ON OPTIMAL ARBITRAGE

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Abstract

In a Markovian model for a financial market, we characterize the best arbitrage with respect to the market portfolio that can be achieved using non-anticipative investment strategies, in terms of the smallest positive solution to a parabolic partial differential inequality; this is determined entirely on the basis of the covariance structure of the model. The solution is also used to generate the investment strategy that realizes the best possible arbitrage. Some extensions to non-Markovian situations are also presented.

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1 Introduction

In a Markovian model for an equity market with mean rates of return $\mathrm{b}_i(\mathfrak{X}(t))$ and covariance rates $\mathrm{a}_{ij}(\mathfrak{X}(t))$, $1 \le i,j \le n$ for its asset prices $\mathfrak{X}(t) = (X_1(t), \cdots, X_n(t))' \in (0,\infty)^n$ at time t, what is the smallest amount of initial capital starting with which, and using non-anticipative investment strategies, one can match or exceed the performance of the market by the end of a given time-horizon $[0,T]$? What are the weights assigned to the different assets by an investment strategy that accomplishes this?

Answers: under appropriate conditions, $U(T, \mathfrak{X}(0)) \big(X_1(0) + \cdots + X_n(0) \big)$ and

$$X_i(t)\, D_i \log U\big(T-t, \mathfrak{X}(t)\big) \,+\, \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}\,, \qquad i = 1, \cdots, n\,, \quad t \in [0,T]$$

respectively, where $U : [0,\infty) \times (0,\infty)^n \to (0,1]$ is the smallest non-negative solution of the linear parabolic partial differential inequality

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{x}) \,\ge\, \widehat{\mathcal{L}}\, U(\tau,\mathbf{x})\,, \quad (\tau,\mathbf{x}) \in (0,\infty) \times (0,\infty)^n$$

subject to the initial condition $U(0+, \cdot\,) \equiv 1$, for the linear, second-order differential operator

$$\widehat{\mathcal{L}} f \,:=\, \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n x_i x_j\, \mathrm{a}_{ij}(\mathbf{x})\, D^2_{ij} f \,+\, \sum_{i=1}^n x_i \left(\sum_{j=1}^n \frac{x_j\, \mathrm{a}_{ij}(\mathbf{x})}{x_1 + \cdots + x_n} \right) D_i f\,.$$

Furthermore, $U(T, \mathbf{x})$ is the probability that the $[0,\infty)^n-$valued diffusion process $\mathfrak{Y}(\cdot) = (Y_1(\cdot), \cdots, Y_n(\cdot))'$ with infinitesimal generator $\widehat{\mathcal{L}}$ as above, and starting with the initial configuration $\mathfrak{Y}(0) = \mathfrak{X}(0) \in (0,\infty)^n$, does not hit the boundary of the non-negative orthant $[0,\infty)^n$ by time $t = T$. It is worth noting that the answers involve only the covariance structure of the market, not the actual mean rates of return; the only rôle these latter play is to ensure that the diffusion $\mathfrak{X}(\cdot)$ lives in $(0,\infty)^n$.

Strong arbitrage relative to the market portfolio exists on the horizon $[0, T]$, if and only if $U(T, \mathfrak{X}(0)) < 1$; this amounts to failure of uniqueness for the Cauchy problem

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{x}) = \widehat{\mathcal{L}}\, U(\tau, \mathbf{x}), \quad (\tau, \mathbf{x}) \in (0,\infty) \times (0,\infty)^n \qquad \text{and} \qquad U(0+, \cdot\,) \equiv 1\,.$$

Sufficient conditions for such failure of uniqueness (equivalently, necessary conditions for uniqueness) for solutions to such Cauchy problems are provided.

Consider an "auxiliary market", whose asset prices are given by $\mathfrak{Y}(\cdot) = (Y_1(\cdot), \cdots, Y_n(\cdot))'$. The probabilistic significance of the change of drift inherent in the definition of the operator $\widehat{\mathcal{L}}$, from $\mathrm{b}_i(\mathbf{x})$ for $\mathfrak{X}(\cdot)$ to $\sum_{j=1}^n (x_j\, \mathrm{a}_{ij}(\mathbf{x}))/(x_1 + \cdots + x_n)$ for $\mathfrak{Y}(\cdot)$, is that it corresponds to a change of measure which makes the weights $\nu_i(\cdot) := Y_i(\cdot)/(Y_1(\cdot) + \cdots + Y_n(\cdot))$, $i = 1, \cdots, n$ of the auxiliary market portfolio martingales. The financial significance of this change of measure is that it bestows to the auxiliary market portfolio $\underline{\nu}(\cdot) = \big(\nu_1(\cdot), \cdots, \nu_1(\cdot)\big)'$ the so-called *numéraire property:* the ratio of any strategy's performance, relative to the new market with prices $\mathfrak{Y}(\cdot)$, is a supermartingale. This change of measure does not come necessarily from a Girsanov-type (absolutely continuous) transformation; rather, it corresponds to, and represents, the *exit measure* of Föllmer (1972) for an appropriate supermartingale.

Sections 2 and 3 set up the exact model we shall be dealing with, whereas sections 4 and 5 review extant theory in an effort to make this paper as self-contained as possible. Section 6 formulates the problem and section 7 presents some preliminary results, actually in some modest generality (which includes the non-Markovian case). Section 8 sets up the Markovian model; the results are presented in earnest in sections 9-11, and a few open questions are raised in section 12.

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2 The Model

We shall consider a financial market model of the form

$$dB(t) = B(t)\, r(t)\, dt\,, \qquad B(0) = 1\,,$$

$$dX_i(t) = X_i(t)\Big(\beta_i(t)dt + \sum_{k=1}^K \sigma_{ik}(t)\, dW_k(t)\Big), \qquad X_i(0) = x_i > 0,\; i = 1, \ldots, n\,, \tag{2.1}$$

consisting of a money-market $B(\cdot)$, and of n stocks with prices $X_1(\cdot), \cdots, X_n(\cdot)$. These are driven by the Brownian motion $W(\cdot) = (W_1(\cdot), \cdots, W_K(\cdot))'$, whose K independent components are the "factors" of the model; we assume $K \geq n$.

To simplify our analysis somewhat, we shall assume throughout that the interest rate process of the money-market is $r(\cdot) \equiv 0$, identically equal to zero: investing in the money-market will thus be tantamount to hoarding, whereas borrowing from the money-market will incur no interest. Furthermore, we shall assume that the vector-valued process $\mathfrak{X}(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))'$ of prices, the vector-valued process $\beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_n(\cdot))'$ of *mean rates of return* for the various stocks, and the $(n \times K)-$matrix-valued process $\sigma(\cdot) = (\sigma_{ik}(\cdot))_{1\le i\le n,\, 1\le k\le K}$ of *volatilities*, are all progressively measurable with respect to a right-continuous filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ which represents the "flow of information" in the market. We shall denote by $\mathbb{G} = \{\mathcal{G}(t)\}_{0\le t<\infty}$ the augmentation of the filtration $\mathbb{F}$ by the null sets of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$; we shall assume that the Brownian motion $W(\cdot)$ is adapted to $\mathbb{G}$, and that $\mathcal{G}(0) = \{\emptyset, \Omega\}$ holds mod. $\mathbb{P}$.

Let $\alpha(\cdot) := \sigma(\cdot)\sigma'(\cdot)$ be the *covariance process* of the stocks in the market, with elements

$$\alpha_{ij}(t) \;=\; \sum_{k=1}^{K} \sigma_{ik}(t)\sigma_{jk}(t) \;=\; \frac{1}{X_i(t)X_j(t)} \cdot \frac{d}{d\,t}\,\langle X_i, X_j\rangle(t)\,, \quad \text{for} \quad 1 \le i,j \le n\,, \tag{2.2}$$

and impose for every $T \in (0,\infty)$ the integrability condition

$$\sum_{i=1}^{n} \int_0^T \Big(\big|\beta_i(t)\big| + \alpha_{ii}(t) \Big)\, dt < \infty\,, \qquad \text{a.s.} \tag{2.3}$$

Under this condition the stock prices $X_1(t), \cdots, X_n(t)$ can be expressed for $0 \le t < \infty$ as

$$X_i(t) \;=\; x_i \cdot \exp\Big\{ \int_0^t \Big(\beta_i(s) - \frac{1}{2}\,\alpha_{ii}(s) \Big) ds + \sum_{k=1}^{K} \int_0^t \sigma_{ik}(s)\, dW_k(s) \Big\} \;>\; 0\,, \quad i = 1, \cdots, n\,.$$

3 Strategies and Portfolios

Let us consider now a *small investor,* whose actions cannot affect market prices. Such an investor selects, at each time t and for every $i = 1, \cdots, n$, which proportion $\pi_i(t)$ of his current wealth $V(t)$ to invest in the i^{th} stock. We shall require that each random variable $\pi_i(t)\,,\, 1 \le i \le n$ be $\mathcal{G}(t)-$measurable; the proportion $1 - \sum_{i=1}^n \pi_i(t)$ gets invested in the money market.

Thus, the wealth $V(\cdot) \equiv V^{v,\pi}(\cdot)$ corresponding to an initial capital $v \in (0,\infty)$ and an investment strategy $\pi(\cdot) = \big(\pi_1(\cdot), \cdots, \pi_n(\cdot)\big)'$ satisfies the linear stochastic equation

$$\frac{d\,V(t)}{V(t)} \;=\; \sum_{i=1}^{n} \pi_i(t)\,\frac{dX_i(t)}{X_i(t)} + \left(1 - \sum_{i=1}^{n} \pi_i(t) \right) \frac{dB(t)}{B(t)} \;=\; \pi'(t)\big[\,\beta(t)dt + \sigma(t)\,dW(t)\,\big] \tag{3.1}$$

and the initial condition $V(0) = v$, since we are assuming $r(\cdot) \equiv 0$.

• We shall call *investment strategy* a $\mathbb{G}-$progressively measurable process $\pi : [0,\infty) \times \Omega \to \mathbb{R}^n$ which satisfies for each $T \in (0,\infty)$ the analogue of the integrability condition

$$\int_0^T \Big(\big|\pi'(t)\beta(t)\big| + \pi'(t)\alpha(t)\pi(t) \Big)\, dt \;<\; \infty \tag{3.2}$$

in (2.3), almost surely. The collection of all investment strategies will be denoted by $\mathcal{H}$.

An investment strategy $\pi(\cdot)$ with $\sum_{i=1}^n \pi_i(t,\omega) = 1$ for all $(t,\omega) \in [0,\infty) \times \Omega$ will be called *portfolio.* A portfolio never invests in the money market, and never borrows from it. We shall say that a portfolio is *bounded*, if there exists a real constant $C > 0$ such that $\|\pi(t,\omega)\| \le C$ holds for all $(t,\omega) \in [0,\infty) \times \Omega$. We shall call *long-only portfolio* one that satisfies almost surely

$$\pi_1(t,\omega) \;\ge\; 0\,, \;\cdots\;, \pi_n(t,\omega) \;\ge\; 0\,, \qquad \forall\;\; (t,\omega) \in [0,\infty) \times \Omega\,;$$

such a portfolio never sells any stock short. Clearly, a long-only portfolio is also bounded.

• Corresponding to an investment strategy $\pi(\cdot)$ and an initial capital $v > 0$, the associated wealth process, i.e., the solution of the equation (3.1), is given as

$$V^{v,\pi}(t) \;=\; v \cdot \exp\left\{ \int_0^t \pi'(s)\beta(s)ds + \int_0^t \pi'(s)\sigma(s)dW(s) - \frac{1}{2}\int_0^t \pi'(s)\alpha(s)\pi(s)ds \right\}. \tag{3.3}$$

• The strategy $\varrho(\cdot) \equiv 0$ invests only in the money market at all times; it results in $V^{v,\varrho}(\cdot) \equiv v$, i.e., in hoarding the initial wealth under the mattress.

3.1 The Market Portfolio

An important long-only portfolio is the *market portfolio*

$$\mu_i(t) := \frac{X_i(t)}{X(t)}, \qquad i = 1, \ldots, n, \qquad \text{where} \qquad X(t) := X_1(t) + \ldots + X_n(t). \tag{3.4}$$

This invests in all stocks of the market in proportion to their relative capitalization weights (we are assuming from here onwards that prices are consistently being normalized, so that each stock has only one share outstanding at all times). It is not hard to see that investing according to the market portfolio amounts to "owning the entire market", in proportion of course to the initial capital $v > 0$: the wealth process of (3.3) becomes then $V^{v,\mu}(\cdot) = vX(\cdot)/X(0)$.

The resulting vector process $\underline{\mu}(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_n(\cdot)\big)'$ of market weights takes values in the positive simplex $\Delta^n_+ := \big\{ (m_1, \cdots, m_n)' \in (0,1)^n \mid \sum_{i=1}^n m_i = 1 \big\}$ of $\mathbb{R}^n$. An application of Itô's rule gives, after some computation, the dynamics of this process as

$$d\mu_i(t) \;=\; \mu_i(t) \left[\gamma_i^\mu(t)\, dt \;+\; \sum_{k=1}^K \tau_{ik}^\mu(t)\, dW_k(t) \right], \qquad i = 1, \cdots, n; \tag{3.5}$$

here $\tau^\mu(t) := \big\{ \tau^\mu_{ik}(t) \big\}_{1 \le i \le n,\, 1 \le k \le K}$ is the matrix with entries $\tau^\mu_{ik}(t) \;:=\; \sigma_{ik}(t) - \sum_{j=1}^n \mu_j(t)\, \sigma_{jk}(t)$, and $\gamma^\mu(t) := \big(\gamma_1^\mu(t), \cdots, \gamma_n^\mu(t)\big)'$ the vector with elements

$$\gamma_i^\mu(t) \;:=\; \beta_i(t) - \sum_{j=1}^n \mu_j(t)\, \beta_j(t) \;+\; \sum_{j=1}^n \sum_{\ell=1}^n \mu_j(t)\mu_\ell(t)\alpha_{j\ell}(t) - \sum_{j=1}^n \mu_j(t)\alpha_{ij}(t). \tag{3.6}$$

Using the notation of this section and with the help of Itô's rule, it can be checked that the performance of a portfolio $\pi(\cdot)$ relative to the market $\mu(\cdot)$ has the dynamics

$$d\left(\frac{V^{v,\pi}(t)}{V^{v,\mu}(t)}\right) \;=\; \left(\frac{V^{v,\pi}(t)}{V^{v,\mu}(t)}\right) \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)}\; d\mu_i(t) \;=\; \left(\frac{V^{v,\pi}(t)}{V^{v,\mu}(t)}\right) \pi'(t) \left[\gamma^\mu(t)\, dt \;+\; \tau^\mu(t)\, dW(t) \right]. \tag{3.7}$$

Remark 1: The expression (3.6) becomes $\gamma_i^\mu(\cdot) \equiv 0$, identically equal to zero, when

$$\beta_i(\cdot) = \sum_{j=1}^n \alpha_{ij}(\cdot)\mu_j(\cdot) + \kappa(\cdot) \qquad \text{holds for all} \quad i = 1, \cdots, n \tag{3.8}$$

for some scalar, $\mathbb{F}-$progressively measurable and locally integrable process $\kappa(\cdot)$. Models that satisfy this so-called "perfect balance" condition were studied fairly recently by Kardaras (2008). Under (3.8), the market weights $\mu_1(\cdot), \cdots, \mu_n(\cdot)$ are all positive martingales, and it follows from (3.5), (3.7) that the market portfolio $\mu(\cdot)$ has the following "numéraire property": *The ratio* $V^{v,\pi}(\cdot)/V^{v,\mu}(\cdot)$ *is a positive local martingale, thus a* supermartingale, *for every portfolio* $\pi(\cdot)$.

4 Relative Arbitrage

The following notion was introduced and studied in Fernholz (2002): given a real number $T > 0$ and any two investment strategies $\pi(\cdot)$ and $\rho(\cdot)$, we shall say that $\pi(\cdot)$ is an *arbitrage relative to* $\rho(\cdot)$ over the time-horizon $[0, T]$, if we have

$$\mathbb{P}\big(V^{1,\pi}(T) \geq V^{1,\rho}(T)\big) = 1 \quad \text{and} \quad \mathbb{P}\big(V^{1,\pi}(T) > V^{1,\rho}(T)\big) > 0\,. \tag{4.1}$$

We call such relative arbitrage *strong*, if

$$\mathbb{P}\big(V^{1,\pi}(T) > V^{1,\rho}(T)\big) = 1\,. \tag{4.2}$$

Arbitrage (resp., strong arbitrage) relative to the strategy $\varrho(\cdot) \equiv 0$ that invests only in the money market, will be called just that, without the qualifier "relative".

