M in the interval $[\alpha;\beta]$ by

$$\operatorname*{var}_{[\alpha;\beta]}{}_{\mathcal{T}}\, M_{[\alpha;\beta]} := \sum_{\mathfrak{x} \in [\alpha;\beta]} \|(\Delta^{\gamma_\mathfrak{x}} M_{[\alpha;\beta]})(\mathfrak{x})\|_{\mathcal{T}}, \tag{3.1}$$

where $\gamma_\mathfrak{x} = (\gamma_{\mathfrak{x}_1}, \ldots, \gamma_{\mathfrak{x}_d})$ with

$$\gamma_{\mathfrak{x}_j} := \begin{cases} 1 & : \quad \mathfrak{x}_j \neq \alpha_j, \\ 0 & : \quad \mathfrak{x}_j = \alpha_j. \end{cases}$$

Of course if $\mathcal{T}$ is the unit ball of $\mathscr{B}(X,Y)$, we have the usual notation of bounded variation, which we simply denote by $\operatorname{var}_{[\alpha;\beta]} M_{[\alpha;\beta]}$ without the subscript $\mathcal{T}$.

The next result is a practical tool to estimate the $\mathcal{T}$-variation of a (discrete) function.

3.3. **Lemma.** *Let $\alpha_n, \beta_n \in \mathbb{Z}^d, \alpha_n \leq \beta_n$ and $([\alpha_n;\beta_n])_{n\in\mathbb{N}}$ be a (disjoint) decomposition of $\mathbb{Z}^d$. If $F : \mathbb{R}^d \to \mathscr{B}(X,Y)$ is a sufficiently smooth function and the collection*

$$\mathcal{T} := \bigcup_{n \in \mathbb{N}} \{(\beta_n - \alpha_n)^\gamma (D^\gamma F)(x) : x \in [\alpha_n, \beta_n], \ \gamma \leq (1, \ldots, 1)\}$$

is bounded, then the restriction of F to $\mathbb{Z}^d$ satisfies

$$\sup_{n\in\mathbb{N}} \operatorname*{var}_{[\alpha_n;\beta_n]}{}_{\mathcal{T}}\, F_{[\alpha_n;\beta_n]} \ \leq \ 2^d.$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. If we rearrange the sum in Definition 3.1 we obtain

$$\operatorname*{var}_{[\alpha_n;\beta_n]}{}_{\mathcal{T}}\, F_{[\alpha_n;\beta_n]} \ = \ \sum_{\gamma \leq (1,\ldots,1)} \ \sum_{\{\mathfrak{x} : \gamma_\mathfrak{x} = \gamma\}} \|(\Delta^\gamma F_{[\alpha_n;\beta_n]})(\mathfrak{x})\|_{\mathcal{T}}, \tag{3.2}$$

where the sum over γ has 2^d summands.

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1. CASE: $\gamma = (0, \dots, 0)$
In this case only $\mathfrak{x} = \alpha$ has the property $\gamma_{\mathfrak{x}} = \gamma$ and therefore

$$\sum_{\{\mathfrak{x}:\gamma_{\mathfrak{x}}=\gamma\}} \|(\Delta^\gamma F_{[\alpha_n;\beta_n]})(\mathfrak{x})\|_{\mathfrak{T}} \;=\; \|F_{[\alpha_n;\beta_n]}(\alpha_n)\|_{\mathfrak{T}} \;=\; \|F(\alpha_n)\|_{\mathfrak{T}} \;\leq\; 1.$$

2. CASE: $(0, \dots, 0) \neq \gamma \leq (1, \dots, 1)$
Take $\mathfrak{x} \in [\alpha_n; \beta_n]$ with $\gamma_{\mathfrak{x}} = \gamma$. Let $q_1, \dots, q_r$ be the coordinate directions for which $\gamma_{\mathfrak{x}_q} \neq 0$. Now by definition of the difference operator and the fundamental theorem of calculus we get

$$\begin{aligned}(\Delta^{\gamma_{\mathfrak{x}}} F_{[\alpha_n;\beta_n]})(\mathfrak{x}) \;&=\; \int_{[\mathfrak{x}_{q_1}-1,\mathfrak{x}_{q_1}]} \cdots \int_{[\mathfrak{x}_{q_r}-1,\mathfrak{x}_{q_r}]} (D^\gamma F)(\xi)\, d\xi_{q_r} \dots d\xi_{q_1} \\ &=\; \frac{1}{(\beta_n-\alpha_n)^\gamma} \int_{[\mathfrak{x}_{q_1}-1,\mathfrak{x}_{q_1}]} \cdots \int_{[\mathfrak{x}_{q_r}-1,\mathfrak{x}_{q_r}]} (\beta_n-\alpha_n)^\gamma (D^\gamma F)(\xi)\, d\xi_{q_r} \dots d\xi_{q_1}.\end{aligned}$$

This immediately implies $(\Delta^\gamma F_{[\alpha_n;\beta_n]})(\mathfrak{x}) \in \frac{1}{(\beta_n-\alpha_n)^\gamma} \cdot \operatorname{aco}(\mathfrak{T})$ and thus

$$\sum_{\{\mathfrak{x}:\gamma_{\mathfrak{x}}=\gamma\}} \|(\Delta^\gamma F_{[\beta_n;\alpha_n]})(\mathfrak{x})\|_{\mathfrak{T}} \;\leq\; \sum_{\{\mathfrak{x}:\gamma_{\mathfrak{x}}=\gamma\}} \frac{1}{(\beta_n-\alpha_n)^\gamma} \;=\; 1. \tag{3.3}$$

The last equality holds because there are exactly $(\beta_n - \alpha_n)^\gamma$ different $\mathfrak{x}$ in $[\alpha_n; \beta_n]$ with $\gamma_{\mathfrak{x}} = \gamma$. Thus the first case, (3.3) and (3.2) yield to the desired result. □

The proof of the following result, which extends Stečkin‘s multilplier theorem, illustrates how the notion of bounded variation allows us to write a multiplier function as a sum of characteristic functions (cf. (3.6) below).

Let $I_n := [\alpha_n; \beta_n]$ with $\alpha_n = (-n, -n, \dots, -n), \beta_n = (n, n, \dots, n)$ and

$$I(\gamma, n) := \{\mathfrak{x} \in I_n : \mathfrak{x}_i = -n \text{ if } \gamma_i = 0\}.$$

3.4. **Theorem.** *Let X be a UMD-space, Y an arbitrary Banach space and $1 < p < \infty$. Assume that the function $M : \mathbb{Z}^d \to \mathscr{B}(X, Y)$ satisfies*

$$\sum_{\mathfrak{x}\in I(\gamma,n)} \|(\Delta^\gamma M_{I_n})(\mathfrak{x})\| \leq C \;< \infty \tag{3.4}$$

for all $\gamma \leq (1, \dots, 1)$ and all $n \in \mathbb{N}$. Then $M \in \mathscr{M}_p(\mathbb{Z}^d; X, Y)$.

