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# ON OPERATOR-VALUED SOLUTIONS OF D'ALEMBERT'S FUNCTIONAL EQUATION, I

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Let $X$ be a Banach space and let $\mathscr{L}(X;X)$ denote the Banach algebra of all bounded linear operators of $X$ into $X$. A mapping $\mathscr{C}\colon (-\infty,\infty)\to\mathscr{L}(X;X)$ will be called *cosine function* if it satisfies the d'Alembert's functional equation

$$\mathscr{C}(t+s)+\mathscr{C}(t-s) = 2\mathscr{C}(t)\mathscr{C}(s), \qquad -\infty < s, t < \infty, \tag{1}$$

and if, moreover,

$$\mathscr{C}(0) = 1. \tag{2}$$

According to Sova [3], the infinitesimal generator of such a cosine function is the linear operator $A$ from $X$ into $X$, defined by the conditions

$$D(A) = \{x\colon x\in X,\ s-\lim_{h\to 0} h^{-2}\big(\mathscr{C}(h)x-x\big)\text{exists}\},$$
$$Ax = s-\lim_{h\to 0} 2h^{-2}\big(\mathscr{C}(h)x-x\big) \text{ for } x\in D(A).$$

As Sova proved, if $\mathscr{L}(X;X)$-valued cosine function $\mathscr{C}(t)$ is strongly continuous on $(-\infty,\infty)$, then $A$ is closed and $D(A)$ is dense in $X$. Moreover, $A\in\mathscr{L}(X;X)$ if and only if $\mathscr{C}(t)$ is continuous on $(-\infty,\infty)$ in the sense of the norm in $\mathscr{L}(X;X)$ and then

$$\mathscr{C}(t) = 1+\sum_{n=1}^{\infty}\frac{t^{2n}A^n}{(2n)!}. \tag{3}$$

In the numerical case, i.e. if $X = C$ (the field complex numbers), then, as is known, all continuous solutions of (1)-(2) have the form

$$\mathscr{C}(t) = \tfrac{1}{2}e^{bt}+\tfrac{1}{2}e^{-bt},$$

where $b$ is a complex number. This suggests the problem of a representation of general $\mathscr{L}(X;X)$-valued cosine function in the form

$$\mathscr{C}(t) = \tfrac{1}{2}G(t)+\tfrac{1}{2}G(-t), \tag{4}$$

where $G(t)$, $-\infty < t < \infty$, is a one-parameter group of operators belonging to $\mathscr{L}(X; X)$. The paper may be considered as a step in this direction. It gives three simple theorems and two examples which, perhaps, are even more interesting.

THEOREM 1. *Let $X$ be a complex Banach space and $\mathscr{C}(t)$ an $\mathscr{L}(X; X)$--valued cosine function bounded on $(-\infty, \infty)$ and continuous there in the sense of the norm in $\mathscr{L}(X; X)$. Then there is an operator $B \in \mathscr{L}(X; X)$ such that*

$$\mathscr{C}(t) = \tfrac{1}{2}\exp(tB) + \tfrac{1}{2}\exp(-tB), \quad -\infty < t < \infty. \tag{5}$$

Proof. In view of (3) it is sufficient to show that the infinitesimal generator $A \in \mathscr{L}(X; X)$ of $\mathscr{C}(t)$ has a square root $B \in \mathscr{L}(X; X)$. Since $\mathscr{C}(t)$ is bounded, for every complex $\lambda$ with $\operatorname{Re}\lambda > 0$ we have

$$\int_0^\infty e^{-\lambda t}\mathscr{C}(t)\,dt = \lambda(\lambda^2 - A)^{-1} \in \mathscr{L}(X; X)$$

and, therefore,

$$\|\lambda(\lambda^2 - A)^{-1}\| \leqslant M(\operatorname{Re}\lambda)^{-1}, \quad M = \text{const}, \quad \text{for } \operatorname{Re}\lambda > 0. \tag{6}$$

Consequently, $\|(\lambda - A)^{-1}\| \leqslant M/\lambda$ for every real $\lambda > 0$ and the existence of $B$ follows from the theorem on fractional powers of closed operators given in the book of S. G. Krein [3], chapter I, § 5. Similar theorems are to be found in the book of Yosida [4], chapter IX, § 11.