4.1 Market Price of Risk

We shall assume from now onward that there exists a *market price of risk* (or "relative risk") $\vartheta : [0,\infty) \times \Omega \to \mathbb{R}^K$, an $\mathbb{F}-$progressively measurable process that satisfies for each $T \in (0,\infty)$ the conditions

$$\sigma(t)\vartheta(t) = \beta(t)\,, \quad \forall \;\; 0 \leq t \leq T \qquad \text{and} \qquad \int_0^T \big\|\vartheta(t)\big\|^2 \, dt < \infty \tag{4.3}$$

almost surely. If the covariance matrix $\alpha(t)$ is invertible for every $t \in [0,\infty)$, then we can take $\vartheta(t) = \sigma'(t)\alpha^{-1}(t)\,\beta(t)$ in (4.3).

The existence of the market-price-of-risk process $\vartheta(\cdot)$ allows us to introduce the exponential local martingale

$$Z(t) \,:=\, \exp\Big\{ - \int_0^t \vartheta'(s)\, dW(s) - \frac{1}{2}\int_0^t \big\|\vartheta(s)\big\|^2\, ds\Big\}, \quad 0 \leq t < \infty\,. \tag{4.4}$$

This process is also a supermartingale; it is a martingale, if and only if $\mathbb{E}(Z(T)) = 1$ holds for all $T \in (0,\infty)$. For the purposes of this paper, it is important to allow this exponential process to be a strict local martingale; that is, not to exclude the possibility $\mathbb{E}(Z(T)) < 1$ for some $T \in (0,\infty)$.

From (4.4) and (3.1), now written in the form

$$d\,V^{v,\pi}(t) \;=\; V^{v,\pi}(t)\,\pi'(t)\sigma(t)\, d\widehat{W}(t)\,, \qquad \widehat{W}(t) \,:=\, W(t) + \int_0^t \vartheta(s)ds\,, \tag{4.5}$$

and in conjunction with the product rule of the stochastic calculus, it follows that

$$Z(t)V^{v,\pi}(t) \;=\; v + \int_0^t Z(s)V^{v,\pi}(s)\,\big(\sigma'(s)\pi(s) - \vartheta(s)\big)'\, dW(s)\,, \qquad 0 \leq t < \infty \tag{4.6}$$

is also a positive local martingale and a supermartingale, for every investment strategy $\pi(\cdot)$.

- If (3.8) holds with $\kappa(\cdot) \equiv 0$, then we can take $\vartheta(\cdot) = \sigma'(\cdot)\mu(\cdot)$ as a market price of risk in (4.3). In this case the equation (4.6) gives directly both $Z(\cdot) \equiv v/V^{v,\mu}(\cdot) \equiv X(0)/X(\cdot)$ and the numéraire property for the market portfolio.
- If the matrix $\sigma(t)$ is invertible for every $t \in [0,T]$, and if the resulting process $\sigma^{-1}(\cdot)$ is $\mathbb{F}-$progressively measurable, then $\varpi(\cdot) = (\sigma^{-1}(\cdot))'\vartheta(\cdot)$ defines a similarly measurable strategy in $\mathcal{H}$. Again, the equation (4.6) gives both $Z(\cdot) \equiv v/V^{v,\varpi}(\cdot)$, and the numéraire property for the strategy $\varpi(\cdot)$: the ratio $V^{v,\pi}(\cdot)/V^{v,\varpi}(\cdot)$ is a positive local martingale, thus a supermartingale, for every investment strategy $\pi(\cdot)$.

4.2 Strict Local Martingales

Suppose that the covariance process $\alpha(\cdot)$ satisfies the *boundedness condition*

$$\xi'\alpha(t,\omega)\xi \;=\; \xi'\sigma(t,\omega)\sigma'(t,\omega)\xi \;\le\; D\,\|\xi\|^2, \quad \forall\;\; (t,\omega)\in[0,\infty)\times\Omega \quad \text{and} \quad \xi\in\mathbb{R}^n \tag{4.7}$$

for some real number $D>0$. Then, if (4.1) holds for two bounded portfolios $\pi(\cdot)$ and $\rho(\cdot)$, it can be seen (e.g., Fernholz & Karatzas (2008)) that the positive local martingales $Z(\cdot)$ and $Z(\cdot)V^{v,\rho}(\cdot)$ of (4.4), (4.6) are strict: $\mathbb{E}\big(Z(T)\big)<1$, $\mathbb{E}\,[\,Z(T)V^{v,\rho}(T)\,]\;<\;v$.

In particular, if (4.7) holds and (4.1) is satisfied for some bounded portfolio $\pi(\cdot)$ and with $\rho(\cdot)\equiv\mu(\cdot)$ the market portfolio, then we have the strict local martingale properties

$$\mathbb{E}\big(Z(T)\big)<1\,, \quad \mathbb{E}\,[\,Z(T)X(T)\,]\;<\;X(0) \quad \text{and} \quad \mathbb{E}\,[\,Z(T)X_i(T)\,]\;<\;X_i(0)\,, \qquad i=1,\cdots,n \tag{4.8}$$

in the notation of (3.4). (These assumptions are satisfied, for instance, in the "diverse market" example of section 6 in Fernholz et al. (2005); that is, under (4.7) and (4.10), (4.14) below.)

Example 1: 3-D Bessel Process. It is perfectly possible, indeed very simple, for $Z(\cdot)$ to be a strict local martingale and $Z(\cdot)X(\cdot)$ to be a martingale. In other words, to have the second and third inequalities in (4.8) fail, while the first stands.

Consider the following example from Karatzas & Kardaras (2007), p. 469: With $n=1$, take $\beta(t)=1/X^2(t)$ and $\sigma(t)=1/X(t)$ in (2.1), where the process $X(\cdot)$ satisfies the equation

$$dX(t) \;=\; \frac{1}{X(t)}\,dt \,+\, dW(t)\,, \qquad X(0)=1\,. \tag{4.9}$$

This is a Bessel process in dimension three – the radial part of a 3-D Brownian motion started at unit distance from the origin – and takes values in $(0,\infty)$. We have then $\vartheta(t)\;=\;1/X(t)$, $Z(t)\;=\;1/X(t)$ for $0\le t<\infty$ in (4.3), (4.4), so $Z(\cdot)X(\cdot)$ is a very solid martingale. However, $Z(\cdot)$ is the prototypical example of a *strict* local martingale – we have $\mathbb{E}(Z(T))\;<\;1$ for every $T\in(0,\infty)$ (see, for instance, Exercise 3.36, p. 168 in Karatzas & Shreve (1991)). In fact, with

$$F(\tau,x)=\frac{\Phi\big(x/\sqrt{\tau}\,\big)}{\Phi\big(1/\sqrt{T}\,\big)}\,, \quad (\tau,x)\in(0,\infty)^2 \quad \text{and} \quad \Phi(\xi)=\int_{-\infty}^{\xi}\varphi(u)\,du\,, \quad \varphi(u)=\frac{e^{-u^2/2}}{\sqrt{2\pi}}\,,$$

a simple application of Itô's rule to the process $V_*(t)=F(T-t,X(t))$, $0\le t\le T$ shows

$$V_*(\cdot)\equiv V^{1,\pi_*}(\cdot)\,, \qquad \text{where} \quad \pi_*(t)=\Pi\left(\frac{X(t)}{\sqrt{T-t}}\right), \quad \Pi(\xi):=\frac{\xi\varphi(\xi)}{\Phi(\xi)}$$

for $0\le t<T$, and $\pi_*(T)=0$. Note also that we have $V_*(0)=1$ and $V_*(T)=1/\Phi\big(1/\sqrt{T}\,\big)>1$, so the strategy $\pi_*(\cdot)$ is a strong arbitrage on $[0,T]$.

4.3 Examples of Arbitrage Relative to the Market

With the notation introduced so far, suppose that there exists a real constant $h>0$ for which the condition

$$\sum_{i=1}^{n}\mu_i(t)\alpha_{ii}(t)-\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i(t)\alpha_{ij}(t)\mu_j(t)\;\ge\;h\,, \qquad \forall\;\; 0\le t<\infty \tag{4.10}$$

holds almost surely. It can be shown that the long-only portfolio

$$\pi_i(t)\;=\;\frac{\mu_i(t)\big(c-\log\mu_i(t)\big)}{\sum_{j=1}^{n}\mu_j(t)\big(c-\log\mu_j(t)\big)}\,, \qquad i=1,\cdots,n \tag{4.11}$$

is then, for a sufficiently large real constant $c > 0$, a strong arbitrage relative to the market portfolio $\mu(\cdot)$ over any time-horizon $[0,T]$ with $T > (2 \log n)/h$.

It is still an open question whether, in the setting of sections 2-4, such relative arbitrage can be constructed over *arbitrary* time-horizons under the condition (4.10).

• Another condition guaranteeing the existence of arbitrage relative to the market, is that there exist a real constant $h > 0$ with

$$\big(\mu_1(t) \cdots \mu_n(t)\big)^{1/n} \left[\sum_{i=1}^{n} \alpha_{ii}(t) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}(t) \right] \geq h\,, \qquad \forall \;\; 0 \leq t < \infty \tag{4.12}$$

almost surely; then for a sufficiently large real constant $c > 0$, the long-only portfolio

$$\pi_i(t) = \frac{c}{c + \big(\mu_1(t) \cdots \mu_n(t)\big)^{1/n}} \cdot \frac{1}{n} + \frac{\big(\mu_1(t) \cdots \mu_n(t)\big)^{1/n}}{c + \big(\mu_1(t) \cdots \mu_n(t)\big)^{1/n}} \cdot \mu_i(t) \tag{4.13}$$

for $i = 1, \cdots, n$ (convex combination of the equally-weighted and market portfolios) is a strong arbitrage relative to the market over any time-horizon $[0,T]$ with $T > \big(2n^{1-(1/n)}\big)/h$.

• Consider the a.s. *strong non-degeneracy condition*

$$\xi' \alpha(t,\omega) \xi \;=\; \xi' \sigma(t,\omega) \sigma'(t) \xi \;\geq\; \varepsilon \|\xi\|^2, \quad \forall \;\; (t,\omega) \in [0,\infty) \times \Omega \quad \text{and} \quad \xi \in \mathbb{R}^n \tag{4.14}$$

for some real number $\varepsilon > 0$, on the covariance process $\alpha(\cdot)$. If (4.7), (4.14) and (4.10) all hold, then it can be shown that for any given constant $p \in (0,1)$, the

$$\mu_i^{(p)}(t) \;=\; \frac{(\mu_i(t))^p}{\sum_{j=1}^{n} (\mu_j(t))^p}\,, \qquad i = 1, \cdots, n \tag{4.15}$$

defines a long-only portfolio which leads again to strong arbitrage relative to the market over sufficiently long time-horizons. Even more importantly, it can be shown that appropriate modifications of the portfolio (4.15) yield such arbitrage over *any* time-horizon $[0,T]$; see Fernholz (2002) for extensive discussion of the properties and performance of (4.15).

Let us also note that under the assumptions (4.7) and (4.10), the requirement (4.14) is equivalent to the "diversity condition" $\mathbb{P}\big[\, \max_{1 \leq i \leq n} \mu_i(t) \;\leq\; 1 - \eta\,, \;\; \forall \;\; 0 \leq t < \infty \big] = 1$, for some $\eta \in (0,1)$.

• The choice of covariance matrix

$$\alpha_{ij}(t) \;=\; \frac{1}{\mu_i(t)}\, \delta_{ij}\,, \qquad 1 \leq i,j \leq n \tag{4.16}$$

was studied by Fernholz & Karatzas (2005) under the rubric of "stabilization by volatility". It assigns to the various stocks variances which are inversely proportional to the stocks' relative weights in the market. For this choice, and with $n \geq 2$, both (4.10) and (4.12) hold with $h = n-1$ (the first as equality), so the portfolio (4.11) (resp., (4.13)) leads to strong arbitrage relative to the market over any time-horizon of length $T > (2 \log n)/(n-1)$ (resp., $T > \big(2n^{1-(1/n)}\big)/(n-1)$).

It was shown recently by Banner & Fernholz (2008) that such arbitrage is then possible over *any* time horizon, through appropriate modifications of the portfolio (4.11).

• For proofs of the claims made in this section, the reader can consult the survey paper Fernholz & Karatzas (2008), especially Examples 11.1, 11.2 and Remark 11.4. Additional conditions leading to strong arbitrage relative to the market portfolio can be found in Fernholz & Karatzas (2005, 2008), Fernholz et al. (2005). These references also contain examples which demonstrate that the existence of a market price of risk $\vartheta(\cdot)$ as in (4.3) is compatible with all of (4.7), (4.10), (4.14).

5 Hedging

We shall impose from now on the following structural assumption on the filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$, the "flow of information" in the market.

Assumption A: *Every local martingale of the filtration $\mathbb{F}$ can be represented as a stochastic integral with respect to the driving Brownian motion $W(\cdot)$ in (2.1) of some $\mathbb{G}-$progressively measurable integrand.*

This is the case, for instance, if $\mathbb{F}$ coincides with $\mathbb{F}^W$, the filtration generated by the Brownian motion $W(\cdot)$. Another possibility is when all coëfficients $\beta_i(\cdot)$, $\sigma_{i\nu}(\cdot)$ (thus also $\alpha_{ij}(\cdot)$) are progressively measurable with respect to

$$\mathbb{F}^{\mathfrak{X}} = \{\mathcal{F}^{\mathfrak{X}}(t)\}_{0 \le t < \infty} \qquad \text{with} \qquad \mathcal{F}^{\mathfrak{X}}(t) := \sigma(\mathfrak{X}(s),\ 0 \le s \le t), \tag{5.1}$$

the filtration generated by the vector of past and present stock-prices (from (2.2), this is not much of an assumption as far as the $\alpha_{ij}(\cdot)$ are concerned); the resulting system of functional SDE's (2.1) has a solution which is unique in distribution (Jacod (1977, 1979)); and $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ is $\mathcal{F}(t) = \bigcap_{\varepsilon > 0} \mathcal{F}^{\mathfrak{X}}(t+\varepsilon)$, namely, the right-continuous version $\mathbb{F}^{\mathfrak{X}}_{+}$ of the filtration $\mathbb{F}^{\mathfrak{X}}$ in (5.1).

Assumption B: *We have $K = n$, and the volatility matrix $\sigma(t)$ is invertible for every $t \in [0, T]$.*

Let us consider now the following situation: fix a number $T \in (0, \infty)$, consider an $\mathcal{F}(T)-$measurable random variable $Y : \Omega \to (0, \infty)$, and think of this variable as a (random) liability that the investor faces at time $t = T$. Then the investor is interested in computing the quantity

$$\mathcal{U}^Y(T) := \inf \big\{ v > 0 \mid \exists\, \pi(\cdot) \in \mathcal{H} \ \text{ s.t. } V^{v,\pi}(T) \ge Y, \ \text{a.s.} \big\}. \tag{5.2}$$

This is the smallest amount of initial capital at time $t = 0$ that allows the investor – via judicious choice of investment strategy $\pi(\cdot) \in \mathcal{H}$ – to amass enough wealth by time $t = T$ so as to cover the liability almost surely, that is, without risk. The quantity of (5.2) is usually called "upper hedging price" for Y (e.g., Karatzas & Shreve (1998)). It is easy to see that we have

$$\mathcal{U}^Y(T) \ \ge\ \mathbb{E}[\, Z(T) Y \,].$$

Indeed, denote by $\mathcal{R}$ the set on the right-hand side of (5.2); suppose that this set is non-empty, and pick an arbitrary $v \in \mathcal{R}$. For this element of $\mathcal{R}$ there exists an investment strategy $\pi(\cdot) \in \mathcal{H}$ for which $V^{v,\pi}(T) \ge Y$ holds almost surely, and the supermartingale property of the process $Z(\cdot) V^{v,\pi}(\cdot)$ in (4.6) gives

$$v \ \ge\ \mathbb{E}[\, Z(T) V^{v,\pi}(T) \,] \ \ge\ \mathbb{E}[\, Z(T) Y \,].$$

The inequality $\mathcal{U}^Y(T) \ \ge\ \mathbb{E}[\, Z(T) Y \,]$ follows from this, and from the arbitrariness of $v \in \mathcal{R}$. If the set $\mathcal{R}$ is empty we have $\mathcal{U}^Y(T) = \infty$ in (5.2), and the inequality holds trivially.