Proof. Using the same rearrangement as in (3.2) and the assumption (3.4) we have that

$$\operatorname*{var}_{I_n} M_{I_n} \;=\; \sum_{\gamma\leq(1,\dots,1)} \sum_{\mathfrak{x}\in I(\gamma,n)} \|(\Delta^\gamma M_{I_n})(\mathfrak{x})\| \;\leq\; 2^d C. \tag{3.5}$$

The point of this notation of bounded variation is that M can be written as

$$M_{[\alpha_n;\beta_n]} \;=\; \sum_{\mathfrak{x}\in[\alpha_{i,n};\beta_{i,n}]} ((\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_n;\beta_n]})(\mathfrak{x}))\chi_{[\mathfrak{x};\beta_n]}\,. \tag{3.6}$$

For scalar valued M this was checked in [Zi], Lemma 1.3 (ii) and we apply this identity to $y^*(M_{[\alpha_n;\beta_n]}(\mathfrak{x})x)$ for all $x \in X, y^* \in Y^*$.

Now for an arbitrary $f : \mathbb{T}^d \to X$ with $\operatorname{supp} \hat{f} \subset I_n$ we obtain from (3.6)

$$\mathscr{K}_{M_{I_n}} f \;=\; \sum_{\mathfrak{x}\in I_n} ((\Delta^{\gamma_{\mathfrak{x}}} M_{I_n})(\mathfrak{x}))^\sim \circ \mathscr{K}_{\chi_{[\mathfrak{x};\beta_n]}} f\,, \tag{3.7}$$

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where $((\Delta^{\gamma_{\mathfrak{x}}} M_{I_n})(\mathfrak{x}))^{\sim}$ is the operator which arises from $(\Delta^{\gamma_{\mathfrak{x}}} M_{I_n})(\mathfrak{x})$ in the same way as it was done in Lemma 2.6. Since X is a UMD-space we know by [Zi] that there exists a constant D such that

$$\|\mathscr{K}_{\chi_{[\alpha;\beta]}} f\| \ \le \ D\|f\|$$

for all intervals $[\alpha;\beta]$. Hence by (3.7) and (3.5)

$$\|\mathscr{K}_M f\| \ = \ \|\mathscr{K}_{M_{I_n}} f\| \ \le \ D\|f\| \sum_{\mathfrak{x}\in I_n} \|(\Delta^{\gamma_{\mathfrak{x}}} M_{I_n})(\mathfrak{x}))\|$$

$$= \ D \operatorname*{var}_{I_n} M_{I_n} \|f\| \ \le \ 2^d C D\|f\|.$$

Since the functions f with compact support $\hat{f}$ are dense in $L^p(\mathbb{T}^d;X)$, the claim follows. □

3.5. **Corollary.** *If X and Y are arbitrary Banach spaces, and $M : \mathbb{Z}^d \to \mathscr{B}(X,Y)$ satisfies* (3.4)*, then $\{M(\mathfrak{x}) : \mathfrak{x} \in \mathbb{Z}^d\}$ is R-bounded.*

Proof. This follows from [We], Lemma 2.4, and the representations (3.6) and (3.7). □

To obtain more refined multiplier theorems that generalize the Marcinkiewicz multiplier theorem, we assume that M is not of bounded variation on all of $\mathbb{Z}^d$ but only uniformly on certain partitions of $\mathbb{Z}^d$. As one might guess from Corollary 3.5, the R-boundedness will then be needed. The partitions we will use are the following ones:

(a) *The coarse decomposition.* Set $\mathbf{D}_0 := \{0\} \subset \mathbb{Z}^d$ and for $n = dr + j$, $r \in \mathbb{N}_0$, $j \in \{1,\dots,d\}$

$$\mathbf{D}_n := \{\mathfrak{x} = (\mathfrak{x}_1,\dots,\mathfrak{x}_d) \in \mathbb{Z}^d \,:\, |\mathfrak{x}_1|,\dots,|\mathfrak{x}_{j-1}| < 2^{r+1},\, 2^r \le |\mathfrak{x}_j| < 2^{r+1},\, |\mathfrak{x}_{j+1}|,\dots,|\mathfrak{x}_d| < 2^r\}.$$

(b) *The fine decomposition.* For $\nu = (\nu_1,\dots,\nu_d) \in \mathbb{N}_0^d$ we define

$$\mathbf{D}_\nu := I_{\nu_1} \times \ldots \times I_{\nu_d},$$

where $I_0 = \{0\}$ and $I_n = \{k \in \mathbb{Z} \,:\, 2^{n-1} \le |k| < 2^n\}$ $(n \in \mathbb{N})$.

Since $\mathbf{D} = \mathbf{D}_n$ (resp. $\mathbf{D}_\nu$) are unions of $s = 2$ (resp. $s = 2^d$) intervals, we can moreover define the $\mathcal{T}$-*variation* of M with respect to the decompositions by

$$\operatorname*{var}_{\mathbf{D}}{}_{\mathcal{T}} M \ := \ \sum_{i=1}^{s} \operatorname*{var}_{[\alpha_i;\beta_i]}{}_{\mathcal{T}} M_{[\alpha_i;\beta_i]} \ .$$

We are now able to define the generalized Marcinkiewicz conditions.

3.6. **Definition.** A function $M : \mathbb{Z}^d \to \mathscr{B}(X,Y)$ is said to be of bounded $\mathcal{T}$-variation with respect to the decompositions $(\mathbf{D}_k)$ (resp. $(\mathbf{D}_\nu)$), if there exist an R-bounded collection $\mathcal{T} \subset \mathscr{B}(X,Y)$ such that the condition $(\mathcal{M}_{cD}^{\mathcal{T}})$

$$\sup_{k\in\mathbb{N}_0} \operatorname*{var}_{\mathbf{D}_k}{}_{\mathcal{T}} M \ < \infty\,,$$

respectively $(\mathcal{M}_{fD}^{\mathcal{T}})$

$$\sup_{\nu\in\mathbb{N}_0^d} \operatorname*{var}_{\mathbf{D}_\nu}{}_{\mathcal{T}} M \ < \infty$$

is fullfilled.

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3.7. **Theorem** (operator-valued Marcinkiewicz theorem for $(\mathbf{D}_k)$)**.** *Let X, Y be UMD-spaces and $1 < p < \infty$. If the function $M : \mathbb{Z}^d \to \mathscr{B}(X,Y)$ possesses condition $(\mathcal{M}^{\mathcal{T}}_{cD})$, then $M \in \mathscr{M}_p(\mathbb{Z}^d; X, Y)$.*

3.8. **Theorem** (operator-valued Marcinkiewicz theorem for $(\mathbf{D}_\nu)$)**.** *Let X, Y be UMD-spaces with the property (α) and $1 < p < \infty$. If the function $M : \mathbb{Z}^d \to \mathscr{B}(X,Y)$ possesses condition $(\mathcal{M}^{\mathcal{T}}_{fD})$, then $M \in \mathscr{M}_p(\mathbb{Z}^d; X, Y)$.*