However, in our case it is possible to prove the existence of a square root of $A$ by more simple reasonings. One of them is the following. It follows from (6) that the spectrum of $A$ is contained in a subinterval $[-R, 0]$, $R > 0$, of the real axis, and

$$\|(\lambda - A)^{-1}\| \leqslant \frac{M}{|\lambda|\cos(\frac{1}{2}\operatorname{Arg}\lambda)} \quad \text{for } -\pi < \operatorname{Arg}\lambda < \pi. \tag{7}$$

Let $C$ be the oriented rectangular contour with edges $-i$, $i$, $-R-1+i$, and $-R-1-i$. Then, for every $\varepsilon \in (0, \frac{1}{2})$, the operator $B_\varepsilon \in \mathscr{L}(X; X)$ defined by Dunford integral

$$B_\varepsilon = \frac{1}{2\pi i}\int_C i\sqrt{-\lambda}\,(\lambda + \varepsilon - A)^{-1}d\lambda,$$

where $\sqrt{\ }$ denotes the main branch of the square root, satisfies the equality $B_\varepsilon^2 = A - \varepsilon$.

From (7) and from the Lebesgue bounded convergence theorem it follows that there exists the limit in the sense of the norm in $\mathscr{L}(X; X)$,

$$B = \lim_{\varepsilon \to +0} B_\varepsilon = \frac{1}{2\pi i}\int_C i\sqrt{-\lambda}(\lambda - A)^{-1}d\lambda,$$

and, obviously, we have $B^2 = A$.

Another proof of the existence of a square root of $A$ in theorem 1 may be obtained by an application to $a = -A$ of the following theorem, which, together with the proof, was communicated to the author by Professor C. Ryll-Nardzewski.

THEOREM 2. *Let $A$ be a Banach algebra with the unit $e$ over real field. Let $a \in A$ be such that $(\lambda e + a)^{-1}$ exists in $A$ for every $\lambda > 0$ and that*

$$\|(\lambda e + a)^{-1}\| \leqslant \frac{M}{\lambda}, \quad M = \text{const}, \textit{ for } \lambda > 0. \tag{8}$$

*Then the integral*

$$b = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (\lambda e + a)^{-1} a \, d\lambda$$

*converges and $b^2 = a$.*

Proof. We may assume that $a \neq 0$, because in the contrary case there is nothing to prove. For any $t \geqslant 0$ and $\varepsilon \geqslant 0$ consider the integral

$$b(t, \varepsilon) = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} \big((\lambda + \varepsilon)e + ta\big)^{-1} (\varepsilon e + ta) \, d\lambda.$$

Since

$$\big\|\big((\lambda + \varepsilon)e + ta\big)^{-1}\big\| = t^{-1} \left\| \left( \frac{\lambda + \varepsilon}{t} e + a \right)^{-1} \right\| \leqslant \frac{M}{\lambda + \varepsilon} \tag{9}$$

and

$$\big((\lambda + \varepsilon)e + ta\big)^{-1} (\varepsilon e + ta) = e - \lambda \big((\lambda + \varepsilon)e + ta\big)^{-1}$$

for $t \geqslant 0$, $\varepsilon \geqslant 0$ and $\lambda > 0$, we infer that the integral converges at infinity as well as at zero, and that $b(t, \varepsilon)$ is a continuous function of $t$ and $\varepsilon$ for $t \geqslant 0$ and $\varepsilon \geqslant 0$.

Obviously, $b = b\,(1, 0)$ and therefore it is sufficient to show that

$$b^2(t, \varepsilon) = \varepsilon e + ta \tag{10}$$

for every $\varepsilon > 0$ and $t \geqslant 0$. If $\varepsilon > 0$ and $t_0 \geqslant 0$ are fixed then, by (9), for $h \in (\max\{-t_0, -\varepsilon/\|a\|\}, \varepsilon/\|a\|)$ we have

$$b(t_0 + h, \varepsilon) = \big((t_0 + h)a + \varepsilon e\big) \frac{1}{\pi} \sum_{n=0}^\infty (-ha)^n \int_0^\infty \lambda^{-1/2} \big((\lambda + \varepsilon)e + t_0 a\big)^{-n-1} d\lambda$$