The following result is well known when the exponential process $Z(\cdot)$ of (4.4) is a martingale, but not nearly as well known when $Z(\cdot)$ is just a local martingale, possibly strict.

Proposition 1: *Under Assumptions A and B, we have* $\mathcal{U}^Y(T) = \mathbb{E}\,[\, Z(T) Y \,]$.
The infimum of (5.2) is attained, if $y := \mathbb{E}[\, Z(T) Y \,] < \infty$; *in fact, there exists in this case an investment strategy $\widehat{\pi}(\cdot) \in \mathcal{H}$ such that*

$$V^{\,y,\widehat{\pi}}(T) \ =\ Y \quad \textit{holds almost surely.} \tag{5.3}$$

Proof: We only need to establish the inequality $\mathcal{U}^Y(T) \le y$; of course there is nothing to prove if $y = \infty$, so let us assume $y < \infty$. Then

$$M(t) := \mathbb{E}\big[\, Z(T) Y \,|\, \mathcal{F}(t) \,\big], \qquad 0 \le t \le T \tag{5.4}$$

is a martingale with $M(0) = y$, $M(T) = YZ(T)$ a.s. Because Y takes values in $(0,\infty)$ by assumption, so does this martingale $M(\cdot)$; and Assumption A guarantees the existence of a $\mathbb{G}-$progressively measurable $\psi : [0,T] \times \Omega \to \mathbb{R}^n$ with $\int_0^T \|\psi(t)\|^2 \, dt < \infty$ and

$$M(t) \,=\, y + \int_0^t M(s)\, \psi'(s)\, dW(s) \,=\, y + \sum_{k=1}^n \int_0^t M(s)\, \psi_k(s)\, dW_k(s)\,, \qquad 0 \le t \le T \tag{5.5}$$

valid almost surely. All that remains is to invoke Assumption B, and to make the identifications

$$\widehat{\pi}(\cdot) \,\equiv\, \sigma^{-1}(\cdot)\, \big(\vartheta(\cdot) + \psi(\cdot)\big)\,, \qquad V^{\,y,\widehat{\pi}}(\cdot) \,\equiv\, M(\cdot)/Z(\cdot) \tag{5.6}$$

in conjunction with (4.6); the claims follow. $\square$

In the terminology of mathematical finance, Proposition 1 provides conditions under which the market of (2.1) is complete – meaning that a large class of $\mathcal{F}(T)-$measurable random variables can be replicated exactly, in the manner of (5.3), via appropriate choice of initial capital and investment strategy – even in situations where no equivalent martingale measure exists.

6 Optimal Arbitrage Relative to the Market

The possibility of strong arbitrage relative to the market portfolio, defined and exemplified in section 4, raises an obvious question: *What is the best possible arbitrage of this kind?*

One way to formulate this question can be cast as follows: On a given time-horizon $[0,T]$, what is the *smallest* relative amount

$$\mathfrak{u}(T) \,:=\, \inf \big\{ r > 0 \mid \exists\ \pi(\cdot) \in \mathcal{H} \ \text{ s.t. } V^{\,rX(0),\pi}(T) \ge X(T), \ \text{a.s.} \big\} \tag{6.1}$$

of initial capital, starting with which one can match or exceed at time $t = T$ the market capitalization $X(T)$? Clearly,

$$0 \,<\, \mathfrak{u}(T) \,\le\, 1\,; \tag{6.2}$$

and with $0 < r < \mathfrak{u}(T)$ no strategy can beat the market with probability one. That is,

$$\mathbb{P}\left[\, V^{\,rX(0),\,\pi}(T) \ge X(T) \,\right] \,<\, 1 \quad \text{ for every investment strategy } \pi(\cdot)\,, \text{ if } \ 0 < r < \mathfrak{u}(T)\,.$$

The inequality on the right-hand side of (6.1) amounts to the a.s. requirement $V^{\,r,\pi}(T) \ge Y$ for the strictly positive and $\mathcal{F}(T)-$measurable random variable $Y := V^{\,1,\mu}(T) = X(T)/X(0)$, and under the conditions of Proposition 1 we obtain

$$\mathfrak{u}(T) \,=\, \mathcal{U}^Y(T) \,=\, \frac{1}{X(0)} \cdot \mathbb{E}\left[\, Z(T) X(T) \,\right]. \tag{6.3}$$

If $\mathfrak{u}(T) = 1$, that is, if $Z(\cdot)X(\cdot)$ is a martingale on $[0,T]$, no arbitrage relative to the market portfolio is possible on this time-horizon. If, on the other hand, $Z(\cdot)X(\cdot)$ is a strict local martingale on $[0,T]$, then arbitrage relative to the market *is* possible; and $\mathfrak{u}(T) < 1$ is the smallest relative amount required for this. In section 7 we obtain characterizations of the quantity $\mathfrak{u}(T)$ in terms of the *Föllmer exit measure* and of a "generalized martingale measure" (equations (7.2) and (7.8)).

A drawback of Proposition 1 is that it provides no information about the investment strategy $\widehat{\pi}(\cdot)$ which implements this 'best possible' arbitrage relative to the market portfolio, apart from ascertaining its existence. In the section 8 we shall specialize the model of (2.1) to a Markovian context; this will enable us to describe $\widehat{\pi}(\cdot)$ somewhat more precisely, in terms of partial differential equations (section 11). It will also enable us to characterize the quantity $\mathfrak{u}(T)$ further, in terms of the smallest solution to a parabolic partial differential inequality (Proposition 2), and as the probability of non-absorption by time T of a suitable diffusion process (Proposition 4).

6.1 Related Questions

Knowledge of the quantity (6.1) allows us to answer some related questions as well. Suppose we are interested in the maximal relative return

$$\mathfrak{W}(T) \;:=\; \sup\big\{ w > 0 \mid \exists\; \pi(\cdot) \in \mathcal{H} \;\text{ s.t. }\; V^{\,X(0),\pi}(T) \geq w\, X(T), \;\text{a.s.} \big\}\,, \tag{6.4}$$

in excess of the market, that can be obtained by investment strategies over the given time-horizon $[0,T]$. It is almost immediate to see that $\mathfrak{W}(T) = 1/\mathfrak{u}(T)$.

Now let us turn this question on its head, and ask: For a given exceedance level $w \geq 1$, what is the shortest length of time

$$\mathbf{T}(w) \;:=\; \inf\big\{ T > 0 \mid \exists\; \pi(\cdot) \in \mathcal{H} \;\text{ s.t. }\; V^{\,X(0),\pi}(T) \geq w\, X(T), \;\text{a.s.} \big\} \tag{6.5}$$

required to guarantee the existence of an investment strategy with return of at least w times the market? It can be shown that the answer is given by the inverse of the non-increasing function $T \mapsto \mathfrak{u}(T)$ evaluated at $1/w$, namely $\mathbf{T}(w) = \inf\{T > 0 \mid \mathfrak{u}(T) \leq 1/w\}$; see the survey by Fernholz & Karatzas (2008) for the details.

• The questions raised in this section and the next are related to the work of Delbaen & Schachermayer (1995.b). They bear an even closer connection with issues raised in the Finance literature under the general rubric of "bubbles". The literature on this letter topic is large, so let us mention the papers by Lowenstein & Willard (2000), Pal & Protter (2007) and, most significantly, Heston et al. (2007), as the closest in spirit to our approach here. Let us also call attention to the recent preprint by Hugonnier (2007), which demonstrates that arbitrage opportunities can arise also in equilibrium models; we also refer to this preprint and to Heston et al. (2007) for an up-to-date survey of the literature on this and related topics.

6.2 Generalized Likelihood Ratios

The positive local martingale $Z(\cdot)X(\cdot)$, whose expectation appears in (6.3), can be expressed as

$$Z(t)X(t) \;=\; X(0)\cdot \exp\Big\{ -\int_0^t \big(\widetilde{\vartheta}(s)\big)'\, dW(s) - \frac{1}{2}\int_0^t \big\|\widetilde{\vartheta}(s)\big\|^2\, ds \Big\}\,, \quad 0 \leq t \leq T\,, \tag{6.6}$$

where we have solved equation (4.6) and set

$$\widetilde{\vartheta}(\cdot) \;:=\; \vartheta(\cdot) - \sigma'(\cdot)\mu(\cdot)\,, \qquad \widetilde{W}(\cdot) \;:=\; W(\cdot) + \int_0^{\cdot} \widetilde{\vartheta}(t)\, dt \tag{6.7}$$

so that $\sigma(\cdot)\widetilde{\vartheta}(\cdot) = \beta(\cdot) - \alpha(\cdot)\mu(\cdot)$, from (4.3). Thus, we can re-cast the equation (2.1) as

$$dX_i(t) \;=\; X_i(t)\left[\frac{\sum_{j=1}^n \alpha_{ij}(t) X_j(t)}{X_1(t) + \cdots + X_n(t)}\, dt \;+\; \sum_{k=1}^K \sigma_{ik}(t)\, d\widetilde{W}_k(t) \right], \quad i = 1, \cdots, n\,, \tag{6.8}$$

and encounter again the drifts $\sum_{j=1}^n \alpha_{ij}(\cdot)\mu_j(\cdot)$ appearing on the right-hand side of (3.8).

On the other hand, we note from (6.6), (6.7) that the reciprocal of the exponential local martingale $Z(\cdot)X(\cdot)/X(0)$ can be expressed as

$$\Lambda(t) \;:=\; \frac{X(0)}{Z(t)X(t)} \;=\; \exp\Big\{ \int_0^t \big(\widetilde{\vartheta}(s)\big)'\, d\widetilde{W}(s) - \frac{1}{2}\int_0^t \big\|\widetilde{\vartheta}(s)\big\|^2\, ds \Big\}\,, \quad 0 \leq t \leq T\,; \tag{6.9}$$

similarly, the reciprocal of the exponential local martingale $Z(\cdot)X_i(\cdot)/X_i(0)$ is

$$\Lambda_i(t) \;:=\; \frac{X_i(0)}{Z(t)X_i(t)} \;=\; \exp\Big\{ \int_0^t \big(\widetilde{\vartheta}^{(i)}(s)\big)'\, d\widetilde{W}^{(i)}(s) - \frac{1}{2}\int_0^t \big\|\widetilde{\vartheta}^{(i)}(s)\big\|^2\, ds \Big\}\,, \quad i = 1, \cdots, n\,, \tag{6.10}$$

where $\widetilde{\vartheta}^{(i)}(\cdot) := \vartheta(\cdot) - \sigma'(\cdot)\mathfrak{e}_i$, $\mathfrak{e}_i$ is the i^{th} unit vector in $\mathbb{R}^n$, and $\widetilde{W}^{(i)}(\cdot) := W(\cdot) + \int_0^\cdot \widetilde{\vartheta}^{(i)}(t)\, dt$. Comparing (6.9) and (6.10), we represent the relative market weights as

$$\frac{\mu_i(t)}{\mu_i(0)} \;=\; \frac{\Lambda(t)}{\Lambda_i(t)} \;= \exp\Big\{ \int_0^t \big(\mathfrak{e}_i - \mu(s)\big)'\sigma(s)\, d\widetilde{W}(s) - \frac{1}{2}\int_0^t \big\|\big(\mathfrak{e}_i - \mu(s)\big)'\sigma(s)\big\|^2\, ds \Big\}\,, \qquad (6.11)$$

and cast the dynamics of (3.5)-(3.6) as

$$d\mu_i(t) \;=\; \mu_i(t)\,\big(\mathfrak{e}_i - \mu(t)\big)'\sigma(t)\, d\widetilde{W}(t)\,, \qquad i = 1, \cdots, n\,. \qquad (6.12)$$

6.3 Absence of Relative Arbitrage

Under the condition (3.8), no portfolio $\pi(\cdot)$ can lead to arbitrage relative to the market, over any time-horizon $[0, T]$ as above. For then Remark 1 gives $\mathbb{E}\big[\, V^{1,\pi}(T)/V^{1,\rho}(T)\,\big] \le 1$ for $\rho(\cdot) \equiv \mu(\cdot)$, a conclusion incompatible with (4.1).

• On the other hand, under the conditions of (4.3) and for any investment strategy $\pi(\cdot)$, the dynamics (4.5), in conjunction with (6.7) and Itô's rule, give

$$d\left(\frac{V^{v,\pi}(t)}{V^{v,\mu}(t)}\right) \;=\; \left(\frac{V^{v,\pi}(t)}{V^{v,\mu}(t)}\right) \cdot \big(\pi(t) - \mu(t)\big)'\, \sigma(t)\, d\widetilde{W}(t)\,.$$

If for a given $T \in (0, \infty)$ the process $Z(t)X(t)$, $0 \le t \le T$ is a martingale, then the measure

$$\widetilde{\mathbb{P}}_T(A) \;:=\; \frac{1}{X(0)} \cdot \mathbb{E}\big[\, Z(T)X(T) \cdot 1_A\,\big]\,, \qquad A \in \mathcal{F}(T) \qquad (6.13)$$

is a probability: $\mathfrak{u}(T) = \widetilde{\mathbb{P}}_T(\Omega) = 1$. Under $\widetilde{\mathbb{P}}_T$ the process $\widetilde{W}(t)$, $0 \le t \le T$ of (6.7) is a Brownian motion by the Girsanov theorem, thus $V^{v,\pi}(t)/V^{v,\mu}(t)$, $0 \le t \le T$ *is a positive local martingale and a supermartingale for each investment strategy* $\pi(\cdot)$ (the numéraire property under this change of probability). We obtain in particular $\mathbb{E}^{\widetilde{\mathbb{P}}_T}\big[\, V^{1,\pi}(T)/V^{1,\mu}(T)\,\big] \le 1$, a conclusion incompatible once again with (4.1) for $\rho(\cdot) \equiv \mu(\cdot)$. No investment strategy can lead then to arbitrage relative to the market on the given time-horizon $[0, T]$.

It is also clear from (6.12) that, under the measure $\widetilde{\mathbb{P}}_T$, the market weights $\mu_1(t), \cdots, \mu_n(t)$, $0 \le t \le T$ are martingales.

7 Exit Measure of a Positive Supermartingale

Throughout this section we shall assume for technical reasons that *the process* $Z(\cdot)$ *of (4.4) is adapted* (thus a local martingale and supermartingale with respect) *to the observations filtration* $\mathbb{F}$, not just to its augmentation $\mathbb{G}$. We shall also assume that this filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$, on the underlying measurable space $(\Omega, \mathcal{F})$, is the right-continuous version $\mathcal{F}(t) = \bigcap_{\varepsilon > 0} \mathcal{F}^o(t + \varepsilon)$ of a *standard system* $\mathbb{F}^o = \{\mathcal{F}^o(t)\}_{0 \le t < \infty}$: Each $(\Omega, \mathcal{F}^o(t))$ is supposed to be a standard Borel space (i.e., $\mathcal{F}^o(t)$ is isomorphic to the $\sigma-$algebra of Borel sets of some complete separable metric space); and for any decreasing sequence of sets $\{A_j\}_{j \in \mathbb{N}}$ such that A_j is an atom of $\mathcal{F}^o(t_j)$, for some increasing sequence $\{t_j\}_{j \in \mathbb{N}} \subset [0, \infty)$, we have $\bigcap_{j \in \mathbb{N}} A_j \neq \emptyset$.

The canonical space of continuous functions $\omega : [0, \infty) \to \mathbb{R}^n$, along with an absorbing point Δ and with the filtration generated by the coördinate mappings, is here the 'standard' example.

Under these conditions we can associate to the positive $\mathbb{F}-$local martingale $Z(\cdot)X(\cdot)$ a positive measure $\mathfrak{P}$, defined on the predictable $\sigma-$algebra of $[0, \infty] \times \Omega$ and with

$$\mathfrak{P}\big((T, \infty] \times A\big) \;=\; \frac{1}{X(0)} \cdot \mathbb{E}\left[\, Z(T)X(T) \cdot 1_A\,\right]\,, \qquad A \in \mathcal{F}(T), \;\; T \in [0, \infty)\,.$$

This is the "exit measure" of the supermartingale $Z(\cdot)X(\cdot)$, as introduced and studied by Föllmer (1972, 1973) in an effort to cast Doob's (1957) *u-path process* in the context of the general theory of processes that was emerging at the time; see also Meyer (1972), Azéma & Jeulin (1976), Delbaen & Schachermayer (1995.a). Note that the right-hand side coincides with that of (6.13), but is now viewed as defining a measure on the product space, rather than a measure on $\mathcal{F}(T)$ for a fixed $T \in [0,\infty)$; note also that $\mathfrak{u}(T) = \mathfrak{P}\big((T,\infty] \times \Omega\big)$.