3.9. **Example.** The Marcinkiewicz multiplier theorems in [Bou2] and [Zi] are special cases of these theorems. In both papers one considers functions M of the form $M(\mathfrak{x}) = a(\mathfrak{x})I$, $a(\mathfrak{x}) \in \mathbb{C}$. If we set $\mathcal{T} := \{M(\mathfrak{x}) : \mathfrak{x} \in \mathbb{Z}^d\}$, then using Remark 2.9 a) we get

$$R(\mathcal{T}) \operatorname*{var}_{\mathbf{D}}{}_{\mathcal{T}} M = R(\mathcal{T}) \sum_i \operatorname*{var}_{[\alpha_i;\beta_i]}{}_{\mathcal{T}} M_{[\alpha_i;\beta_i]} = R(\mathcal{T}) \sum_i \sum_{\mathfrak{x} \in [\alpha_i;\beta_i]} \|(\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_i;\beta_i]})(\mathfrak{x})\|_{\mathcal{T}}$$

$$= R(\mathcal{T}) \sum_i \sum_{\mathfrak{x} \in [\alpha_i;\beta_i]} \|(\Delta^{\gamma_{\mathfrak{x}}} a_{[\alpha_i;\beta_i]})(\mathfrak{x}) I\|_{\mathcal{T}} \le 2 \sum_i \sum_{\mathfrak{x} \in [\alpha_i;\beta_i]} |(\Delta^{\gamma_{\mathfrak{x}}} a_{[\alpha_i;\beta_i]})(\mathfrak{x})| = 2 \operatorname*{var}_{\mathbf{D}} a \,.$$

3.10. *Remark.* Since the elements $\mathbf{D}_k$ are unions of two intervals $[\alpha_{1k}; \beta_{1k}], [\alpha_{2k}; \beta_{2k}]$, the assumption of Theorem 3.7 implies in particular that the collection $\mathcal{S} = \{M(\alpha_{ik}) : i = 1, 2;\ k \in \mathbb{N}_0\}$ has to be R-bounded (use Remark 2.9 b,(ii)).

Before we start to prove both theorems we list some further results, which will simplify the later arguments. For $G \subset \mathbb{Z}^d$ we define a characteristic multiplier function from $\mathbb{Z}^d$ to $\mathscr{B}(E)$ by

$$\chi_G(\mathfrak{x}) := \begin{cases} I & : \ \mathfrak{x} \in G, \\ 0 & : \ \mathfrak{x} \notin G. \end{cases}$$

Now the following results hold

3.11. **Theorem.** *Let E be a UMD-space, $1 < p < \infty$ and $S^E_k := \mathscr{K}_{\chi_{\mathbf{D}_k}}, k \in \mathbb{N}_0$. Then there exist a $C_p > 0$ such that for every trigonometric polynomial f*

$$\frac{1}{C_p} \|f\|_{L^p(\mathbb{T}^d;E)} \ \le\ \|\sum_{k=0}^{\infty} \varepsilon_k S^E_k f\|_{L^p(\Omega;L^p(\mathbb{T}^d;E))} \ \le\ C_p \|f\|_{L^p(\mathbb{T}^d;E)} \,. \tag{3.8}$$

A similar result is true for the fine decomposition. By $(\varepsilon_\nu)_{\nu \in \mathbb{N}_0^d}$ we denote an arbitrary d-dimensional renumeration of $(\varepsilon_k)_{k=0}^{\infty}$.

3.12. **Theorem.** *Let E be a UMD-space with the property (α), $1 < p < \infty$ and $S^E_\nu := \mathscr{K}_{\chi_{\mathbf{D}_\nu}}, \nu \in \mathbb{N}_0^d$. Then there exist a $C_p > 0$ such that for every trigonometric polynomial f*

$$\frac{1}{C_p} \|f\|_{L^p(\mathbb{T}^d;E)} \ \le\ \|\sum_{\nu \in \mathbb{N}_0^d} \varepsilon_\nu S^E_\nu f\|_{L^p(\Omega;L^p(\mathbb{T}^d;E))} \ \le\ C_p \|f\|_{L^p(\mathbb{T}^d;E)} \,. \tag{3.9}$$

The proofs of both lemmas can be found in [Bou2] and [Zi]. Actually Zimmermann assumes for Theorem 3.12 that E is a UMD-space with local unconditional structure. But his proof works also for our weaker assumption. The next result is a consequence of the Definition of a UMD-space and Lemma 2.5. The proof is implicitly in [Zi] (see also Lemma 7 in [Bou2] or Lemma 3.5 in [BG]).

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3.13. **Lemma.** *Let E be a UMD-space and $1 < p < \infty$. Then the collection*

$$\mathcal{K} := \{\mathscr{K}_{\chi_G} : G \text{ is an interval in } \mathbb{Z}^d\} \subset \mathscr{B}(L^p(\mathbb{T}^d; E))$$

is R-bounded.

3.14. *Remark.* The same statement also holds in the case $\mathbb{R}^d$.

Proof of Theorem 3.7. Let $\mathscr{K}_n := \mathscr{K}_{M_{\mathbf{D}_n}}$, $n \in \mathbb{N}_0$. Now for any trigonometric polynomial we get

$$S_n^Y \circ \mathscr{K}_M f = \mathscr{K}_n \circ S_n^X f$$

and thus, using Theorem 3.11,

$$\|\mathscr{K}_M f\|_{L^p(\mathbb{T}^d;Y)} \leq C_p \, \| \sum_{n=0}^{\infty} \varepsilon_n \mathscr{K}_n \circ S_n^X f\|_{L^p(\Omega;L^p(\mathbb{T}^d;Y))}.$$

Now, if we can prove that the collection $\{\mathscr{K}_n : n \in \mathbb{N}_0\}$ is R-bounded, an additional application of Theorem 3.11 would complete this proof. □

3.15. **Lemma.** *The collection*

$$\{\mathscr{K}_n : n \in \mathbb{N}_0\} \subset \mathscr{B}(L^p(\mathbb{T}^d; X), L^p(\mathbb{T}^d; Y))$$

is R-bounded.

Proof of Lemma 3.15. By assumption there exist an R-bounded collection $\mathcal{T}$ with

$$\operatorname*{var}_{\mathbf{D}_k} {}_{\mathcal{T}} M = \sum_{i=1}^{2} \sum_{\mathfrak{r} \in [\alpha_{i,k};\beta_{i,k}]} \|(\Delta^{\gamma_\mathfrak{r}} M_{[\alpha_{i,k};\beta_{i,k}]})(\mathfrak{r})\|_{\mathcal{T}} \leq C < \infty \quad \forall\, k \in \mathbb{N}_0 . \tag{3.10}$$

For the operator $\mathscr{K}_n$ we have the representation

$$\mathscr{K}_n = \mathscr{K}_{M_{[\alpha_{1,n};\beta_{1,n}]}} + \mathscr{K}_{M_{[\alpha_{2,n};\beta_{2,n}]}} . \tag{3.11}$$

Let us define the collection

$$\mathcal{S} := \operatorname{aco}(\tilde{\mathcal{T}})\mathcal{K} ,$$

where $\tilde{\mathcal{T}}$ is the collection from Lemma 2.6 and $\mathcal{K}$ is the one from Lemma 3.13. Using Lemma 2.4, Lemma 2.6 and Lemma 3.13 we get that $\mathcal{S}$ is R-bounded. Now using the representation formulas from (3.6) and (3.7) we obtain

$$\mathscr{K}_{M_{[\alpha_{i,n};\beta_{i,n}]}} = \sum_{\mathfrak{r} \in [\alpha_{i,n};\beta_{i,n}]} ((\Delta^{\gamma_\mathfrak{r}} M_{[\alpha_{i,n};\beta_{i,n}]})(\mathfrak{r}))^{\sim} \circ \mathscr{K}_{\chi_{[\mathfrak{r};\beta_{i,n}]}} . \tag{3.12}$$