and so, for fixed $\varepsilon > 0$, $b(t, \varepsilon)$ is an analytic function of $t$ on $[0, \infty)$. Therefore it is sufficient to verify that (10) holds for every $\varepsilon > 0$ and $t \in \left[0, \dfrac{\varepsilon}{\|a\|}\right)$. But then we have

$$b(t, \varepsilon) = (ta + \varepsilon e) \sum_{n=0}^\infty c_n(\varepsilon)(ta)^n,$$

where

$$c_n(\varepsilon) = \frac{(-1)^n}{\pi}\int\limits_0^\infty \lambda^{-1/2}(\lambda+\varepsilon)^{-n-1}d\lambda = \frac{1}{n!}\,\frac{d^n\varepsilon^{-1/2}}{d\varepsilon^n},$$

so that

$$\sum_{n=0}^{\infty} c_n(\varepsilon)(ta)^n = (\varepsilon e+ta)^{-1/2}.$$

THEOREM 3. *Let $X$ be a Banach space and $\mathscr{C}(t)$ a strongly continuous $\mathscr{L}(X;X)$-valued cosine function. Then there are: a Banach space $Y$, an isomorphic imbedding $\mathscr{I}$ of $X$ into $Y$, and a strongly continuous one parameter group $\{G(t)\colon -\infty<t<\infty\}\subset\mathscr{L}(Y;Y)$, such that*

$$\big(G(t)+G(-t)\big)\mathscr{I}(X)\subset\mathscr{I}(X),\qquad -\infty<t<\infty,$$

*and*

$$\mathscr{C}(t) = \tfrac{1}{2}\mathscr{I}^{-1}\big(G(t)+G(-t)\big)\mathscr{I},\qquad -\infty<t<\infty.$$

Proof. As Sova [2] proved, for every strongly continuous $\mathscr{L}(X;X)$-valued cosine function $\mathscr{C}(t)$ there are constants $M$ and $\omega$ such that

$$\|\mathscr{C}(t)\|\leqslant Me^{\omega|t|},\qquad -\infty<t<\infty.$$

Let $Y$ be the space of all $X$-valued functions $x(s)$ strongly continuous for $-\infty<s<\infty$ and such that

$$\sup_{-\infty<s<\infty} e^{-\omega|s|}\|x(s)\|<\infty.$$

Under the norm

$$\|x\|_Y = \sup_{-\infty<s<\infty} e^{-\omega|s|}\|x(s)\|_X$$

$Y$ is a Banach space. Let the operator $\mathscr{I}\in\mathscr{L}(X;Y)$ be defined by

$$(\mathscr{I}x)(s) = \mathscr{C}(s)x,\ x\in X,\qquad -\infty<s<\infty.$$

Let $G(t)$ be the group of left translations in $Y$. If $x\in X$, then

$$\begin{aligned}\big[\big(G(t)+G(-t)\big)\mathscr{I}x\big](s) &= \mathscr{C}(s+t)x+\mathscr{C}(s-t)x\\ &= \mathscr{C}(s)[2\mathscr{C}(t)x] = [\mathscr{I}(2\mathscr{C}(t)x)](s).\end{aligned}$$

Consequently,

$$\big(G(t)+G(-t)\big)\mathscr{I} = 2\mathscr{I}\mathscr{C}(t),\qquad -\infty<t<\infty,$$

which is all we have to prove.

Example 1. Let $X = C^2$, the two dimensional complex space, and let the $\mathscr{L}(C^2;C^2)$-valued cosine function be represented by the matrices

$$\mathscr{C}(t) = \begin{pmatrix}1 & \frac{1}{2}t^2\\ 0 & 1\end{pmatrix}.$$

Then its infinitesimal generator is a matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which has no square root. Hence there is no $(2\times 2)$-matrix $B$ satisfying (5), because otherwise we should have $B^2 = A$. But if we put $Y = C^3$ and $\mathscr{J}(x_1, x_2) = (x_1, x_2, 0)$ for every pair $(x_1, x_2)\in C^2$, then

$$\mathscr{C}(t) = \tfrac{1}{2}\mathscr{J}^{-1}\left[\exp\left(t\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right) + \exp\left(-t\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right)\right]\mathscr{J}.$$

This example shows, that the assumption of boundedness in theorem 1 is essential.