Föllmer (1972) obtained a characterization of the (process-theoretic) properties of supermartingales, such as $Z(\cdot)X(\cdot)$ here, in terms of the (measure-theoretic) properties of $\mathfrak{P}$. It follows from his work, in particular, that $Z(\cdot)X(\cdot)$ is a

- *martingale*, if and only if $\mathfrak{P}$ in concentrated on $\{\infty\} \times \Omega$;
- *potential* (i.e., $\mathfrak{u}(\infty) = 0$), if and only if $\mathfrak{P}$ in concentrated on $(0,\infty) \times \Omega$.

7.1 A Representation of the Föllmer Measure

Let us recall now the exponential process $\Lambda(\cdot)$ of (6.9). It is a continuous, non-negative $\mathbb{F}-$martingale under a probability measure $\mathbb{Q}$ defined on a suitable canonical space; the measure $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ with $d\mathbb{P} = \Lambda(T)\, d\mathbb{Q}$ on each $\mathcal{F}(T)$; and the process $\widetilde{W}(\cdot)$ of (6.7) is $\mathbb{Q}-$Brownian motion. Define the first time

$$\mathcal{T} := \inf\{\, t \geq 0 \,|\, \Lambda(t) = 0 \,\} \tag{7.1}$$

this process $\Lambda(\cdot)$ hits the origin (setting $\mathcal{T} = \infty$ if it never does). We have of course $\mathbb{P}(\mathcal{T} < \infty) = 0$, but $\mathbb{Q}(\mathcal{T} < \infty)$ may well be positive; and $\int_0^{\mathcal{T}} \big\|\widetilde{\vartheta}(t)\big\|^2 dt = \infty$ holds $\mathbb{Q}-$a.s.

With this setting $Z(\cdot)X(\cdot) = X(0)/\Lambda(\cdot)$ is, under the original probability measure $\mathbb{P}$, a

- *strict local martingale* (i.e., $\mathfrak{u}(\infty) < 1$), if and only if we have $\mathbb{Q}\big(\mathcal{T} < \infty\big) > 0$ (a potential, if and only if $\mathbb{Q}\big(\mathcal{T} < \infty\big) = 1$); and

$$\mathfrak{u}(T) \,=\, \mathfrak{P}\big((T,\infty] \times \Omega\big) \,=\, \mathbb{E}\left(\frac{1_{\{\mathcal{T}>T\}}}{\Lambda(T)}\right) \,=\, \mathbb{E}^{\mathbb{Q}}\left(\frac{1_{\{\mathcal{T}>T\}}}{\Lambda(T)} \cdot \Lambda(T)\right) \,=\, \mathbb{Q}\big(\mathcal{T} > T\big)\,. \tag{7.2}$$

See in this connection Theorem 4 in Delbaen & Schachermayer (1995.a), as well as Theorem 1 and Lemma 4 of Pal & Protter (2007).

We observe also from (6.11) that $1/\Lambda(t) = \sum_{i=1}^n (\mu_i(0)/\Lambda_i(t))$, therefore

$$\mathcal{T} \,=\, \min_{1\leq i\leq n} \widetilde{\mathcal{T}}_i\,, \quad \text{where } \widetilde{\mathcal{T}}_i \text{ is the first time the process } \Lambda_i(\cdot) \text{ of (6.10) hits the origin.} \tag{7.3}$$

But let us note that $\Lambda_i(\mathcal{T})\mu_i(\mathcal{T}) = 0$ holds $\mathbb{Q}-$a.e. on the event $\{\mathcal{T} < \infty\}$ for *every* integer $i = 1, \cdots, n$. Thus, unless both $\widetilde{\mathcal{T}}_1 = \cdots = \widetilde{\mathcal{T}}_n$ and $\mu_1(\widetilde{\mathcal{T}}_1)\cdots\mu_n(\widetilde{\mathcal{T}}_n) > 0$ hold $\mathbb{Q}-$a.e. on this event, we shall have also

$$\mathcal{T} \,=\, \min_{1\leq i\leq n} \mathcal{T}_i\,, \quad \text{where } \mathcal{T}_i \text{ is the first time the market weight } \mu_i(\cdot) \text{ hits the origin.} \tag{7.4}$$

In subsection 9.3 we obtain a characterization of the type (7.4) for $\mathfrak{u}(T)$ in a Markovian context, with the help of an appropriate partial differential equation and in terms of properties of an auxiliary diffusion – whose nature is more "intrinsic" to the setting under consideration than the processes $\Lambda(\cdot)$, $\Lambda_i(\cdot)$ of (6.9), (6.10). This characterization will enable us eventually to describe the investment strategy that realizes the optimal arbitrage in this Markovian context (section 11).

Example 2: Volatility-Stabilized Model. For the model (4.16) of 'stabilization by volatility' introduced in Fernholz & Karatzas (2005), we have

$$\beta_i(t) \,=\, \frac{1+\zeta}{2}\,\frac{X(t)}{X_i(t)}\,, \qquad \sigma_{ik}(t) \,=\, \delta_{ik}\left(\frac{X(t)}{X_i(t)}\right)^{1/2} \tag{7.5}$$

thus $\vartheta_i(t) = \big(X(t)/X_i(t)\big)^{1/2}(1+\zeta)/2\,,\ 1 \le i,k \le n$, for some constant $\zeta \in [0,1]$. That is,

$$dX_i(t) \;=\; \frac{1+\zeta}{2}\,\big(X_1(t)+\cdots+X_n(t)\big)\,dt \;+\; \sqrt{X_i(t)\,\big(X_1(t)+\cdots+X_n(t)\big)}\;dW_i(t) \tag{7.6}$$

and

$$dX_i(t) = X_i(t) + \sqrt{X_i(t)\,\big(X_1(t)+\cdots+X_n(t)\big)}\;d\widetilde{W}_i(t) \tag{7.7}$$

in the notation of (2.1) and (6.8), (6.7), respectively. In particular, $dX(t) = X(t)[\,dt + d\widetilde{B}(t)\,]$ for the $\mathbb{Q}-$Brownian motion $\widetilde{B}(\cdot) := \sum_{i=1}^n \int_0^\cdot \sqrt{\mu_i(t)}\,d\widetilde{W}_i(t)$.

With $\zeta = 1$, one computes $Z(t) = \prod_{j=1}^n (X_i(0)/X_i(t))$, therefore

$$\Lambda(t) \;=\; \left(\frac{X(t)}{X(0)}\right)^{n-1}\cdot\prod_{j=1}^n\left(\frac{\mu_i(t)}{\mu_i(0)}\right), \qquad \Lambda_i(t) \;=\; \left(\frac{X(t)}{X(0)}\right)^{n-1}\cdot\prod_{j\neq i}\left(\frac{\mu_i(t)}{\mu_i(0)}\right), \quad i=1,\cdots n\,.$$

Both representations (7.3) and (7.4) hold for the first hitting time of (7.1) in this case.

7.2 A Generalized Martingale Measure

In a similar vein, there exists on a suitable canonical space a probability measure $\widehat{\mathbb{Q}}$ under which

$$L(t) \;:=\; \frac{1}{Z(t)} \;=\; \exp\left\{\int_0^t \vartheta'(s)\,d\widehat{W}(s) - \frac{1}{2}\int_0^t \big\|\vartheta(s)\big\|^2\,ds\right\}, \quad 0 \le t < \infty$$

is a continuous non-negative martingale, and with respect to which the original measure $\mathbb{P}$ is absolutely continuous with $d\mathbb{P} = L(T)\,d\widehat{\mathbb{Q}}$ on each $\mathcal{F}(T)$; whereas the process $\widehat{W}(\cdot)$ of (4.5) is $\widehat{\mathbb{Q}}-$Brownian motion. The nonnegative processes $X_i(\cdot)$, $i=1,\cdots,n$ and $X(\cdot)$ are local martingales (and supermartingales) under $\widehat{\mathbb{Q}}$: in particular, $dX_i(t) = X_i(t)\sum_{k=1}^K \sigma_{ik}(t)\,d\widehat{W}_k(t)$.

Defining $\mathcal{S} := \inf\{\,t \ge 0 \,|\, L(t) = 0\,\}$, we have $\mathbb{P}(\mathcal{S} < \infty) = 0$; we also note that $Z(\cdot)$ *is a strict local martingale under* $\mathbb{P}$, *if and only if* $\widehat{\mathbb{Q}}\big(\mathcal{S}<\infty\big) > 0$ (and is a potential, if and only if $\widehat{\mathbb{Q}}\big(\mathcal{S}<\infty\big) = 1$). The quantity of Proposition 1 can be written then as

$$\mathcal{U}^Y(T) = \mathbb{E}\left[\,Z(T)Y\,1_{\{\mathcal{S}>T\}}\right] = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\,Y\,1_{\{\mathcal{S}>T\}}\right];$$

and for $Y = X(T)/X(0)$ we obtain the expression of (6.1), (6.3) in the form

$$\mathfrak{u}(T) \;=\; \frac{1}{X(0)}\cdot\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\,X(T)\,1_{\{\mathcal{S}>T\}}\right]. \tag{7.8}$$

Thus $\mathfrak{u}(T) = 1$ (to wit, no arbitrage is possible on $[0,T]$ relative to the market), if and only if: *$X(\cdot)$ is a $\widehat{\mathbb{Q}}-$martingale on $[0,T]$, and $X(T)1_{\{\mathcal{S}\le T\}} = 0$ holds $\widehat{\mathbb{Q}}-$a.e.*

- When $X(\cdot)$ is a $\widehat{\mathbb{Q}}-$martingale, (7.8) takes the more appealing form

$$\mathfrak{u}(T) \;=\; 1 - \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\,X(\mathcal{S})\,1_{\{\mathcal{S}\le T\}}\right];$$

from (4.5) we see that this will be the case if the Novikov condition $\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\,e^{\,(1/2)\int_0^T \mu'(t)a(t)\mu(t)\,dt}\,\right] < \infty$ holds for every $T \in (0,\infty)$; or, a bit more concretely, if the boundedness condition (4.7) is satisfied.

Example 2 *(cont'd):* For the volatility-stabilized model of (7.5) we have

$$dX_i(t) = \sqrt{X_i(t)\,\big(X_1(t)+\cdots+X_n(t)\big)}\;d\widehat{W}_i(t)\,, \qquad i=1,\cdots,n$$

in the notation of (4.5), thus $dX(t) = X(t)\,d\widehat{B}(t)$ with $\widehat{B}(\cdot) := \sum_{i=1}^n \int_0^\cdot \sqrt{\mu_i(s)}\,d\widehat{W}_i(s)$ a Brownian motion under $\widehat{\mathbb{Q}}$. The process $X(\cdot) = X_1(\cdot)+\cdots+X_n(\cdot)$ is a strictly positive martingale under $\widehat{\mathbb{Q}}$. On the other hand, we have $\mathcal{S} = \mathcal{T} = \min_{1\le i\le n}\mathcal{T}_i$ as in (7.1), (7.4), since

$$L(t) \;=\; \frac{1}{Z(t)} \;=\; \left(\frac{X(t)}{X(0)}\right)^n\cdot\prod_{j=1}^n\left(\frac{\mu_i(t)}{\mu_i(0)}\right).$$

8 A Diffusion Model

We shall assume from now on that $K = n$, and that the processes $\beta_i(\cdot)$, $\sigma_{ik}(\cdot)$ in (2.1) are

$$\beta_i(t) \;=\; \mathrm{b}_i(\mathfrak{X}(t))\,, \qquad \sigma_{ik}(t) \;=\; \mathrm{s}_{ik}(\mathfrak{X}(t))\,, \qquad 1 \le i, k \le n\,, \quad 0 \le t < \infty\,. \tag{8.1}$$

Here $\mathfrak{X}(t) = (X_1(t), \cdots, X_n(t))'$ is the vector of stock prices at time t, and $\mathrm{b}_i : (0,\infty)^n \to \mathbb{R}$, $\mathrm{s}_{ik} : (0,\infty)^n \to \mathbb{R}$ suitable continuous functions. We shall denote by $\mathrm{b}(\cdot) = (\mathrm{b}_1(\cdot), \cdots, \mathrm{b}_n(\cdot))'$ and $\mathrm{s}(\cdot) = (\mathrm{s}_{ik}(\cdot))_{1\le i\le n,\, 1\le k\le n}$ the vector and matrix, respectively, of these local rate-of-return and local volatility functions. With this setup, the vector process $\mathfrak{X}(t),\ 0 \le t < \infty$ of stock-prices becomes a diffusion, with values in $(0,\infty)^n$ and dynamics

$$dX_i(t) \;=\; \mathfrak{b}_i(\mathfrak{X}(t))\,dt \,+\, \sum_{k=1}^n \mathfrak{s}_{ik}(\mathfrak{X}(t))\,dW_k(t) \;=\; X_i(t)\Big[\mathrm{b}_i(\mathfrak{X}(t))\,dt \,+\, \sum_{k=1}^n \mathrm{s}_{ik}(\mathfrak{X}(t))\,dW_k(t)\Big] \tag{8.2}$$

for $i = 1, \cdots, n$, where

$$\mathfrak{b}_i(\mathrm{x}) \;:=\; x_i\,\mathrm{b}_i(\mathbf{x})\,, \quad \mathfrak{s}_{ik}(\mathbf{x}) \;:=\; x_i\,\mathrm{s}_{ik}(\mathbf{x}) \qquad \text{for} \qquad \mathbf{x} = (x_1, \cdots, x_n)' \in (0,\infty)^n\,. \tag{8.3}$$

We shall assume further that these functions can be extended by continuity on all of $[0,\infty)^n$.

Let us introduce also the infinitesimal generator

$$\mathcal{L}f \;:=\; \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \mathfrak{a}_{ij}(\mathbf{x})\,D^2_{ij}f \,+\, \sum_{i=1}^n \mathfrak{b}_i(\mathbf{x})\,D_i f \tag{8.4}$$

of this diffusion, where we use the notation $D_i = \partial/\partial x_i$, $D = \big(\partial/\partial x_1 \cdots, \partial/\partial x_n\big)'$ and $D^2_{ij} = \partial^2/\partial x_i \partial x_j$, as well as

$$\mathfrak{a}_{ij}(\mathbf{x}) \;:=\; x_i x_j\,\mathrm{a}_{ij}(\mathbf{x})\,, \qquad \mathrm{a}_{ij}(\mathbf{x}) \;:=\; \sum_{k=1}^n \mathrm{s}_{ik}(\mathbf{x})\,\mathrm{s}_{jk}(\mathbf{x}) \qquad \text{for} \quad 1 \le i, j \le n\,. \tag{8.5}$$

Assumption C: *For every $\mathbf{x} \in (0,\infty)^n$, the local volatility matrix $\mathrm{s}(\mathbf{x}) = (\mathrm{s}_{ij}(\mathbf{x}))_{1\le i,j\le n}$ is invertible, and the system (8.2) of stochastic differential equations has a weak solution with initial condition $\mathfrak{X}(0) = \mathbf{x}$ and values in the positive orthant $(0,\infty)^n$. This solution is unique in distribution and, with $\Theta(\mathbf{x}) := \mathrm{s}^{-1}(\mathbf{x})\mathrm{b}(\mathbf{x})$, the following analogue of the conditions in (2.3), (4.3) is satisfied for each $T \in (0,\infty)$:*

$$\sum_{i=1}^n \int_0^T \Big(\big|\mathrm{b}_i\big(\mathfrak{X}(t)\big)\big| + \mathrm{a}_{ii}\big(\mathfrak{X}(t)\big) + \Theta_i^2\big(\mathfrak{X}(t)\big)\Big)\,dt \;<\; \infty\,, \quad \text{a.s.} \tag{8.6}$$

It follows from this assumption and from equation (8.2) that the Brownian motion $W(\cdot)$ is adapted to the augmentation of the filtration $\mathbb{F}^{\mathfrak{X}}$ of (5.1), and that every local martingale of the filtration $\mathbb{F}^{\mathfrak{X}}$ can be represented as a stochastic integral with respect to the Brownian motion $W(\cdot)$; see, for instance, Jacod (1977, 1979). Assumption C implies also the existence of a market-price of risk process as postulated in (4.3), namely $\vartheta(\cdot) = \Theta\big(\mathfrak{X}(\cdot)\big)$.