Since

$$\begin{aligned}
&\|((\Delta^{\gamma_\mathfrak{r}} M_{[\alpha_{i,n};\beta_{i,n}]})(\mathfrak{r}))^{\sim} \circ \mathscr{K}_{\chi_{[\mathfrak{r};\beta_{i,n}]}}\|_{\mathcal{S}} \\
&\quad = \inf\{t > 0 : ((\Delta^{\gamma_\mathfrak{r}} M_{[\alpha_{i,n};\beta_{i,n}]})(\mathfrak{r}))^{\sim} \circ \mathscr{K}_{\chi_{[\mathfrak{r};\beta_{i,n}]}} \in t \cdot \operatorname{aco}(\mathcal{S})\} \\
&\quad \leq \inf\{t > 0 : ((\Delta^{\gamma_\mathfrak{r}} M_{[\alpha_{i,n};\beta_{i,n}]})(\mathfrak{r}))^{\sim} \circ \mathscr{K}_{\chi_{[\mathfrak{r};\beta_{i,n}]}} \in t \cdot \operatorname{aco}(\tilde{\mathcal{T}})\mathcal{K}\} \\
&\quad \leq \inf\{t > 0 : ((\Delta^{\gamma_\mathfrak{r}} M_{[\alpha_{i,n};\beta_{i,n}]})(\mathfrak{r}))^{\sim} \in t \cdot \operatorname{aco}(\tilde{\mathcal{T}})\} \\
&\quad = \inf\{t > 0 : (\Delta^{\gamma_\mathfrak{r}} M_{[\alpha_{i,n};\beta_{i,n}]})(\mathfrak{r}) \in t \cdot \operatorname{aco}(\mathcal{T})\} \\
&\quad = \|(\Delta^{\gamma_\mathfrak{r}} M_{[\alpha_{i,n};\beta_{i,n}]})(\mathfrak{r})\|_{\mathcal{T}},
\end{aligned}$$

we thus obtain from (3.12) and Remark 2.9 b,(i) that

$$\|\mathscr{K}_{M_{[\alpha_{i,n};\beta_{i,n}]}}\|_{\mathcal{S}} \leq \sum_{\mathfrak{r} \in [\alpha_{i,n};\beta_{i,n}]} \|(\Delta^{\gamma_\mathfrak{r}} M_{[\alpha_{i,n};\beta_{i,n}]})(\mathfrak{r})\|_{\mathcal{T}} .$$

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Taking (3.11), (3.10) and again Remark 2.9 b,(i) this yields

$$\|\mathscr{K}_n\|_{\mathcal{S}} \leq C < \infty \quad \forall n \in \mathbb{N}_0.$$

Applying Remark 2.9 b,(ii) the proof is complete. □

3.16. *Remark.* The proofs of Theorem 3.7 and Lemma 3.15 showed that the operator norm of $\mathscr{K}_M$ can be estimated by

$$\|\mathscr{K}_M\|_{\mathscr{B}(L^p(\mathbb{T}^d;X),L^p(\mathbb{T}^d;Y))} \leq CR(\mathcal{T}) \sup_{k\in\mathbb{N}_0} \operatorname{var}_{\mathcal{T}}_{\mathbf{D}_k} M,$$

where the constant C only depends on p and the dimension d, but not on the collection $\mathcal{T}$ and the multiplier M.

3.17. *Remark.* The proof of Theorem 3.8 works in the same way.

4. The continuous case

In the beginning of this section we'd like to present a criterion for the R-boundedness of an operator-valued function on $\mathbb{R}^d$. This will be done by using Corollary 3.5 of the preceding section, which already gave a tool on how to decide whether an operator-valued function on $\mathbb{Z}^d$ is R-bounded. Before stating the result, we need some additional notation.

Let $\xi \in \mathbb{R}^d$, γ be a multiindex with $0 \neq \gamma \leq (1,\ldots,1)$ and $q_1,\ldots,q_r$ be the coordinate directions for which $\gamma_{q_i} = 1$. In this case we set $\xi_\gamma = (\xi_{q_1},\ldots,\xi_{q_r}) \in \mathbb{R}^r$.

4.1. **Theorem.** *Let X, Y be arbitrary Banach spaces and $M : \mathbb{R}^d \to \mathscr{B}(X,Y)$ a bounded function with continuous derivatives $D^\gamma M$, $\gamma \leq (1,\ldots,1)$. If moreover*

$$\int_{\mathbb{R}^r} \|(D^\gamma M)(\xi_\gamma)\| \, d\xi_\gamma \;\leq\; C \;< \infty \tag{4.1}$$

for each $0 \neq \gamma \leq (1,\ldots,1)$, then the collection

$$\mathcal{M} \;:=\; \{M(x) \,:\, x \in \mathbb{R}^d\}$$

is R-bounded.

Proof. Set $M_k(x) := M(x/2^k), k \in \mathbb{N}$, and restrict M_k to $\mathbb{Z}^d$. Since M is bounded, we know from Corollary 3.5 that the collection

$$\mathcal{M}_k \;:=\; \{M_k(\mathfrak{x}) \,:\, \mathfrak{x} \in \mathbb{Z}^d\}$$

is R-bounded, if

$$\sum_{\mathfrak{x}\in I(\gamma,n)} \|(\Delta^\gamma M_{k,I_n})(\mathfrak{x})\| \leq \tilde{C}_k \;< \infty$$

for each $0 \neq \gamma \leq (1,\ldots,1)$ holds. Now, in analogy to the proof of Lemma 3.3, the fundamental theorem of calculus states that for $\mathfrak{x} \in I(\gamma,n)$ we have

$$(\Delta^\gamma M_{k,I_n})(\mathfrak{x}) \;=\; \int_{[\mathfrak{x}_{q_1}-1,\mathfrak{x}_{q_1}]} \cdots \int_{[\mathfrak{x}_{q_r}-1,\mathfrak{x}_{q_r}]} (D^\gamma M_k)(\xi)\, d\xi_{q_r} \ldots d\xi_{q_1}$$

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and moreover

$$\sum_{\mathfrak{x}\in I(\gamma,n)} \|(\Delta^\gamma M_{k,I_n})(\mathfrak{x})\| \leq \int_{[-n,n]^r} \|(D^\gamma M_k)(\xi_\gamma)\|\,d\xi_\gamma$$
$$= \int_{[-n/2^k,n/2^k]^r} \|(D^\gamma M)(\xi_\gamma)\|\,d\xi_\gamma$$
$$\leq \int_{\mathbb{R}^r} \|(D^\gamma M)(\xi_\gamma)\|\,d\xi_\gamma \;\leq\; C.$$

So by now we have proved that

$$\sum_{\mathfrak{x}\in I(\gamma,n)} \|(\Delta^\gamma M_{k,I_n})(\mathfrak{x})\| \;\leq\; C\,,$$

where the constant is independent of k. Since $\mathcal{M}_k \subset \mathcal{M}_{k+1}$, we obtain from Corollary 3.5 that the collection

$$\bigcup_{k=1}^{\infty} \mathcal{M}_k \;=\; \{M(\mathfrak{x}/2^k)\,:\,\mathfrak{x}\in\mathbb{Z}^d, k\in\mathbb{N}\}$$

is R-bounded. An application of Lemma 2.3 completes the proof. □

The remaining part of this section is concerned with extensions of the Mikhlin multiplier theorems from [Bou2], [McC] and [Zi] for scalar-valued multipliers to new theorems with operator-valued multiplier functions. In [We] the second author already considered the one-dimensional case and used it to give a new characterization of maximal L^p-regularity of abstract differential equations. In this paragraph we will generalize this result to the higher-dimensional setting using the Marcinkiewicz theorems from section 3.