Example 2. Let $Y$ denote the space of all complex functions of real variable, continuous on $(-\infty, \infty)$ and periodic with period $2\pi$. We consider $Y$ with the norm $\|x\| = \sup\limits_{0\leqslant s\leqslant 2\pi} |x(s)|$. Let $X$ be the subspace of $Y$ formed by all impair functions. Let $\mathscr{G}(t)$, $-\infty < t < \infty$, be the group of left translations in $Y$, i.e.

$$\big(\mathscr{G}(t)x\big)(s) = x(s+t), \quad x\in X, \ -\infty < s, t < \infty.$$

Let

$$\mathscr{C}(t) = \tfrac{1}{2}\big(\mathscr{G}(t)+\mathscr{G}(-t)\big)\big|_X, \quad -\infty < t < \infty.$$

Then $\mathscr{C}(t)$ is an $\mathscr{L}(X; X)$-valued cosine function bounded and strongly continuous on $(-\infty, \infty)$. We shall show that there is no strongly continuous one parameter group $\{G(t)\colon -\infty < t < \infty\} \subset \mathscr{L}(X; X)$ satisfying (4). Indeed, the set $\{x_n\colon n = 1, 2, \ldots\}$, where $x_n(s) = \sin(ns)$, is linearly dense in $X$ and if the projection operators $P_n$, $n = 1, 2, \ldots$, are defined by the formula

$$P_n x = \left(\frac{2}{\pi}\int_0^{\pi} x_n(s)x(s)\,ds\right)x_n,$$

then, since $[\mathscr{C}(t)x_n](s) = \cos nt\cdot\sin ns$, we have

$$P_n = \frac{2}{\pi}\int_0^{\pi}\cos(nt)\mathscr{C}(t)\,dt.$$

Consequently, for any group $G(t)$ satisfying (4), we should have

$$P_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nt)G(t)\,dt$$

and hence

$$P_n G(t) = G(t)P_n, \quad -\infty < t < \infty, \ n = 1, 2, \ldots \tag{11}$$

Since

$$\mathscr{C}(t)P_n = \cos(nt)P_n,$$

it follows from (4) and (11) that

$$G(t)P_n = e^{in\varepsilon(n)t}P_n$$

for every $n = 1, 2, \ldots$, where $\varepsilon(n) = 1$ or $\varepsilon(n) = -1$, and, consequently,

$$\big(G(t)x_n\big)(s) = \cos nt\cdot\sin ns - \frac{i\varepsilon(n)}{2}[\cos n(s+t)-\cos n(s-t)]$$

$$= \big(\mathscr{C}(t)x_n\big)(s)+\tfrac{1}{2}(T*x_n)(s+t)-\tfrac{1}{2}(T*x_n)(s-t),$$

where $T$ is a periodic distribution with Fourier series

$$(12)\qquad T\sim\sum_{n=1}^{\infty}2i\varepsilon(n)\sin(ns) = \sum_{n=-\infty}^{\infty}(\operatorname{sign} n)\varepsilon(|n|)e^{ins}.$$

It follows that for every infinitely differentiable function $x\in X$ we have

$$\big(G(t)x\big)(s) = \big(\mathscr{C}(t)x\big)(s)+\tfrac{1}{2}(T*x)(s+t)-\tfrac{1}{2}(T*x)(s-t)$$

and so

$$\|T*x\|\leqslant\inf_{0\leqslant s\leqslant 2\pi}|(T*x)(s)|+2\sup_{0\leqslant t\leqslant 2\pi}\|G(t)x\|+2\sup_{0\leqslant t\leqslant 2\pi}\|\mathscr{C}(t)x\|.$$

Since

$$\inf_{0\leqslant s\leqslant 2\pi}|(T*x)(s)|\leqslant\left(\frac{1}{2\pi}\int_0^{2\pi}|(T*x)(s)|^2ds\right)^{1/2} = \left(\frac{1}{2\pi}\int_0^{2\pi}|x(s)|^2ds\right)^{1/2}\leqslant\|x\|,$$

it follows that there is a constant $C<\infty$ such that

$$|\langle T, x\rangle| = |(T*x)(0)|\leqslant C\sup_{0\leqslant\sigma\leqslant 2\pi}|x(\sigma)|$$

for every infinitely differentiable function $x\in X$. Since $T$ is impair, it follows that $T$ is a measure. But this is in contradiction to a result of Helson [1], which says that if the sequence of Fourier coefficients of a measure takes only a finite number of distinct values, then this sequence can be made periodic by a change of values of a finite number of its elements. Of course, the sequence $\varepsilon(n)\operatorname{sign} n$, $n = 0, \pm 1, \pm 2, \ldots$, which is impair and has only one element equal zero, cannot be made periodic by such a change.