The requirement (8.6) is there to ensure that the solution $\mathfrak{X}(\cdot)$ of the system (8.2) takes values in the positive orthant $(0,\infty)^n$, not just the non-negative orthant $[0,\infty)^n$; and that the exponential local martingale $Z(\cdot)$ of (4.4) is well-defined.

• The following conditions from Bass & Perkins (2002), in particular their Theorem 1.2 and Corollary 1.3, are sufficient for the existence of a weak solution for (8.2) which is unique in distribution:

the functions $\mathfrak{b}_i(\cdot)$ and $\mathfrak{f}_{ij}(\mathbf{x}) := \sqrt{x_i x_j}\, \mathrm{a}_{ij}(\mathbf{x}) > 0$ are Hölder continuous on compact subsets of $(0,\infty)^n$; and we have

$$\mathfrak{b}_i(\mathbf{x}) \geq 0 \quad \text{whenever } x_i = 0\,, \qquad \|\, \mathfrak{b}(\mathbf{x})\| \leq C\big(1 + \|\mathbf{x}\|\big) \quad \text{for all } \mathbf{x} \in [0,\infty)^n \tag{8.7}$$

as well as $\mathfrak{f}_{ij}(\mathbf{x}) = 0$ for $i \neq j$ and $\mathbf{x} \in \mathcal{O}^n$, where $\mathcal{O}^n$ is the boundary of $[0,\infty)^n$. (Consult also Dawson & Perkins (2006). The Hölder continuity condition can be dropped, and we might assume instead that $\mathfrak{b}_i(\cdot)$, $\mathfrak{f}_{ij}(\cdot)$ are simply continuous, at the expense of strengthening the first requirement in (8.7) to $\mathfrak{b}_i(\mathbf{x}) > 0$ for $x_i = 0,\ \mathbf{x} \neq \mathbf{0}$; see Athreya et al. (2002), or Theorem A in Bass & Perkins (2002).)

Assumption D: *There exists a function* $H : (0,\infty)^n \to \mathbb{R}$ *of class* $\mathcal{C}^2$ *, such that*

$$\mathfrak{b}(\mathbf{x}) \;=\; \mathfrak{a}(\mathbf{x})\, DH(\mathbf{x})\,, \qquad \forall\ \ \mathbf{x} \in (0,\infty)^n\,. \tag{8.8}$$

Under this assumption, the market price of risk function $\Theta(\cdot) = \mathrm{s}^{-1}(\cdot)\mathrm{b}(\cdot)$ can be taken as

$$\Theta(\mathbf{x}) = \mathfrak{s}'(\mathbf{x})\, DH(\mathbf{x}) \qquad \text{and satisfies} \qquad \mathfrak{s}(\mathbf{x})\Theta(\mathbf{x}) \;=\; \mathfrak{b}(\mathbf{x})\,, \quad \mathbf{x} \in (0,\infty)^n\,. \tag{8.9}$$

Example 2 *(cont'd):* For the volatility-stabilized model of (7.5), we have

$$\mathrm{b}_i(\mathbf{x}) = \frac{1+\zeta}{2}\left(\frac{x_1 + \cdots + x_n}{x_i}\right), \qquad \mathrm{s}_{ik}(\mathbf{x}) = \delta_{ik}\left(\frac{x_1 + \cdots + x_n}{x_i}\right)^{1/2} \tag{8.10}$$

and $\Theta_i(\mathbf{x}) = (1+\zeta)\,\mathrm{s}_{ii}(\mathbf{x})/2$, as well as

$$\mathfrak{b}_i(\mathbf{x}) \;=\; \frac{1+\zeta}{2}\big(x_1 + \cdots + x_n\big)\,, \qquad \mathfrak{f}_{ij}(\mathbf{x}) \;=\; \delta_{ij}\,\big(x_1 + \cdots + x_n\big)\,, \qquad \mathfrak{a}_{ij}(\mathbf{x}) \;=\; x_i\, \mathfrak{f}_{ij}(\mathbf{x})$$

for $1 \leq i, j \leq n$. Thus, (8.8) is satisfied with

$$D_i H(\mathbf{x}) \;=\; \frac{1+\zeta}{2\, x_i}\,, \qquad H(\mathbf{x}) \;=\; \frac{1+\zeta}{2} \sum_{i=1}^n \log x_i\,, \qquad \mathbf{x} \in (0,\infty)^n\,.$$

The conditions of Bass & Perkins (2002) clearly hold for the system of stochastic differential equations (7.6); the unique-in-distribution solution to this system can be described in terms of independent Bessel processes $\mathfrak{R}_1(\cdot), \cdots, \mathfrak{R}_n(\cdot)$ in dimension $2(1+\zeta)$ as

$$X_i(t) \;=\; \mathfrak{R}_i^2\big(A(t)\big) > 0 \quad i = 1, \cdots, n\,, \qquad \text{with} \qquad A(t) = \frac{1}{4}\int_0^t X(s)\, ds\,. \tag{8.11}$$

In particular, the process $\mathfrak{X}(\cdot)$ takes values in the positive orthant $(0,\infty)^n$. It is also straightforward to check that the condition (8.6) is satisfied in this example, so Assumption C also holds.

Furthermore, with $\delta = \big(n(1+\zeta) - 1\big)/2$ we have the Lamperti-type description

$$X(t) \;=\; \mathfrak{R}^2\big(A(t)\big) \;=\; X(0)\, e^{\,\delta t + \mathfrak{B}(t)}\,, \qquad \mathfrak{R}(\cdot) := \sqrt{\mathfrak{R}_1^2(\cdot) + \cdots + \mathfrak{R}_n^2(\cdot)} \tag{8.12}$$

for the total capitalization $X(\cdot)$ of (3.4) in terms of a suitable Brownian motion $\mathfrak{B}(\cdot)$ and of the Bessel process $\mathfrak{R}(\cdot)$ in dimension $2n(1+\zeta)$. This process $\mathfrak{R}(\cdot)$ is *independent* of the market weight processes $\big(\mu_1(\cdot), \cdots, \mu_n(\cdot)\big)$, and we have the representations

$$\mathfrak{R}_i^2(u) \;=\; \mathfrak{R}^2(u) \cdot \mu_i\big(A^{-1}(u)\big)\,, \quad 0 \leq u < \infty \qquad \text{with} \qquad A^{-1}(u) = 4\int_0^u \frac{dv}{\mathfrak{R}^2(v)}$$

for $i = 1, \cdots, n$. This is in the spirit of Warren & Yor's (1997) "skew-representation" for processes of Bessel and Jacobi type. □

Throughout the remainder of this paper, Assumptions C, D will be in force, and we shall take as our observations filtration $\mathbb{F} \equiv \mathbb{F}_+^{\mathfrak{X}}$ *, the right-continuous version of the natural filtration in (5.1);* this is of course a very natural choice and, as we have seen, consistent with Assumption A.

9 A Parabolic PDE for the function $U(\tau, \mathbf{x})$

Because of the uniqueness in distribution posited in Assumption B, the process $\mathfrak{X}(\cdot)$ is (strongly) Markovian; we shall denote by $\mathbb{P}^{\mathbf{x}}$ the distribution of this process started at $\mathfrak{X}(0) = \mathbf{x} \in (0,\infty)^n$. Our objective now is to study

$$U(T, \mathbf{x}) := \frac{1}{x_1 + \cdots + x_n} \cdot \mathbb{E}^{\mathbb{P}^{\mathbf{x}}} \big[Z(T) X(T) \big], \tag{9.1}$$

the quantity of (6.1), (6.3) in this diffusion context.

We start by observing that for the function $H(\cdot)$ of Assumption D, Itô's rule gives

$$H\big(\mathfrak{X}(T)\big) - H\big(\mathfrak{X}(0)\big) - \int_0^T \mathcal{L}H\big(\mathfrak{X}(t)\big)dt = \int_0^T \big(DH\big(\mathfrak{X}(t)\big)\big)' \mathfrak{s}(\mathfrak{X}(t)) dW(t) = \int_0^T \Theta'\big(\mathfrak{X}(t)\big) dW(t)$$

in the notation of (8.4) and (8.9). Thus, the exponential local martingale $Z(\cdot)$ of (4.4) becomes

$$Z(T) = \exp\left\{ H\big(\mathfrak{X}(0)\big) - H\big(\mathfrak{X}(T)\big) - \int_0^T k\big(\mathfrak{X}(t)\big)\, dt \right\}, \tag{9.2}$$

where

$$k(\mathbf{x}) := -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n x_i\, x_j\, \mathrm{a}_{ij}(\mathbf{x}) \Big[D^2_{ij} H(\mathbf{x}) + D_i H(\mathbf{x})\, D_j H(\mathbf{x}) \Big], \qquad \mathbf{x} \in (0,\infty)^n. \tag{9.3}$$

(In particular, the process $Z(\cdot)$ is adapted to the filtration $\mathbb{F}^{\mathfrak{X}}$ of (5.1).) Then (9.1) becomes

$$U(T, \mathbf{x}) = \frac{e^{H(\mathbf{x})}}{x_1 + \cdots + x_n} \, \mathbb{E}^{\mathbb{P}^{\mathbf{x}}} \left[e^{-H(\mathfrak{X}(T))} \big(X_1(T) + \cdots + X_n(T) \big) \exp\left\{ - \int_0^T k\big(\mathfrak{X}(t)\big)\, dt \right\} \right]$$

or equivalently

$$U(T, \mathbf{x}) = \frac{G(T, \mathbf{x})}{g(\mathbf{x})}, \qquad T \in (0,\infty), \quad \mathbf{x} \in (0,\infty)^n \tag{9.4}$$

where we have set

$$g(\mathbf{x}) := \frac{x_1 + \cdots + x_n}{e^{H(\mathbf{x})}}, \qquad G(T, \mathbf{x}) := \mathbb{E}^{\mathbb{P}^{\mathbf{x}}} \left[g\big(\mathfrak{X}(T)\big) \exp\left\{ - \int_0^T k\big(\mathfrak{X}(t)\big)\, dt \right\} \right]. \tag{9.5}$$

A bit more generally, these same considerations – coupled with the Markov property of the process $\mathfrak{X}(\cdot)$ – lead for any given $0 \le t \le T$ to the almost sure identity

$$\frac{\mathbb{E}^{\mathbb{P}^{\mathbf{x}}} \big[X(T) Z(T) \,|\, \mathcal{F}(t) \big]}{X(t) Z(t)} = \frac{G(T-t, \mathbf{y})}{g(\mathbf{y})} \bigg|_{\mathbf{y} = \mathfrak{X}(t)} = U\big(T - t, \mathfrak{X}(t)\big). \tag{9.6}$$

Assumption E: *The function $G(\,\cdot\,,\cdot\,)$ of (9.5) takes values in $(0,\infty)$, is of class $\mathcal{C}^{1,2}$ on $(0,\infty) \times (0,\infty)^n$, and solves the Cauchy problem*

$$\frac{\partial G}{\partial \tau}(\tau, \mathbf{x}) = \mathcal{L}G(\tau, \mathbf{x}) - k(\mathbf{x}) G(\tau, \mathbf{x}), \qquad \tau \in (0,\infty), \quad \mathbf{x} \in (0,\infty)^n \tag{9.7}$$

$$G(0+, \mathbf{x}) = g(\mathbf{x}), \qquad \mathbf{x} \in (0,\infty)^n. \tag{9.8}$$

This Cauchy problem is exactly the one arising in classical Feynman-Kac theory; see, for instance, sections 5.6, 6.5 in Friedman (1975), as well as section 5.7 in Karatzas & Shreve (1991).

The difficulty here comes from the unboundedness of the coëfficients $\mathfrak{b}_i(\cdot)$, $\mathfrak{a}_{ij}(\cdot)$, and from the degeneracy they may exhibit on the boundary $\mathcal{O}^n$ of the orthant $[0,\infty)^n$.

The partial differential equation (9.7) is satisfied under the Bass & Perkins (2002) conditions preceding, following and including (8.7), provided that the continuous function $k(\cdot)$ of (9.3) is bounded from below and locally Hölder continuous; that the vector-valued function $\mathfrak{b}(\cdot) = \big(\mathfrak{b}_1(\cdot), \cdots, \mathfrak{b}_n(\cdot)\big)'$ and the matrix-valued function $\mathfrak{s}(\cdot) = \big(\mathfrak{s}_{ik}(\cdot)\big)_{1 \le i,k \le n}$ of (8.3) are locally Hölder continuous and satisfy the linear growth condition

$$\| \, \mathfrak{b}(\mathbf{x})\| + \| \, \mathfrak{s}(\mathbf{x})\| \le C\big(1 + \|\mathbf{x}\|\big) , \qquad \forall \;\; \mathbf{x} \in [0,\infty)^n$$

for some $C \in (0,\infty)$; and that the following non-degeneracy condition holds: *For every compact subset* $\mathcal{K}$ *of* $(0,\infty)^n$, *there exists a number* $\varepsilon = \varepsilon_{\mathcal{K}} > 0$ *such that*

$$\sum_{i=1}^n \sum_{j=1}^n \mathfrak{a}_{ij}(\mathbf{x})\, \xi_i \xi_j \;\ge\; \varepsilon \, \|\xi\|^2 , \quad \forall \;\; \mathbf{x} \in \mathcal{K} , \;\; \xi \in \mathbb{R}^n . \tag{9.9}$$

The initial condition (9.8) is then satisfied if the continuous function $g(\cdot)$ of (9.5) has polynomial growth of the type

$$0 \;<\; g(\mathbf{x}) \;\le\; \mathcal{C}\big(1 + \| \, \mathbf{x}\|^p + \| \, \mathbf{x}^{-1}\|^q\big) , \qquad \forall \;\; \mathbf{x} \in (0,\infty)^n$$

for some real constants $\mathcal{C} > 0$, $p > 0$ and $q > 0$, where $\mathbf{x}^{-1} = (1/x_1, \cdots, 1/x_n)'$.

For justification of these claims, see Friedman (1964), Chapter 1 and Janson & Tysk (2006), Theorem 2.7.

Example 2 *(cont'd):* For the volatility-stabilized model of (8.10), we have

$$e^{\,H(\mathbf{x})} \;=\; \big(x_1 \cdots x_n\big)^{(1+\zeta)/2} , \qquad k(\mathbf{x}) \;=\; \frac{1-\zeta^2}{8}\, \big(x_1 + \cdots + x_n\big) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) .$$

The function $k(\cdot)$ is non-negative, whereas

$$g(\mathbf{x}) \;=\; (x_1 + \cdots + x_n)\, \big(x_1 \cdots x_n\big)^{-(1+\zeta)/2}$$

satisfies $0 < g(\mathbf{x}) \le n^{-\kappa}\, (x_1 + \cdots + x_n)\big(x_1^{-1} + \cdots + x_n^{-1}\big)^{\kappa}$ with $\kappa := n(1+\zeta)/2$.