To be able to use the results of the preceeding section we apply the Poisson summation formula, as in section 2 of [Zi], and the following two lemmas.

4.2. **Lemma.** *Let E be a Banach space, $1 \leq p < \infty$ and $\varphi \in \mathscr{S}(\mathbb{R}^d;E)$. Then we have*

$$\|\varphi\|_{L^p(\mathbb{R}^d;E)} \;=\; \lim_{k\to\infty} \|\varphi_{k,p}\|_{L^p(\mathbb{T}^d;E)},$$

where

$$\varphi_{k,p}(\mathfrak{x}) \;=\; 2^{-\frac{dk}{p'}} \sum_{\mathfrak{x}\in\mathbb{Z}^d} \hat{\varphi}(\mathfrak{x}/2^k)\otimes \mathfrak{e}_{\mathfrak{x}} \qquad (1/p+1/p'=1).$$

4.3. **Lemma.** *Let $(M_n)_{n\in\mathbb{N}} \subset \mathscr{M}_p(\mathbb{R}^d;X,Y)$ be a sequence of Fourier multipliers that converges almost everywhere to M. Then*

$$\|\mathscr{K}_M\|_{\mathscr{B}(L^p(\mathbb{T}^d;X),L^p(\mathbb{T}^d;Y))} \;\leq\; \sup\{\|\mathscr{K}_{M_n}\|_{\mathscr{B}(L^p(\mathbb{T}^d;X),L^p(\mathbb{T}^d;Y))} \,:\, n\in\mathbb{N}\}.$$

We now state the first of two Mikhlin-type Fourier multiplier theorems

4.4. **Theorem** (First operator-valued Mikhlin theorem). *Let X, Y be UMD spaces and $1 < p < \infty$. If the function $M : \mathbb{R}^d\setminus\{0\} \to \mathscr{B}(X,Y)$ has the property that their distributional derivatives $D^\gamma M$ of order $\gamma \leq (1,\dots,1)$ are represented by functions and moreover*

$$R(\{|x|^{|\gamma|}(D^\gamma M)(x)\,:\, x\in\mathbb{R}^d\setminus\{0\},\, \gamma\leq(1,\dots,1)\}) \;<\; \infty$$

holds, then $M \in \mathscr{M}_p(\mathbb{R}^d;X,Y)$.

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Proof. We will divide the proof into three steps.

Step 1: $M \in \mathscr{S}(\mathbb{R}^d; \mathscr{B}(X,Y))$

For $f \in \mathscr{S}(\mathbb{R}^d; X)$ we now have

$$\mathscr{K}_M f = (M(\cdot)\hat{f}(\cdot))^\vee = \check{M} * f \ \in \mathscr{S}(\mathbb{R}^d; Y).$$

By applying Lemma 4.2 we thus obtain

$$\begin{aligned} \|\mathscr{K}_M f\|_{L^p(\mathbb{R}^d;Y)} \ &= \ \lim_k \|\mathscr{K}_{M_k} f_k\|_{L^p(\mathbb{T}^d;Y)} \\ &\le \sup\{\|\mathscr{K}_{M_k}\|_{\mathscr{B}(L^p(\mathbb{T}^d;X),L^p(\mathbb{T}^d;Y))} : k \in \mathbb{N}\}\|f\|_{L^p(\mathbb{R}^d;X)}, \end{aligned} \tag{4.2}$$

where

$$M_k(\mathfrak{x}) \ := \ M(\mathfrak{x}/2^k), \quad f_k(\mathfrak{x}) \ = \ 2^{-\frac{dk}{p'}} \sum_{\mathfrak{x}\in\mathbb{Z}^d} \hat{f}(\mathfrak{x}/2^k) \otimes \mathfrak{e}_{\mathfrak{x}}, \qquad \mathfrak{x} \in \mathbb{Z}^d.$$

The goal of the next calculation is to show that

$$\|\mathscr{K}_{M_k}\|_{\mathscr{B}(L^p(\mathbb{T}^d;X),L^p(\mathbb{T}^d;Y))} \ \le \ C \quad \forall\, k \in \mathbb{N}.$$

This will be done by using Theorem 3.7. For that reason we have to secure that the Marcinkiewicz conditions $(\mathcal{M}_{cD}^{\mathcal{T}_k})$ hold for each M_k with Marcinkiewicz constants that can be estimated independently of k (see Remark 3.16). For the R-bounded collection $\mathcal{T}_k$ we choose

$$\mathcal{T}_k \ := \bigcup_{i\in\{1,2\},n\in\mathbb{N}_0} \{(\beta_{i,n} - \alpha_{i,n})^\gamma (D^\gamma M(\cdot/2^k))(x) \ : \ x \in [\alpha_{i,n}, \beta_{i,n}],\ \gamma \le (1,\dots,1)\}.$$

Here $([\alpha_{i,n};\beta_{i,n}])_{i,n}$ is the coarse decomposition. By definition of the $\mathcal{T}$-variation we have $(n \in \mathbb{N}_0)$

$$\operatorname*{var}_{\mathbf{D}_n}{}_{\mathcal{T}_k} M_k = \sum_{i=1}^{2} \sum_{\mathfrak{x}\in[\alpha_{i,n};\beta_{i,n}]} \|(\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n};\beta_{i,n}]}(\cdot/2^k))(\mathfrak{x})\|_{\mathcal{T}_k}.$$

Using Lemma 3.3, we obtain

$$\sup_{n\in\mathbb{N}_0} \operatorname*{var}_{\mathbf{D}_n}{}_{\mathcal{T}_k} M_k \ \le \ 2^d\,. \tag{4.3}$$

To apply Theorem 3.7 and Remark 3.16 we have to estimate $R(\mathcal{T}_k)$. If we define $\Delta_0 = \{0\} \subset \mathbb{R}^d$ and for $n = dr + j$, $r \in \mathbb{N}_0$, $j \in \{1,\dots,d\}$

$$\begin{aligned} \Delta_n := \{x = (x_1,\dots,x_d) \in \mathbb{R}^d \ : \ |x_1|,\dots,|x_{j-1}| < 2^{r+1}, 2^r \le |x_j| < 2^{r+1},& \\ |x_{j+1}|,\dots,|x_d| < 2^r\},& \end{aligned}$$

then from the definition of the coarse decomposition we know that (note that the following constant r depends on n)

- The sizes of the edges of the two subcubes $[\alpha_{1,n};\beta_{1,n}], [\alpha_{2,n};\beta_{2,n}]$ of $\mathbf{D}_n$ are not greater than 2^{r+1},
- $|x|_\infty \ge 2^r$ for all $x \in \Delta_n$ $(n \ge 1)$.