Let us remark that also without the theorem of Helson we may prove that (12) cannot be Fourier series of a measure. Namely, it follows from (12) that

$$T*T = \delta_0-\frac{1}{2\pi}m,$$

where $\delta_0$ is the unit mass at zero and $m$ is the Lebesgue measure. If $T$ would be a measure and $T_a$ its atomic part, then $T_a$ would be impair and so $T_a * 1 = 0$. But on the other hand the atomic part of $T*T$ is $T_a * T_a$, so that $T_a * T_a = \delta_0$ and $\langle T_a, T_a * 1\rangle = \langle \delta_0, 1\rangle = 1$.

The example presented above shows that the assumption of continuity of $\mathscr{C}(t)$ in the sense of the norm in $\mathscr{L}(X; X)$ is essential in theorem 1.

A similar example may be constructed in the space of functions almost periodic in the sense of Bohr. Namely, let $Z$ be the space of all complex, almost periodic, impair functions on $(-\infty, \infty)$ under the norm

$$\|x\| = \sup_{-\infty<s<\infty} |x(s)|, \qquad x \in Z,$$

and let $\mathscr{C}(t)$ be the strongly continuous cosine operator function in the space $Z$ defined by the formula

$$\big(\mathscr{C}(t)x\big)(s) = \tfrac{1}{2}x(s+t) + \tfrac{1}{2}x(s-t), \qquad x \in Z, \; -\infty < s, t < \infty.$$

Then, for any $x \in Z$ fixed, $\mathscr{C}(t)x$ is a strongly almost periodic $Z$-valued function of $t$ on $(-\infty, \infty)$. For each $\tau > 0$ let $x_\tau \in Z$ be defined by

$$x_\tau(s) = \sin \tau s, \qquad -\infty < s < \infty,$$

and let the projection operator $P_\tau$ be defined by

$$P_\tau x = \left(\lim_{T\to\infty} \frac{1}{T} \int_{T}^{T} (\sin \tau u) x(u)\, du\right) x_\tau, \qquad x \in Z.$$

Since

$$\mathscr{C}(t) x_\tau = (\cos \tau t) x_\tau,$$

it follows that if $p \in Z$ is a trigonometric polynomial, i.e. if $p = \sum_{k=1}^{n} a_k x_{\tau_k}$, then

$$P_\tau p = \sum_{k=1}^{n} a_k \left(\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} \cos \tau t \cos \tau_k t\, dt\right) x_{\tau_k} = s-\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} (\cos \tau t) \mathscr{C}(t) p\, dt.$$

Since the set of all impair trigonometric polynomials is dense in $Z$, it follows that

$$P_\tau x = s-\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} (\cos \tau t) \mathscr{C}(t) x\, dt, \qquad \tau > 0, \; x \in Z. \tag{13}$$

Using (13), we shall show that there is no strongly continuous one parameter group $G(t)$ of bounded linear operators in the space $Z$ such that

$$\mathscr{C}(t) = \tfrac{1}{2}G(t) + \tfrac{1}{2}G(-t).$$

Indeed, if such a group $G(t)$ would exist, then by (13) we should have

$$P_\tau x = \mathrm{s}-\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} (\cos \tau t) G(t) x\, dt, \qquad \tau > 0, \; x \in Z,$$

from which we would get

$$P_\tau G(t) = G(t) P_\tau, \qquad \tau > 0, \; -\infty < t < +\infty,$$

and, consequently,

$$G(t) X \in X, \qquad -\infty < t < +\infty,$$

where $X$ is the space of all periodic, continuous, impair complex-valued functions having period $2\pi$. Hence $G(t)|_X$ would be a one-parameter group of operators in the space $X$ with the properties which are impossible by the antecedent reasoning.

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