In particular, with $\zeta = 1$ the function $U(T,\mathbf{x})$ of (9.1) becomes

$$U(T,\mathbf{x}) \;=\; \frac{x_1 \cdots x_n}{x_1 + \cdots + x_n} \cdot \mathbb{E}^{\mathbb{P}^{\mathbf{x}}} \left[\frac{X_1(T) + \cdots + X_n(T)}{X_1(T) \cdots X_n(T)} \right] . \tag{9.10}$$

It would be of considerable interest to achieve as explicit a computation of this expression as possible, using the theory of Bessel processes and of exponential functionals of Brownian motion that has been developed by Yor (2001) and his collaborators. For some first steps in this regard, see the computations in the Appendix. □

Assumption E will also be imposed from now onwards. It implies that the function $U : [0,\infty) \times (0,\infty)^n \to (0,1]$ of (9.1) is of class $\mathcal{C}^{1,2}$ on $(0,\infty) \times (0,\infty)^n$ as well; and some sustained computation shows that the Cauchy problem of (9.7), (9.8) for the function $G(\,\cdot\,,\,\cdot\,)$, leads to a corresponding Cauchy problem for the function $U(\,\cdot\,,\,\cdot\,)$, namely

$$\frac{\partial U}{\partial \tau}(\tau,\mathbf{x}) \;=\; \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \mathfrak{a}_{ij}(\mathbf{x})\, D^2_{ij} U(\tau,\mathbf{x}) \;+\; \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\mathfrak{a}_{ij}(\mathbf{x})}{x_1 + \cdots + x_n} \right) D_i U(\tau,\mathbf{x}) \tag{9.11}$$

$$\;=\; \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n x_i x_j \, \mathrm{a}_{ij}(\mathbf{x})\, D^2_{ij} U(\tau,\mathbf{x}) \;+\; \sum_{i=1}^n x_i \left(\sum_{j=1}^n \frac{x_j \, \mathrm{a}_{ij}(\mathbf{x})}{x_1 + \cdots + x_n} \right) D_i U(\tau,\mathbf{x})$$

for $(\tau,\mathbf{x}) \in (0,\infty) \times (0,\infty)^n$, as well as to the initial condition

$$U(0+,\mathbf{x}) \;=\; 1 , \qquad \mathbf{x} \in (0,\infty)^n . \tag{9.12}$$

9.1 An Informal Derivation of (9.11), (9.12)

Rather than including the computations which lead from (9.7) to the equation (9.11), we present here a rather simple, informal argument that we shall find useful also in the next subsection, in a more formal setting. We start by casting (6.6) as

$$\frac{d\,(X(t)Z(t))}{X(t)Z(t)} = \sum_{k=1}^{n}\left(\sum_{i=1}^{n}\mu_i(t)\sigma_{ik}(t)-\vartheta_k(t)\right)d\,W_k(t) = -\sum_{k=1}^{n}\widetilde{\Theta}_k\big(\mathfrak{X}(t)\big)\,d\,W_k(t)\,,$$

where we have set

$$\widetilde{\Theta}_k(\mathbf{x}) := \Theta_k(\mathbf{x})-\sum_{i=1}^{n}\left(\frac{x_i\,\mathrm{s}_{ik}(\mathbf{x})}{x_1+\cdots+x_n}\right), \qquad k=1,\cdots,n \tag{9.13}$$

by analogy with (6.7). On the other hand, Itô's rule gives

$$d\,U\big(T-t,\mathfrak{X}(t)\big) = \left(\mathcal{L}U-\frac{\partial U}{\partial\tau}\right)\big(T-t,\mathfrak{X}(t)\big)\,dt + \sum_{k=1}^{n}R_k\big(T-t,\mathfrak{X}(t)\big)\,d\,W_k(t)\,,$$

where

$$R_k(\tau,\mathbf{x}) := \sum_{i=1}^{n}x_i\,\mathrm{s}_{ik}(\mathbf{x})\,D_iU(\tau,\mathbf{x})\,, \qquad k=1,\cdots,n\,. \tag{9.14}$$

Thus, for the martingale

$$N(t) := X(t)Z(t)\cdot U\big(T-t,\mathfrak{X}(t)\big) = \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\big[\,X(T)Z(T)\,|\,\mathcal{F}(t)\,\big]\,, \qquad 0\le t\le T \tag{9.15}$$

of (9.6), the product rule of the stochastic calculus gives

$$\frac{d\,N(t)}{X(t)Z(t)} = d\,U\big(T-t,\mathfrak{X}(t)\big)+U\big(T-t,\mathfrak{X}(t)\big)\frac{d\,(X(t)Z(t))}{X(t)Z(t)} - \sum_{k=1}^{n}R_k\big(T-t,\mathfrak{X}(t)\big)\,\widetilde{\Theta}_k\big(\mathfrak{X}(t)\big)\,dt$$

$$= C\big(T-t,\mathfrak{X}(t)\big)dt + \sum_{k=1}^{n}\big[\,R_k\big(T-t,\mathfrak{X}(t)\big)-U\big(T-t,\mathfrak{X}(t)\big)\,\widetilde{\Theta}_k\big(\mathfrak{X}(t)\big)\,\big]d\,W_k(t)\,.$$

We have set

$$C(\tau,\mathbf{x}) := \left(\mathcal{L}U-\frac{\partial U}{\partial\tau}\right)(\tau,\mathbf{x}) - \sum_{k=1}^{n}R_k(\tau,\mathbf{x})\,\widetilde{\Theta}_k(\mathbf{x})$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathfrak{a}_{ij}(\mathbf{x})\,D_{ij}^2U(\tau,\mathbf{x}) + \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\mathfrak{a}_{ij}(\mathbf{x})}{x_1+\cdots+x_n}\right)D_iU(\tau,\mathbf{x}) - \frac{\partial U}{\partial\tau}(\tau,\mathbf{x})\,,$$

where the last equality is checked easily from (8.4) and (8.9).

But the process $N(\cdot)$ is a martingale, so this finite-vatiation term $C(\tau,\mathbf{x})$ should vanish, and thus

$$\frac{d\,N(t)}{N(t)} = \sum_{k=1}^{n}\left[\frac{R_k\big(T-t,\mathfrak{X}(t)\big)}{U\big(T-t,\mathfrak{X}(t)\big)} - \widetilde{\Theta}_k\big(\mathfrak{X}(t)\big)\right]d\,W_k(t)\,. \tag{9.16}$$

In other words, the function $U:[0,\infty)\times(0,\infty)^n\to(0,1]$ of (9.3) must satisfy the parabolic partial differential equation (9.11), as postulated earlier.

Remark 2: This informal derivation raises the possibility that it might be possible to dispense with Assumptions D, E altogether, if it could be shown from first principles that the function $U : [0, \infty) \times (0, \infty)^n \to (0, 1]$ of (9.1) is of class $\mathcal{C}^{1,2}$ on $(0, \infty) \times (0, \infty)^n$.

Indeed, under suitable conditions on the functions $\mathfrak{s}_{ik}(\cdot)$, $\Theta_i(\cdot)$, $1 \le i, k \le n$, one can rely on techniques from the Malliaving calculus and the Hörmander hypoëllipticity theorem (Nualart (1995), pp. 99-124, Bell (1995) or Bass (1998), Ch. 8) to show that the $(n+2)-$dimensional random vector $\big(\mathfrak{X}(T), \Upsilon(T), \Xi(T)\big)$ with

$$\Upsilon(T) := \int_0^T \Theta\big(\mathfrak{X}(t)\big)' \, dW(t) \,, \qquad \Xi(T) := \int_0^T \|\Theta\big(\mathfrak{X}(t)\big)\|^2 \, dt$$

has an infinitely differentiable probability density function, for any given $T \in (0, \infty)$. This provides then the requisite smoothness on the part of

$$U(T, \mathbf{x}) = \frac{1}{x_1 + \cdots + x_n} \cdot \mathbb{E}^{\mathbb{P}^{\mathbf{x}}} \Big[\big(X_1(T) + \cdots + X_n(T)\big) \, e^{\,\Upsilon(T) - (\Xi(T)/2)} \Big] \,.$$

The conditions that are needed for this approach to work are strong; they include the infinite-differentiability on the functions $\mathfrak{s}_{ik}(\cdot)$, $\Theta_i(\cdot)$, $1 \le i, k \le n$, as well as additional algebraic conditions which, in the present context, are somewhat opaque and not very easy to state or verify. For these reasons we have opted for sticking with Assumptions D and E; these are satisfied in the Examples we follow throughout this paper, are easy to test, and allow us to describe Föllmer's exit measure via the explicit representations (9.31), (9.32) which do not involve stochastic integrals.

9.2 Results and Ramifications

Let us note that the equation (9.11) is determined *entirely* from the volatility structure of the model of (2.1). Note also that the Cauchy problem of (9.11), (9.12) admits the trivial solution $U(\tau, \mathbf{x}) \equiv 1$; thus, the existence of arbitrage relative to the market portfolio over a finite time-horizon $[0, T]$ is tantamount to the *failure of uniqueness* for the Cauchy problem of (9.11), (9.12) over the strip $[0, T] \times (0, \infty)^n$.

Remark 3: Assume there exists some real number $h > 0$ such that the continuous functions $\mathrm{a}_{ij}(\cdot)$, $1 \le i, j \le n$ satisfy either the condition

$$(x_1 + \cdots + x_n) \sum_{i=1}^n x_i \mathrm{a}_{ii}(\mathbf{x}) - \sum_{i=1}^n \sum_{j=1}^n x_i x_j \mathrm{a}_{ij}(\mathbf{x}) \ge h \, (x_1 + \cdots + x_n)^2 \,, \tag{9.17}$$

or the condition

$$\big(x_1 \cdots x_n\big)^{1/n} \left[\sum_{i=1}^n \mathrm{a}_{ii}(\mathbf{x}) - \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \mathrm{a}_{ij}(\mathbf{x}) \right] \ge h \, (x_1 + \cdots + x_n) \,, \tag{9.18}$$

for all $\mathbf{x} \in (0, \infty)^n$ (we have just re-written (4.10), (4.12) in the present context). Then from the results of Fernholz & Karatzas (2005, 2008) reviewed in section 4, we deduce that

$$U(T, \mathbf{x}) < 1 \,, \qquad \forall \;\; \mathbf{x} \in (0, \infty)^n$$

holds for all $T > (2 \log n)/h$ under (9.17), and for all $T > \big(2n^{1-(1/n)}\big)/h$ under (9.18).

In particular, under either (9.17) or (9.18), *uniqueness fails for the Cauchy problem of (9.11), (9.12).* To the best of our knowledge, this is the first instance where sufficient conditions for non-uniqueness (equivalently, necessary conditions for uniqueness) are obtained for such Cauchy problems involving linear parabolic partial differential equations. □

Whenever uniqueness fails for the Cauchy problem of (9.11), (9.12), it is important to know how to pick the "right" solution from among all possible solutions, the one which gives the quantity of (9.1) we are interested in. The next result addresses this question.

Proposition 2: *The function* $U : [0,\infty) \times (0,\infty)^n \to (0,1]$ *of (9.1) is the* smallest non-negative *and continuous function, which is of class* $\mathcal{C}^{1,2}$ *on* $(0,\infty)\times(0,\infty)^n$ *and satisfies the initial condition (9.12), as well as the following parabolic partial differential inequality on* $(0,\infty) \times (0,\infty)^n$ *:*

$$\frac{\partial U}{\partial \tau}(\tau,\mathbf{x}) \;\geq\; \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathfrak{a}_{ij}(\mathbf{x})\, D^2_{ij} U(\tau,\mathbf{x}) \;+\; \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\mathfrak{a}_{ij}(\mathbf{x})}{x_1+\cdots+x_n}\right) D_i U(\tau,\mathbf{x})\,. \tag{9.19}$$

Proof: Consider any continuous function $\widetilde{U} : [0,\infty) \times (0,\infty)^n \to [0,\infty)$ of class $\mathcal{C}^{1,2}$ on $(0,\infty) \times (0,\infty)^n$ that satisfies both (9.19) and (9.12), and define as in (9.15) the process

$$\widetilde{N}(t) \;:=\; X(t)Z(t)\cdot \widetilde{U}\big(T-t, \mathfrak{X}(t)\big)\,, \qquad 0 \leq t \leq T\,.$$

Repeat (almost verbatim) the arguments in subsection 8.1, and use (9.19) to conclude that the positive process $\widetilde{N}(\cdot)$ is a local supermartingale. Thus $\widetilde{N}(\cdot)$ is bona-fide supermartingale,

$$(x_1+\cdots+x_n)\,\widetilde{U}(T,\mathbf{x}) = \widetilde{N}(0) \geq \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\big(\widetilde{N}(T)\big) = \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\big(X(T)Z(T)\big)$$

holds for every $T \in (0,\infty)$, $\mathbf{x} \in (0,\infty)^n$, and $\widetilde{U}(T,\mathbf{x}) \geq U(T,\mathbf{x})$ follows from (9.1). □

Proposition 3: *Assume that the continuous functions* $\big(\mathfrak{a}_{ij}(\cdot)\big)_{1\leq i,j \leq n}$ *of (8.5) satisfy the non-degeneracy condition (9.9). Then, if*

$$U(T,\mathbf{x}) \;<\; 1\,, \quad \forall \;\; \mathbf{x} \in (0,\infty)^n \tag{9.20}$$

holds for some $T \in (0,\infty)$, *we have*

$$U(T,\mathbf{x}) \;<\; 1\,, \quad \forall \;\; (T,\mathbf{x}) \in (0,\infty)\times(0,\infty)^n\,. \tag{9.21}$$

Proof: For every $\tau > 0$, consider the set $\mathcal{S}(\tau) := \{\mathbf{x} \in (0,\infty)^n \,|\, U(\tau,\mathbf{x}) = 1\}$ and define

$$\tau_* \;:=\; \sup\{\,\tau \in (0,\infty) \mid \mathcal{S}(\tau) \neq \emptyset\,\}$$

(with the usual convention $\tau_* = 0$ if the indicated set is empty). The assumption (9.20) of the Proposition amounts to $\tau_* < \infty$, and its claim (9.21) to $\tau_* = 0$; we shall prove this claim by contradiction.

Suppose $\tau_* > 0$; then $U(\tau_* - \delta, \mathbf{x}_*) = 1$ for any given $\delta \in (0, \tau_*/2)$, and some $\mathbf{x}_* \in (0,\infty)^n$. For *any* given $\mathbf{x} \in (0,\infty)^n$, consider an open, connected set D which contains both $\mathbf{x}$ and $\mathbf{x}_*$, and whose closure $\overline{D}$ is a compact subset of $(0,\infty)^n$; in particular, we have $\inf\{\|\mathbf{y}-\mathbf{z}\| \mid \mathbf{y} \in \overline{D}\,, \mathbf{z} \in \mathcal{O}^n\,\} > 0$. Then the function $U(\,\cdot\,,\cdot\,)$ attains its maximum value over the cylindrical domain $\mathfrak{C} = \big\{\,(\tau,\xi) \,|\, 0 < \tau < \tau_* + 1\,,\; \xi \in D\,\big\}$ at the point $(\tau_* - \delta, \mathbf{x}_*)$, which lies in the interior of this domain. By assumption, the operator

$$\widehat{\mathcal{L}} f \;:=\; \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathfrak{a}_{ij}(\mathbf{x})\, D^2_{ij} f \;+\; \sum_{i=1}^{n} \widehat{\mathfrak{b}}_i(\mathbf{x})\, D_i f \tag{9.22}$$

with

$$\widehat{\mathfrak{b}}_i(\mathbf{x}) \;:=\; x_i\, \widehat{\mathrm{b}}_i(\mathbf{x})\,, \qquad \widehat{\mathrm{b}}_i(\mathbf{x}) \;:=\; \sum_{j=1}^{n} \frac{x_j\, \mathrm{a}_{ij}(\mathbf{x})}{x_1+\cdots+x_n}\,, \quad i = 1,\cdots,n\,, \tag{9.23}$$

is uniformly parabolic and has bounded, continuous coëfficients on $\mathfrak{E}$. The maximum principle for parabolic partial differential operators (Friedman (1964), Chapter 2; Protter & Weinberger (1967), Chapter 3, section 3) implies then

$$U(\tau, \mathbf{x}) \,=\, 1\,, \quad \forall \;\; 0 \le \tau \le \tau_* - \delta\,. \tag{9.24}$$

Now let us recall the $\mathbb{P}^{\mathbf{x}}-$a.s. equality $\mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\big[\,X(T)Z(T)\,|\,\mathcal{F}(t)\,\big] = U\big(T-t, \mathfrak{X}(t)\big)\cdot X(t)Z(t)$ from (9.6); we apply it with $0 \le t \le \tau_* - \delta$, $0 \le T - t \le \tau_* - \delta$, then take expectations with respect to the probability measure $\mathbb{P}^{\mathbf{x}}$ and use (9.24) to obtain

$$U(T, \mathbf{x}) \,=\, \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\big[\,X(T)Z(T)\,\big] \,=\, \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\big[\,X(t)Z(t)\,\big] \,=\, U(t, \mathbf{x}) \,=\, 1$$

for every $T \in [\,0,\, 2(\tau_* - \delta)\,]$ and $\mathbf{x} \in (0, \infty)^n$. But since $2(\tau_* - \delta) > \tau_*$, this contradicts the definition of τ_*. □

Corollary: Under the non-degeneracy condition (9.9), and with either (9.17) or (9.18), the inequality of (9.21) holds. To wit, arbitrage with respect to the market exists then over *any* time-horizon $[0, T]$ with $T \in (0, \infty)$.