Now let $x \in [\alpha_{i,n};\beta_{i,n}]$ be arbitrary. Thus

$$(\beta_{i,n} - \alpha_{i,n})^\gamma (D^\gamma M(\cdot/2^k))(x) \ = \ 2^{|\gamma|} \frac{(\beta_{i,n} - \alpha_{i,n})^\gamma}{2^{(r+1)|\gamma|}} 2^{(r-k)|\gamma|} (D^\gamma M)(x/2^k)$$

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and therefore (use Remark 2.2 (d))

$$\begin{aligned} R(\mathcal{T}_k) &\le C_1 R(\bigcup_{i,n}\{2^{(r-k)|\gamma|}(D^\gamma M)(x/2^k) : x \in [\alpha_{i,n},\beta_{i,n}],\, \gamma \le (1,\dots,1)\}) \\ &= C_1 R(\bigcup_{n}\{2^{(r-k)|\gamma|}(D^\gamma M)(x/2^k) : x \in \Delta_n,\, \gamma \le (1,\dots,1)\}) \\ &= C_1 R(\bigcup_{n}\{\frac{2^{(r-k)|\gamma|}}{|x|^{|\gamma|}} \cdot |x|^{|\gamma|}(D^\gamma M)(x) : x \in \frac{1}{2^k}\Delta_n,\, \gamma \le (1,\dots,1)\}) \\ &\le C_2 R(\bigcup_{n}\{|x|^{|\gamma|}(D^\gamma M)(x) : x \in \frac{1}{2^k}\Delta_n,\, \gamma \le (1,\dots,1)\}) \\ &\le C_2 R(\{|x|^{|\gamma|}(D^\gamma M)(x) : x \in \mathbb{R}^d\setminus\{0\},\, \gamma \le (1,\dots,1)\}). \end{aligned}$$

So (4.3), (4.2) and Remark 3.16 yield

$$\|\mathscr{K}_M\| \le CR(\{|x|^{|\gamma|}(D^\gamma M)(x) : x \in \mathbb{R}^d\setminus\{0\},\, \gamma \le (1,\dots,1)\}), \tag{4.4}$$

where C does not depend on the multiplier function M.

Step 2: M is infinitely often differentiable.

Fix an infinitely often differentiable (scalar) function ϱ with compact support such that $\varrho(0) = 1$. Define for all $\varepsilon > 0$, $\varrho_\varepsilon(\cdot) := \varrho(\varepsilon\cdot)$. Now $M_\varepsilon := \varrho_\varepsilon M \in \mathscr{S}(\mathbb{R}^d;\mathscr{B}(X,Y))$ converges pointwise to M as ε goes to 0. By Lemma 4.3 and (4.4) we get

$$\begin{aligned} \|\mathscr{K}_M\| &\le \sup\{\|\mathscr{K}_{M_\varepsilon}\| : \varepsilon > 0\} \\ &\le C\sup_\varepsilon R(\{|x|^{|\gamma|}(D^\gamma M_\varepsilon)(x) : x \in \mathbb{R}^d\setminus\{0\},\, \gamma \le (1,\dots,1)\}). \end{aligned} \tag{4.5}$$

By Leibniz's formula we have

$$|x|^{|\gamma|}(D^\gamma M_\varepsilon)(x) = \sum_{\alpha+\beta=\gamma} C_{\alpha,\beta}|x|^{|\alpha|}(D^\alpha M)(x)|x|^{|\beta|}(D^\beta \varrho_\varepsilon)(x).$$

The R-bound of each term in the sum can be estimated by

$$\begin{aligned} &R(\{|x|^{|\alpha|}(D^\alpha M)(x)|x|^{|\beta|}(D^\beta\varrho_\varepsilon)(x) : x \in \mathbb{R}^d\setminus\{0\}\}) \\ &\qquad \le R(\{|x|^{|\alpha|}(D^\alpha M)(x) : x \ne 0\}) \cdot \sup_\varepsilon\{|x|^{|\beta|}(D^\beta\varrho_\varepsilon)(x) : x \ne 0\}, \end{aligned}$$

where the supremum is independent of ε. Therefore Remark 2.2 (d) and (4.5) yield

$$\begin{aligned} \|\mathscr{K}_M\| &\le C\sup_\varepsilon R(\{|x|^{|\gamma|}(D^\gamma M_\varepsilon)(x) : x \in \mathbb{R}^d\setminus\{0\},\, \gamma \le (1,\dots,1)\}) \\ &\le C_1 \sup_{\varepsilon,\gamma}\sum_{\alpha+\beta=\gamma} C_{\alpha,\beta} R(\{|x|^{|\alpha|}(D^\alpha M)(x)|x|^{|\beta|}(D^\beta\varrho_\varepsilon)(x) : x \in \mathbb{R}^d\setminus\{0\}\}) \\ &\qquad \le C_2 R(\{|x|^{|\gamma|}(D^\gamma M)(x) : x \in \mathbb{R}^d\setminus\{0\},\, \gamma \le (1,\dots,1)\}). \end{aligned}$$

Here C_2 is again independent of M.

Step 3: M arbitrary as in the assumption.

Choose an infinitely often differentiable (scalar) function ϱ with $\varrho \ge 0$, $\|\varrho\|_1 = 1$ and $\operatorname{supp}\varrho \subset [-1,1]^d$. For $\varepsilon > 0$ define $\varrho_\varepsilon(\cdot) := \varepsilon^{-d}\varrho(\cdot/\varepsilon)$. Now $M_\varepsilon := M * \varrho_\varepsilon$ is

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infinitely often differentiable and converges almost everywhere to M as ε converges to 0. The result from the second step in connection with Lemma 4.3 lead to

$$\text{(4.6)}\quad \begin{aligned} \|\mathscr{K}_M\| &\le \sup\{\|\mathscr{K}_{M_\varepsilon}\| : \varepsilon > 0\} \\ &\le C \sup_\varepsilon R(\{|x|^{|\gamma|}(D^\gamma[M * \varrho_\varepsilon])(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \le (1,\dots,1)\}). \end{aligned}$$

Since for arbitrary $\gamma \le (1,\dots,1)$

$$\begin{aligned} & R(\{|x|^{|\gamma|}(D^\gamma[M * \varrho_\varepsilon])(x) : 0 < |x|_\infty \le 2\varepsilon\}) \\ = \ & R(\{\frac{|x|^{|\gamma|}}{(2\varepsilon)^{|\gamma|}} \cdot (2\varepsilon)^{|\gamma|} \cdot \|D^\gamma \varrho_\varepsilon\|_1 \cdot (M * \frac{D^\gamma \varrho_\varepsilon}{\|D^\gamma \varrho_\varepsilon\|_1})(x) : 0 < |x|_\infty \le 2\varepsilon\}) \\ \le \ & C_1 (2\varepsilon)^{|\gamma|} \cdot \|D^\gamma \varrho_\varepsilon\|_1 \cdot R(\{M(x) : x \in \mathbb{R}^d \setminus \{0\}\}) \\ = \ & C_1 2^{|\gamma|} \cdot \|D^\gamma \varrho\|_1 \cdot R(\{M(x) : x \in \mathbb{R}^d \setminus \{0\}\}), \end{aligned}$$

we thus in particular obtain

$$\text{(4.7)}\quad \begin{aligned} & R(\{|x|^{|\gamma|}(D^\gamma[M * \varrho_\varepsilon])(x) : 0 < |x|_\infty \le 2\varepsilon, \gamma \le (1,\dots,1)\}) \\ & \qquad \le C_2 R(\{|x|^{|\gamma|}(D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \le (1,\dots,1)\}). \end{aligned}$$

The above estimations are consequences of Remark 2.2 (d) and Lemma 2.3.

For arbitrary $\varepsilon > 0$, ϱ_ε has its support in $[-\varepsilon,\varepsilon]^d$ and so for any $\alpha \le (1,\dots,1)$ and each $|x|_\infty \ge 2\varepsilon$

$$\begin{aligned} |x|^{|\alpha|}(D^\alpha[M * \varrho_\varepsilon])(x) &= |x|^{|\alpha|} \int_{\{\xi : |x-\xi| \le \varepsilon\}} (D^\alpha M)(\xi) \varrho_\varepsilon(x-\xi)\, d\xi \\ &= \int_{\{\xi : |x-\xi| \le \varepsilon\}} |\xi|^{|\alpha|}(D^\alpha M)(\xi) \cdot \frac{|x|^{|\alpha|}}{|\xi|^{|\alpha|}} \varrho_\varepsilon(x-\xi)\, d\xi \\ &= C_1 \int_{\{\xi : |x-\xi| \le \varepsilon\}} |\xi|^{|\alpha|}(D^\alpha M)(\xi) \cdot \frac{|x|^{|\alpha|}}{C_1 |\xi|^{|\alpha|}} \varrho_\varepsilon(x-\xi)\, d\xi, \end{aligned}$$

where C_1 is a constant which fulfills for all α and $\varepsilon > 0$

$$\sup\{\frac{|x|^{|\alpha|}}{|\xi|^{|\alpha|}} : |x-\xi| \le \varepsilon, |x|_\infty \ge 2\varepsilon\} \le C_1.$$

Since $\|\varrho_\varepsilon\|_1 = 1$ we obtain that for each x with $|x|_\infty \ge 2\varepsilon$

$$\int\limits_{\{\xi : |x-\xi| \le \varepsilon\}} |\xi|^{|\alpha|}(D^\alpha M)(\xi) \cdot \frac{|x|^{|\alpha|}}{C_1 |\xi|^{|\alpha|}} \varrho_\varepsilon(x-\xi)\, d\xi \in \operatorname{aco}(\{|x|^{|\alpha|}(D^\alpha M)(x) : x \in \mathbb{R}^d \setminus \{0\}\})$$

and thus by Lemma 2.3

$$\text{(4.8)}\quad \begin{aligned} & R(\{|x|^{|\gamma|}(D^\gamma M * \varrho_\varepsilon)(x) : |x|_\infty \ge 2\varepsilon, \gamma \le (1,\dots,1)\}) \\ & \qquad \le C_2 R(\{|x|^{|\gamma|}(D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \le (1,\dots,1)\}). \end{aligned}$$

Using (4.6), (4.7) the theorem is proved. □

4.5. **Theorem** (Second operator-valued Mikhlin theorem)**.** *If X and Y are UMD-spaces with the property (α) and where the function $M : \mathbb{R}^d \setminus \{0\} \to \mathscr{B}(X,Y)$ has*

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the property that their distributional derivatives $D^\gamma M$ of order $\gamma \leq (1,\dots,1)$ are represented by functions which fulfill

$$R(\{x^\gamma (D^\gamma M)(x) \,:\, x \in \mathbb{R}^d \setminus \{0\},\, \gamma \leq (1,\dots,1)\}) \;<\; \infty\,,$$

then $M \in \mathscr{M}_p(\mathbb{R}^d; X, Y)$ $(1 < p < \infty)$.

Proof. Again we divide the proof into three steps.

Step 1: $M \in \mathscr{S}(\mathbb{R}^d; \mathscr{B}(X,Y))$

Arguing as in the first step of the proof of Theorem 4.4 we obtain in analogy to (4.3)

$$\sup_{\nu \in \mathbb{N}_0^d} \operatorname{var}_{\mathcal{T}_k}_{\mathbf{D}_\nu} M_k \;\leq\; 2^d\,,$$

where the collection $\mathcal{T}_k$ is defined by

$$\mathcal{T}_k \;:=\; \bigcup_{\substack{i \in \{1,\dots,2^d\} \\ \nu \in \mathbb{N}_0^d}} \{(\beta_{i,\nu} - \alpha_{i,\nu})^\gamma (D^\gamma M(\cdot/2^k))(x) \,:\, x \in [\alpha_{i,\nu}, \beta_{i,\nu}],\, \gamma \leq (1,\dots,1)\}.$$

Here $([\alpha_{i,\nu}; \beta_{i,\nu}])_{i,\nu}$ is the fine decomposition. Again it remains to show that $\mathcal{T}_k$ is R-bounded. Now we define for $\nu = (\nu_1,\dots,\nu_d) \in \mathbb{Z}^d$

$$\Delta_\nu := \{x \in \mathbb{R}^d \setminus \{0\} \,:\, 2^{\nu_i - 1} \leq |x_i| < 2^{\nu_i} \text{ for } i \in \{1,\dots,d\}\}.$$

For the fine decomposition we know

- The sizes of the edges of the 2^d subcuboids $[\alpha_{i,\nu}; \beta_{1,\nu}]$ of $\mathbf{D}_\nu$ in the j-th coordinate direction are not larger than 2^{ν_j}.
- For $x \in \Delta_\nu$, $(\nu \in \mathbb{N}^d)$ we have $|x_j| \geq 2^{\nu_j - 1}$.

If $x \in [\alpha_{i,\nu}; \beta_{i,\nu}]$ is arbitrary, we use the identity $((2^\alpha) := (2^{\alpha_1},\dots,2^{\alpha_d}),\ \alpha \in \mathbb{Z}^d)$

$$(\beta_{i,\nu} - \alpha_{i,\nu})^\gamma (D^\gamma M(\cdot/2^k))(x) \;=\; \frac{(\beta_{i,n} - \alpha_{i,n})^\gamma}{(2^\nu)^\gamma} \cdot (2^\nu)^\gamma \cdot 2^{-k|\gamma|} (D^\gamma M)(x/2^k)$$