Example 2 *(cont'd):* For the volatility-stabilized model of (8.10), the equation (9.11) becomes

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{x}) \,=\, \frac{1}{2}\big(x_1 + \cdots + x_n\big) \sum_{i=1}^{n} x_i\, D^2_{ii} U(\tau, \mathbf{x}) \,+\, \sum_{i=1}^{n} x_i\, D_i U(\tau, \mathbf{x})$$

and both (9.17), (9.18) are satisfied, the first as equality, with $h = n - 1$; from the previous Corollary and from Remark 3, it follows that (9.21) holds.

We recover in this manner the 'Markovian' results of Banner & Fernholz (2008) on the existence of arbitrage relative to the volatility-stabilized market over *arbitrary* time-horizons (their 'non-Markovian' results are not covered by Proposition 3). □

Remark 4: The condition (9.20) of Proposition 3 can be relaxed to the requirement that there exist some $T \in (0, \infty)$ for which

$$U(T, \mathbf{x}) \,<\, 1 \quad \text{holds for some } \; \mathbf{x} \in (0, \infty)^n\,. \tag{9.25}$$

An argument similar to that used in the proof of Proposition 3, based on the maximum principle, shows that (9.25) implies (9.20) under the assumptions in the first sentence of the proposition.

9.3 An Auxiliary Diffusion

Let us consider now the diffusion process $\mathfrak{Y}(t) = \big(Y_1(t), \cdots, Y_n(t)\big)'$, $\; 0 \le t < \infty$ with dynamics

$$dY_i(t) \,=\, Y_i(t) \left[\widehat{\mathrm{b}}_i\big(\mathfrak{Y}(t)\big)\, dt \,+\, \sum_{k=1}^{n} \mathrm{s}_{ik}\big(\mathfrak{Y}(t)\big)\, dW_k(t) \right], \qquad i = 1, \cdots, n \tag{9.26}$$

and infinitesimal generator $\widehat{\mathcal{L}}$ as in (9.22), (9.23). *We shall assume that the system of stochastic differential equations (9.26) admits a weak solution with values in* $[0, \infty)^n \setminus \{\mathbf{0}\}$ *which is unique in distribution.* This will be the case, for instance, if the drift functions $\widehat{\mathfrak{b}}_i(\cdot)$, $1 \le i \le n$ of (9.23) also satisfy the Bass & Perkins (2002) conditions preceding, following and including (8.7).

The resulting process $\mathfrak{Y}(\cdot)$ is then Markovian; we shall denote by $\mathbb{Q}^{\mathbf{y}}$ the distribution of this process started at $\mathfrak{Y}(0) = \mathbf{y} \in [0, \infty)^n$. Unlike our original process $\mathfrak{X}(\cdot)$, which takes values in $(0, \infty)^n$, this new process $\mathfrak{Y}(\cdot)$ is only guaranteed to take values in the non-negative orthant $[0, \infty)^n \setminus \{\mathbf{0}\}$. To wit, the first hitting time

$$\mathfrak{T} \,:=\, \inf\{\, t \ge 0 \mid \mathfrak{Y}(t) \in \mathcal{O}^n \,\} \tag{9.27}$$

of the boundary $\mathcal{O}^n$ of $[0,\infty)^n$ may be finite with positive $\mathbb{Q}^{\mathbf{x}}-$probability, for $\mathbf{x} \in (0,\infty)^n$.

As our next result, Proposition 4, shows, this possibility amounts to the existence of arbitrage relative to the market portfolio, and to the lack of uniqueness for the Cauchy problem of (9.11), (9.12). Conversely, arbitrage relative to the market portfolio is impossible on any given finite time-horizon, if and only if the diffusion process $\mathfrak{Y}(\cdot)$ of (9.26) never reaches the boundary $\mathcal{O}^n$ of $[0,\infty)^n$ when started with initial configuration $\mathfrak{Y}(0) = \mathbf{x} \in (0,\infty)^n$.

Let us observe that, in accordance with Remark 1 following (3.6), the 'relative weights' $\nu_i(t) := Y_i(t)/(Y_1(t) + \cdots + Y_n(t))$, $i = 1, \cdots, n$ with the dynamics of (6.12), namely

$$d\nu_i(t) \,=\, \nu_i(t) \sum_{k=1}^n \Big(\mathrm{s}_{ik}\big(\mathfrak{Y}(t)\big) - \sum_{j=1}^n \nu_j(t)\, \mathrm{s}_{jk}\big(\mathfrak{Y}(t)\big) \Big)\, dW_k(t)\,, \qquad i = 1, \cdots, n\,. \tag{9.28}$$

These weights are non-negative martingales with values in $[0,1]$; in particular, when any one of them hits either boundary point of the unit interval $(0,1)$, it gets absorbed there. Not all can get absorbed at zero, however, because (9.28) gives $\mathbb{Q}^{\mathbf{x}}-$a.e.: $\sum_{i=1}^n \nu_i(t) = 1$, $\forall\ 0 \le t < \infty$. In terms of these weights, the first hitting time of (9.27) can be expressed in the manner of (7.4) as

$$\mathfrak{T} \,=\, \min_{1 \le i \le n} \mathfrak{T}_i\,, \quad \text{ where } \quad \mathfrak{T}_i \,:=\, \inf\{\, t \ge 0 \mid \nu_i(t) = 0\,\}\,. \tag{9.29}$$

Proposition 4: *With the above assumptions and notation, the function* $U : [0,\infty) \times (0,\infty)^n \to (0,1]$ *of (9.1) admits the stochastic representation*

$$U(T,\mathbf{x}) \,=\, \mathbb{Q}^{\mathbf{x}}\left[\, \mathfrak{T} > T\,\right], \qquad (T,\mathbf{x}) \in (0,\infty) \times (0,\infty)^n\,. \tag{9.30}$$

Proof: The function on the right-hand side of (9.30) satisfies the equation (9.11) and the initial condition (9.12). On the other hand, consider any continuous function $V : [0,\infty)\times[0,\infty)^n \to [0,\infty)$ which is of class $\mathcal{C}^{1,2}$ on $(0,\infty) \times (0,\infty)^n$ and satisfies there the parabolic partial differential inequality (9.19), as well as the initial condition $V(0,\,\cdot\,) \equiv 1$ on $(0,\infty)^n$. Then $V(T-t, \mathfrak{Y}(t))$, $0 \le t \le T$ is a non-negative local supermartingale, thus also a bona-fide supermartingale, and

$$V(T,\mathbf{x}) \,\ge\, \mathbb{E}^{\mathbb{Q}^{\mathbf{x}}} \Big[\, V\big(T - (T \wedge \mathfrak{T}),\, \mathfrak{Y}(T \wedge \mathfrak{T})\big) \Big] \,\ge\, \mathbb{E}^{\mathbb{Q}^{\mathbf{x}}} \Big[\, V\big(0,\, \mathfrak{Y}(T)\big) \cdot 1_{\{\mathfrak{T} > T\}} \Big] \,=\, \mathbb{Q}^{\mathbf{x}}\left[\, \mathfrak{T} > T\,\right]$$

holds for every $(T,\mathbf{x}) \in (0,\infty) \times (0,\infty)^n$. The claim follows now from Proposition 2. □

Corollary: *Under the assumptions of this subsection, for any given* $\mathbf{x} \in (0,\infty)^n$ *the* $\mathbb{P}^{\mathbf{x}}-$*supermartingale* $Z(\cdot)X(\cdot)$ *is a* $\mathbb{P}^{\mathbf{x}}-$

- *martingale, if and only if* $\mathbb{Q}^{\mathbf{x}}\big(\mathfrak{T} < \infty\big) = 0$;
- *potential* (*i.e.,* $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\big(Z(T)X(T)\big) = 0$), *if and only if* $\mathbb{Q}^{\mathbf{x}}\big(\mathfrak{T} < \infty\big) = 1$;
- *strict local* (*and super*)*martingale on the time-horizon* $[0,T]$ (*respectively, on* $[0,\infty)$), *if and only if* $\mathbb{Q}^{\mathbf{x}}\big(\mathfrak{T} > T\big) < 1$ (*respectively,* $\mathbb{Q}^{\mathbf{x}}\big(\mathfrak{T} < \infty\big) > 0$).

When $\mathbf{x} \in (0,\infty)^n$ and the quantity of (9.30) is equal to one, the $\mathbb{Q}^{\mathbf{x}}-$distribution of the process $\mathfrak{Y}(t)$, $0 \le t \le T$ in (9.26) is the same as the $\widetilde{\mathbb{P}}^{\mathbf{x}}_T-$distribution of the original stock-price process $\mathfrak{X}(t)$, $0 \le t \le T$; this follows by comparing (9.26), (9.23) with (6.8), and denoting by $\widetilde{\mathbb{P}}^{\mathbf{x}}_T$ the probability measure $\widetilde{\mathbb{P}}_T$ of (6.7) with $\mathfrak{X}(0) = \mathbf{x}$.

More generally, we represent as in (7.2) the exit measure $\mathfrak{P}^{\mathbf{x}}$ of the supermartingale $Z(\cdot)X(\cdot)$

$$\mathfrak{P}^{\mathbf{x}}\big((T,\infty] \times \Omega\big) \,=\, U(T,\mathbf{x}) \,=\, \mathbb{Q}^{\mathbf{x}}\left[\, \mathfrak{T} > T\,\right] \tag{9.31}$$

with initial configuration $\mathfrak{X}(0) = \mathbf{x}$; and from (9.2)-(9.6) we have, for $A \in \mathcal{F}(t)$, $0 \le t \le T$:

$$\mathfrak{P}^{\mathbf{x}}\big((T,\infty] \times A\big) \,=\, \mathbb{E}^{\mathbb{P}^{\mathbf{x}}} \left[\, \frac{g\big(\mathfrak{X}(t)\big)}{g(\mathbf{x})}\, \exp\Big\{ -\int_0^t k\big(\mathfrak{X}(s)\big)\, ds \Big\}\, 1_A \cdot \Big(\mathbb{Q}^{\mathbf{z}}\left[\, \mathfrak{T} > T - t\,\right]\Big)\Big|_{\mathbf{z} = \mathfrak{X}(t)} \right]. \tag{9.32}$$

Remark 5: The probability measure $\mathbb{Q}^{\mathbf{x}}$ corresponds to a change of drift functions, from $\mathrm{b}_i(\mathbf{x})$ in (8.2) to $\widehat{\mathrm{b}}_i(\mathbf{x})$ in (9.23), (9.26). The rôle of this change of drift is to ensure that in this latter, "fictitious" model with asset prices $\mathfrak{Y}(\cdot)$, the relative capitalization weights of (9.28) are $\mathbb{Q}^{\mathbf{x}}-$martingales – and thus the resulting "fictitious market" portfolio $\underline{\nu}(\cdot) = (\nu_1(\cdot), \cdots, \nu_n(\cdot))'$ has the numéraire property of Remark 1. In this sense, $\mathbb{Q}^{\mathbf{x}}$ can be called "martingale measure".

This measure $\mathbb{Q}^{\mathbf{x}}$ does not come from a Girsanov change of probability: for one, it is not necessarily absolutely continuous with respect to $\mathbb{P}^{\mathbf{x}}$. Rather, it affects a *"Föllmer change of measure"*, in that it allows us to represent Föllmer's exit measure for the supermartingale $Z(\cdot)X(\cdot)$ in the diffusion context as we have just seen.

Finally, let us also mention the Elworthy et al. (1997, 1999) representation

$$1-U(T,\mathbf{x}) \,=\, \mathbb{Q}^{\mathbf{x}}\left[\,\mathfrak{T} \le T\,\right] = \lim_{u\to\infty}\left(u\,\mathbb{P}^{\mathbf{x}}\left[\max_{0\le t\le T}\left(g\big(\mathfrak{X}(t)\big)\,e^{-\int_0^t k(\mathfrak{X}(s))ds}\right) \,\ge\, g(\mathbf{x})\,u\right]\right) \tag{9.33}$$

for the "amount by which the martingale property fails on $[0,T]$" on the part of the $\mathbb{P}^{\mathbf{x}}-$local martingale $Z(\cdot)X(\cdot)$ – equivalently, for the $\mathbb{Q}^{\mathbf{x}}-$probability that the process $\mathfrak{Y}(\cdot)$ hits the boundary of the non-negative orthant by time T.

Example 1 *(cont'd):* For the 3-D Bessel process model of (4.9) with $n=1$, $b(x)=1/x^2$ and $s(x)=1/x$, we have $\Theta(x)=1/x$, $H(x)=\log x$ and $k(\cdot)\equiv 0$, $g(\cdot)\equiv 1$, $G(\,\cdot\,,\,\cdot\,)\equiv 1$ in (9.3), (9.5), thus $U(T,x)\equiv 1$ for all $T\in[0,\infty)$, $x\in(0,\infty)$.

Arbitrage relative to $X(\cdot)$ does not exist in this model, despite the existence of classical arbitrage (relative to the money market) and the fact that $Z(\cdot)$ is a strict local martingale. Note also that $\widehat{\mathfrak{b}}(x)=1/x$ in (9.23), so the diffusion of (9.26) is

$$dY(t) \,=\, \frac{1}{Y(t)}\,dt \,+\, dW(t)\,, \qquad Y(0)=y>0$$

again a Bessel process in dimension three. This process never hits the boundary $\{0\}$ of $[0,\infty)$, so the probability in (9.30) is equal to one, for all $T\in[0,\infty)$.

Example 2 *(cont'd):* For the volatility-stabilized model of (8.10) with $n\ge 2$, the $n-$dimensional diffusion $\mathfrak{Y}(\cdot)$ of (9.26) takes the form

$$dY_i(t) \,=\, Y_i(t)\,dt \,+\, \sqrt{Y_i(t)\,\big(Y_1(t)+\cdots+Y_n(t)\big)}\;dW_i(t)\,, \qquad i=1,\cdots,n \tag{9.34}$$

by analogy with (9.35). The conditions of Bass & Perkins (2002) are satisfied again, though one should compare the "weak drift" $\widehat{\mathfrak{b}}_i(\mathbf{x}) = x_i \ge 0$ in (9.34), which vanishes for $x_i=0$, with the "strong drift" $\mathfrak{b}_i(\mathbf{x}) = (1+\zeta)(x_1+\cdots+x_n)/2$ for the the original $n-$dimensional diffusion $\mathfrak{X}(\cdot)$ in (7.6). It is seen from (9.34) that $Y(\cdot):=Y_1(\cdot)+\cdots+Y_n(\cdot)$ satisfies the equation

$$dY(t) \,=\, Y(t)\left[\,dt+dB(t)\,\right], \qquad \text{where} \qquad B(\cdot) \,:=\, \sum_{j=1}^n \int_0^{\cdot} \sqrt{Y_j(t)/Y(t)}\;dW_j(t) \tag{9.35}$$

is standard, real-valued Brownian motion. Thus, $Y(\cdot)$ is geometric Brownian motion with drift.

In contrast to $\mathfrak{X}(\cdot)$ which lives in $(0,\infty)^n$, the new diffusion $\mathfrak{Y}(\cdot)$ of (9.34) lives in $[0,\infty)^n\setminus\{\mathbf{0}\}$, and hits the boundary $\mathcal{O}^n$ of this non-negative orthant with positive probability $\mathbb{Q}^{\mathbf{x}}\left[\,\mathfrak{T}\le T\,\right] = 1-U(T,\mathbf{x})$ for every $T\in(0,\infty)$. On the other hand, the positive $\mathbb{P}^{\mathbf{x}}-$supermartingale $Z(\cdot)X(\cdot)$ is a $\mathbb{P}^{\mathbf{x}}-$potential, for every starting point $\mathbf{x}\in(0,\infty)^n$.

In this case, the three inequalities of (4.8) hold for every $T\in(0,\infty)$: the local martingales $Z(\cdot)$, $Z(\cdot)X(\cdot)$ and $Z(\cdot)X_i(\cdot)$, $i=1,\cdots,n$ are all strict.

We shall see in the next section that the process $\mathfrak{Y}(\cdot)$ of (9.34) eventually hits not just the boundary of the non-negative orthant $[0,\infty)^n$, but also a positive half-line on one of its axes. The first hitting time

$$\mathfrak{T}_* := \inf\big\{ t \ge 0 \mid Y_j(t) = 0 \ \text{ for exactly } n-1 \text{ indices } j \,\big\}$$

of such a half-line is a random variable with finite $\mathbb{Q}^{\mathbf{x}}-$expectation; after this time, the process evolves like geometric Brownian motion with drift on that half-line.