and Remark 2.2 (d) to get

$$\begin{aligned}
R(\mathcal{T}_k) \;&\leq\; C_1 R\Big(\bigcup_{i,\nu} \Big\{\frac{(2^\nu)^\gamma}{2^{k|\gamma|}} (D^\gamma M)(x/2^k) \,:\, x \in [\alpha_{i,\nu}, \beta_{i,\nu}],\, \gamma \leq (1,\dots,1)\Big\}\Big) \\
&=\; C_1 R\Big(\bigcup_\nu \{(2^\nu)^\gamma \cdot 2^{-k|\gamma|} (D^\gamma M)(x/2^k) \,:\, x \in \Delta_\nu,\, \gamma \leq (1,\dots,1)\}\Big) \\
&=\; C_1 R\Big(\bigcup_\nu \Big\{\frac{(2^\nu)^\gamma \cdot 2^{-k|\gamma|}}{x^\gamma} \cdot x^\gamma (D^\gamma M)(x) \,:\, x \in \frac{1}{2^k}\Delta_\nu,\, \gamma \leq (1,\dots,1)\Big\}\Big) \\
&\leq\; C_2 R\Big(\bigcup_\nu \{x^\gamma (D^\gamma M)(x) \,:\, x \in \frac{1}{2^k}\Delta_\nu,\, \gamma \leq (1,\dots,1)\}\Big) \\
&\leq\; C_2 R(\{x^\gamma (D^\gamma M)(x) \,:\, x \in \mathbb{R}^d \setminus \{0\},\, \gamma \leq (1,\dots,1)\})
\end{aligned}$$

and therefore

$$\|\mathscr{K}_M\| \;\leq\; C R(\{x^\gamma (D^\gamma M)(x) \,:\, x \in \mathbb{R}^d \setminus \{0\},\, \gamma \leq (1,\dots,1)\}),$$

where C does not depend on the multiplier function M.

Step 2: The case where M is infinitely often differentiable can be treated in the same way as in Theorem 4.4.

Step 3: M fulfills the assumption of Theorem 4.5.

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For this step we modify the first part of the proof of Theorem 4.4 and obtain similarly to (4.7) for all $0 < \varepsilon \leq 1$

$$(4.9)\quad R(\{x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) \,:\, 0 < |x|_\infty \leq \varepsilon,\, \gamma \leq (1,\dots,1)\}) \\ \leq\; C\,R(\{x^\gamma(D^\gamma M)(x) \,:\, x \in \mathbb{R}^d \setminus \{0\},\, \gamma \leq (1,\dots,1)\}).$$

The remaining part will be treated in the following way:

Choose $\gamma \leq (1,\dots,1)$ arbitrary. Applying the binomial formula we get

$$x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) \;=\; \sum_{\alpha+\beta\leq\gamma} C_{\alpha,\beta}((\xi^\alpha(D^\alpha M)(\xi)) * (\xi^\beta(D^\beta\varrho_\varepsilon)(\xi)))(x).$$

Since $\operatorname{supp}\varrho_\varepsilon \subset [-\varepsilon,\varepsilon]^d$, our assumption on the function M enables us to write the above convolution for each $|x|_\infty > \varepsilon$ as follows:

$$x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) \;=\; \sum_{\alpha+\beta=\gamma} C_{\alpha,\beta}\int_{[-\varepsilon,\varepsilon]^d}(x-\xi)^\alpha(D^\alpha M)(x-\xi)\xi^\beta(D^\beta\varrho_\varepsilon)(\xi)\,d\xi\,.$$

Now a similar argument as used in step 3 of the proof of Theorem 4.3 yields

$$\begin{aligned} &R(\{x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) \,:\, |x|_\infty > \varepsilon\}) \\ &\leq\; C_1 R(\{x^\alpha(D^\alpha M)(x) \,:\, x \in \mathbb{R}^d \setminus \{0\},\, \alpha \leq (1,\dots,1)\}) \\ &\qquad\qquad \cdot \sup_\varepsilon\{\|\xi^\beta(D^\beta\varrho_\varepsilon)(\xi)\| \,:\, \beta \leq (1,\dots,1)\} \\ &=\; C_1 R(\{x^\alpha(D^\alpha M)(x) \,:\, x \in \mathbb{R}^d \setminus \{0\},\, \alpha \leq (1,\dots,1)\}) \\ &\qquad\qquad \cdot \sup\{\|\xi^\beta(D^\beta\varrho)(\xi)\| \,:\, \beta \leq (1,\dots,1)\} \end{aligned}$$

and thus also

$$R(\{x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) \,:\, |x|_\infty > \varepsilon,\, \gamma \leq (1,\dots,1)\}) \\ \leq\; C_2\,R(\{x^\gamma(D^\gamma M)(x) \,:\, x \in \mathbb{R}^d \setminus \{0\},\, \gamma \leq (1,\dots,1)\}).$$

Together with (4.9) this completes the proof. □

Note. Observe that the weight function $|x|^{|\gamma|}$ of Theorem 4.4 is larger than $|x^\gamma|$ from Theorem 4.5.

4.6. *Remark.* i) Theorem 4.4 and Theorem 4.5 are generalizations of Proposition 3 in [Zi].

ii) If X and Y are Hilbert spaces, then the unit ball of $\mathscr{B}(X,Y)$ is R-bounded and both theorems reduce to the result of Schwartz, which assumes that

$$\|x^\gamma(D^\gamma M)(x)\| \;\leq C \;< \infty$$

for all $x \in \mathbb{R}^d \setminus \{0\}$ and $\gamma \leq (1,\dots,1)$.

References

[BG] E. Berkson and T.A. Gillespie, *Spectral decompositions and harmonic analysis on UMD spaces*, Studia Math. 112, (1994), no. 1, 13-49. MR1307598 (96c:42022)

[BL] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer 1976. MR0482275 (58:2349)

[Bl] S. Blunck, *Maximal regularity of discrete and continuous time evolution equations*, Studia Math. 146 (2001), 157–176. MR1853519 (2002g:34125)

[Bou1] J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Arkiv Math. 21 (1983), 163-168. MR0727340 (85a:46011)

[Bou2] ———, *Vector-valued Singular Integrals and the H^1-BMO Duality*, Probability Theory and Harmonic Analysis, Monographs and Textbooks in Pure and Appl. Math. 98, Marcel Dekker, New York, 1986, 1-19. MR0830227 (87j:42049b)

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[Bu] D. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, Proc. of Conf. on Harmonic Analysis in Honor of Antoni Zygmund, Chicago 1981, Wadsworth Publishers, Belmont, Calif., 1983, 270-286. MR0730072 (85i:42020)

[CPSW] Ph. Clément, B. de Pagter, F.A. Sukochev and H. Witvliet, *Schauder Decompositions and Multiplier Theorems*, Studia Mathematica 138 (2000), 135-163. MR1749077 (2002c:47036)

[DJT] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Cambridge University Press, 1995. MR1342297 (96i:46001)

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II (Function Spaces)*, Springer-Verlag 1979. MR0540367 (81c:46001)

[McC] T. Mc Connell, *On Fourier multiplier transformations of Banach-valued functions*, Trans. Amer. Math. Soc. 285 (1984), 739-757. MR0752501 (87a:42033)

[Pi1] G. Pisier, *Les inegalités de Khintchine-Kahane d'après C. Borel*, Seminaire sur la géométrie des espaces de Banach 7, (1977-1978), Ecole Polytechnique, Paris. MR0520209 (81c:60005)

[Pi2] ———, *Some results on Banach spaces without local unconditional structure*, Comp. Math. 37 (1978), 3-19. MR0501916 (80e:46012)

[We] L. Weis, *Operator-Valued Fourier Multiplier Theorems and Maximal L^p-regularity*, Math. Ann. 319 (2001), 735–758. MR1825406 (2002c:42016)

[Zi] F. Zimmermann, *On vector-valued Fourier multiplier theorems*, Studia Mathematica, T. XCIII (1989), 201-222. MR1030488 (91b:46031)

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