10 Markovian Market Weights

Suppose now that the functions of (8.1) are of the form

$$\mathrm{b}_i(\mathbf{x}) = \mathfrak{B}_i\left(\frac{x_1}{\sum_{j=1}^n x_j}, \cdots, \frac{x_n}{\sum_{j=1}^n x_j} \right), \qquad \mathrm{s}_{ik}(\mathbf{x}) = \mathfrak{S}_{ik}\left(\frac{x_1}{\sum_{j=1}^n x_j}, \cdots, \frac{x_n}{\sum_{j=1}^n x_j} \right)$$

for suitable functions $\mathfrak{B}_i(\cdot)$, $\mathfrak{S}_{ik}(\cdot)$ on Δ_+^n, and set $\mathcal{A}_{ij}(\mathrm{m}) = \sum_{k=1}^n \mathfrak{S}_{ik}(\mathrm{m})\mathfrak{S}_{jk}(\mathrm{m})$ for $\mathrm{m} = (m_1, \cdots, m_n)' \in \Delta_+^n$.

The vector of relative market weights $\underline{\mu}(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_n(\cdot)\big)'$ of (3.5)-(3.6) is now a diffusion process with values in the positive simplex Δ_+^n and dynamics

$$d\mu_i(t) \,=\, \mu_i(t) \Big[\Gamma_i\big(\underline{\mu}(t)\big)\, dt \,+\, \sum_{k=1}^n \mathcal{T}_{ik}\big(\underline{\mu}(t)\big)\, dW_k(t) \Big], \qquad i = 1, \cdots, n, \tag{10.1}$$

with the notation $\mathcal{T}_{ik}(\mathrm{m}) := \mathfrak{S}_{ik}(\mathrm{m}) - \sum_{j=1}^n m_j\, \mathfrak{S}_{jk}(\mathrm{m})$, $\mathcal{P}_{ij}(\mathrm{m}) := \sum_{k=1}^n \mathcal{T}_{ik}(\mathrm{m})\mathcal{T}_{jk}(\mathrm{m})$ and

$$\Gamma_i(\mathrm{m}) \,:=\, \mathfrak{B}_i(\mathrm{m}) - \sum_{j=1}^n m_j\, \mathfrak{B}_j(\mathrm{m}) \,-\, \sum_{j=1}^n m_j\, \mathcal{A}_{ij}(\mathrm{m}) \,+\, \sum_{j=1}^n \sum_{k=1}^n m_j\, m_\ell\, \mathcal{A}_{j\ell}(\mathrm{m}) \,. \tag{10.2}$$

In this setup, the function $U(\,\cdot\,,\,\cdot\,)$ of (9.1) is given as

$$U(T, \mathbf{x}) \,=\, Q\left(T\,,\ \frac{x_1}{\sum_{j=1}^n x_j}, \cdots, \frac{x_n}{\sum_{j=1}^n x_j} \right), \tag{10.3}$$

in terms of a function $Q : (0,\infty) \times \Delta_+^n \to (0,1]$ which satisfies the initial condition $Q(0+, \,\cdot\,) \equiv 1$, as well as the equation

$$\frac{\partial Q}{\partial \tau}(\tau, \mathrm{m}) \,=\, \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n m_i\, m_j\, \mathcal{P}_{ij}(\mathrm{m})\, D_{ij}^2 Q(\tau, \mathrm{m})\,, \qquad (\tau, \mathrm{m}) \in (0,\infty) \times \Delta_+^n \tag{10.4}$$

which appears on p. 56 of Fernholz (2002); it can be derived from (9.11) through some computation.

Subject to this same initial condition $Q(0+, \,\cdot\,) \equiv 1$, the function $Q(\,\cdot\,,\,\cdot\,)$ is also the smallest solution of the partial differential inequality

$$\frac{\partial Q}{\partial \tau}(\tau, \mathrm{m}) \,\ge\, \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n m_i\, m_j\, \mathcal{P}_{ij}(\mathrm{m})\, D_{ij}^2 Q(\tau, \mathrm{m})\,, \qquad (\tau, \mathrm{m}) \in (0,\infty) \times \Delta_+^n$$

of class $\mathcal{C}^{1,2}$ on $(0,\infty) \times \Delta_+^n$; this is by analogy with Proposition 2. On the other hand, by analogy with Proposition 4 and (9.28), the quantity $Q(T, \mathrm{m})$ is the probability that the process $\underline{\nu}(\cdot) = \big(\nu_1(\cdot), \cdots, \nu_n(\cdot)\big)'$ with dynamics

$$d\nu_i(t) \,=\, \nu_i(t) \sum_{k=1}^n \mathcal{T}_{ik}\big(\underline{\nu}(t)\big)\, dW_k(t)\,, \qquad i = 1, \cdots, n\,, \tag{10.5}$$

and initial configuration $\underline{\nu}(0) = \mathrm{m} \in \Delta^n_+$ at time $t = 0$, does not hit the boundary of the non-negative simplex $\Delta^n := \big\{ (m_1, \cdots, m_n)' \in [0,1]^n \mid \sum_{i=1}^n m_i = 1 \big\}$ before time $t = T$.

In the present context, the conditions of (4.10) and (4.12) read

$$\sum_{i=1}^n m_i \, \mathcal{A}_{ii}(\mathrm{m}) - \sum_{i=1}^n \sum_{j=1}^n m_i m_j \, \mathcal{A}_{ij}(\mathrm{m}) \, \geq \, h \, , \qquad \forall \ \mathrm{m} \in \Delta^n_+ \, , \tag{10.6}$$

$$\big(m_1 \cdots m_n \big)^{1/n} \left[\sum_{i=1}^n \mathcal{A}_{ii}(\mathrm{m}) - \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \mathcal{A}_{ij}(\mathrm{m}) \right] \geq h \, , \qquad \forall \ \mathrm{m} \in \Delta^n_+ \, , \tag{10.7}$$

respectively, for some real constant $h > 0$. Under either of these conditions, uniqueness fails for the Cauchy problem (10.4) with $Q(0+, \, \cdot \,) \equiv 1$.

Example 2 *(cont'd):* The volatility-stabilized model of (8.10) with $n \geq 2$ can be cast in the form of (10.1) as a *multivariate Jacobi process* (cf. Gouriéroux & Jasiak (2006)), namely

$$d\mu_i(t) \ = \ n(1+\zeta) \left(\frac{1}{n} - \mu_i(t) \right) dt \ + \ \sqrt{\mu_i(t)} \, dW_i(t) \ - \ \mu_i(t) \sum_{k=1}^n \sqrt{\mu_k(t)} \, dW_k(t) \, . \tag{10.8}$$

Note that $\mathcal{A}_{ij}(\mathrm{m}) = \delta_{ij}/m_i$, so (10.6) is satisfied as equality with $h = n-1$ as we have seen; and with appropriate dependent Brownian motions $W_1^\sharp(\cdot), \cdots, W_n^\sharp(\cdot)$, we see that (10.8) gives

$$d\mu_i(t) \ = \ n(1+\zeta)\big(n^{-1} - \mu_i(t)\big)dt + \sqrt{\mu_i(t)(1-\mu_i(t))} \, dW_i^\sharp(t) \, , \qquad i = 1, \cdots, n \, . \tag{10.9}$$

Thus, not only is the vector process $\underline{\mu}(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_n(\cdot)\big)'$ a diffusion, but each of its components $\mu_i(\cdot)$ is also a diffusion process on the unit interval $(0,1)$ with local drift $n(1+\zeta)\big(n^{-1} - y\big)$ and local variance $y(1-y)$ of Wright-Fisher type; see Goia (2008) for a detailed study of this multivariate diffusion.

On the other hand, the process $\underline{\nu}(\cdot) = \big(\nu_1(\cdot), \cdots, \nu_n(\cdot)\big)'$ of (10.5) is related to the diffusion $\mathfrak{Y}(\cdot)$ of (9.34) via $\nu_i(\cdot) = Y_i(\cdot)/Y(\cdot)$, where $Y(\cdot) = Y_1(\cdot) + \cdots + Y_n(\cdot)$. The dynamics of this process $\underline{\nu}(\cdot)$ are straightforward to describe in the manner of (9.28), namely,

$$d\nu_i(t) \ = \ \sqrt{\nu_i(t)} \, dW_i(t) \ - \ \nu_i(t) \sum_{k=1}^n \sqrt{\nu_k(t)} \, dW_k(t) \, , \qquad i = 1, \cdots, n \, , \tag{10.10}$$

or $d\nu_i(t) = \sqrt{\nu_i(t)(1-\nu_i(t))} \, dW_i^\sharp(t)$ in the notation of (10.9).

In this formulation it is very easy to check, using the familiar Feller test for explosions (e.g. Karatzas & Shreve (1991), pp. 348-350), that each $\nu_i(\cdot)$ hits eventually one of the boundary points of the unit interval, with probability one; in fact, the time it takes for this to happen, has finite expectation. But this means that, eventually, all but one of the $\nu_i(\cdot)$'s (equivalently, all but one of the $Y_i(\cdot)$'s) are absorbed at zero, and only one of them remains positive. From that time onwards, the surviving (non-zero) component is equal to the sum $Y(\cdot)$ and behaves like a geometric Brownian motion with drift, as in (9.35); whereas the diffusion $\mathfrak{Y}(\cdot)$ never hits the origin.

Remark: It may be useful here to think of $Y_i(t)$ in (9.34) as representing the number of molecules of compound $i = 1, \cdots, n$ to be found at time t in a chemistry experiment, so $\nu_i(t)$ is the relative density of that compound in the mixture. A compound that vanishes does not re-appear; this is the absorbing nature of the boundaries of the unit interval. From that absorption time onwards, the diffusion process $\mathfrak{Y}(\cdot)$ is governed by the remaining molecules and their stochastic evolution. (We borrow this interpretation from Janson & Tysk (2006).)

11 The Investment Strategy

We substitute now the expressions of (9.13), (9.14) into (9.16), to obtain the dynamics of the martingale $N(\cdot)$ in (9.15), namely

$$N(t) \;=\; N(0) + \sum_{k=1}^{n} \int_0^t N(s)\, \Psi_k\big(\mathfrak{X}(s)\big)\, dW_k(s)\,, \qquad 0 \le t \le T \tag{11.1}$$

where $N(0) = X(0)\, U(T, \mathfrak{X}(0))$ and

$$\Psi_k(\tau, \mathbf{x}) \;:=\; \sum_{i=1}^{n} \mathrm{s}_{ik}(\mathbf{x}) \left(x_i\, D_i \log U(\tau, \mathbf{x}) + \frac{x_i}{x_1 + \cdots + x_n} \right) \;-\; \Theta_k(\mathbf{x})\,, \qquad k = 1, \cdots, n\,. \tag{11.2}$$

Let us compare now (9.6), (9.15) and the dynamics of (11.1), with (5.4) and the dynamics of (5.5); this leads to the obvious identifications

$$Y \equiv X(T)\,, \quad M(\cdot) \equiv N(\cdot)\,, \quad y \equiv X(0)\, U\big(T, \mathfrak{X}(0)\big)\,. \tag{11.3}$$

But these imply then the additional identification

$$\psi_k(t) \;=\; \Psi_k\big(T - t, \mathfrak{X}(t)\big)\,, \quad 0 \le t \le T\,, \quad k = 1, \cdots, n$$

in the notation of (11.2), for the components of the process $\psi = (\psi_1, \cdots, \psi_n)'$. This is the integrand in the stochastic exponential/integral representation of the positive martingale $M(\cdot) \equiv N(\cdot)$, in the manner of (5.5).

We substitute now into (5.6), to obtain the $\sigma(\mathfrak{X}(t))-$measurable weights

$$\widehat{\pi}_i(t) \;=\; X_i(t)\, D_i \log U\big(T - t, \mathfrak{X}(t)\big) \;+\; \frac{X_i(t)}{X(t)} \;=\; X_i(t) \cdot D_i \left[\, \log G\big(T - t, \mathfrak{X}(t)\big) + H\big(\mathfrak{X}(t)\big) \,\right] \tag{11.4}$$

placed at time t in the various stocks $i = 1, \cdots, n$ by $\widehat{\pi}(\cdot)$. This is the investment strategy which generates on the time-horizon $[0, T]$ the optimal arbitrage relative to the market; that is, $V^{y,\widehat{\pi}}(T) = X(T)$ almost surely. In fact, we get in the notation of (11.3), (11.4) the additional a.s. identification

$$V^{y,\widehat{\pi}}(t) \;=\; X(t) \cdot U\big(T - t, \mathfrak{X}(t)\big)\,, \qquad 0 \le t \le T\,. \tag{11.5}$$

It is worth noting that, apart from the function $U(\,\cdot\,,\,\cdot\,)$ – whose determination needs knowledge of the model's volatility structure – the formulae (11.4), (11.5) involve only the vector $\mathfrak{X}(t) = \big(X_1(\cdot), \cdots, X_n(\cdot)\big)'$ of capitalizations at time t, a quantity that is readily observable.

If $U(\,\cdot\,,\,\cdot\,) \equiv 1$ is the unique solution of the Cauchy problem (9.11)-(9.12) – that is, if no arbitrage is possible relative to the market, or equivalently if the diffusion $\mathfrak{Y}(\cdot)$ of (9.26) stays in the positive orthant when started there – then clearly $\widehat{\pi}(\cdot)$ becomes the market portfolio $\mu(\cdot)$.

• In the special case of a Markovian model for the market weights $\underline{\mu}(\cdot) = (\mu_1(\cdot), \cdots, \mu_n(\cdot))'$ as in section 10, the expression (11.4) takes the form

$$\widehat{\pi}_i(t) \;=\; \mu_i(t) \Big(1 + \; D_i \log Q\big(T - t, \underline{\mu}(t)\big) \;-\; \sum_{j=1}^{n} \mu_j(t)\, D_j \log Q\big(T - t, \underline{\mu}(t)\big) \Big) \tag{11.6}$$

of a "functionally-generated portfolio" in the terminology of Fernholz (2002), pp. 55-56; whereas the expression (11.5) becomes

$$V^{y,\widehat{\pi}}(t) \;=\; X(t)\, Q\big(T - t, \underline{\mu}(t)\big)\,, \qquad 0 \le t \le T\,.$$

In this case we have $\sum_{i=1}^{n} \widehat{\pi}_i(\cdot) \equiv 1$: the strategy that implements the best possible arbitrage relative to the equity market never borrows from, or lends into, a money-market.

12 Some Open Questions

What conditions, if any, on the covariance structure $\big(\mathrm{a}_{ij}(\cdot)\big)_{1\le i,j\le n}$ will guarantee that $\widehat{\pi}(\cdot)$ of (9.26) never borrows from the money-market, i.e., that

$$\sum_{i=1}^{n} x_i\, D_i U(T,\mathbf{x}) \;\le\; 0\,, \qquad \forall\;\, (T,\mathbf{x}) \in (0,\infty)\times(0,\infty)^n$$

holds? That it is a portfolio, i.e., that

$$\sum_{i=1}^{n} x_i\, D_i U(T,\mathbf{x}) \;=\; 0\,, \qquad \forall\;\, (T,\mathbf{x}) \in (0,\infty)\times(0,\infty)^n$$

holds? (A partial answer to this question was provided at the end of the previous section.) Or better, that $\widehat{\pi}(\cdot)$ of (9.26) is a long-only portfolio, meaning that both this condition and

$$D_i\,\big(\,G(T,\mathbf{x})\, e^{\,H(\mathbf{x})}\,\big) \;\ge\; 0\,, \qquad \forall\;\, (T,\mathbf{x}) \in (0,\infty)\times(0,\infty)^n$$

for every $i=1,\cdots,n\,,$ should hold?

Can an iterative method be constructed, which converges to the minimal solution of the parabolic differential inequality (9.19), (9.12) and is numerically implementable? How about developing a Monte-Carlo scheme that computes the quantity $U(T,\mathbf{x})$ of (9.30) by generating the paths of the diffusion process $\mathfrak{Y}(\cdot)$, then simulating the probability $\mathbb{Q}^{\mathbf{x}}\left[\,\mathfrak{T} > T\,\right]$ that $\mathfrak{Y}(\cdot)$ does not hit the boundary of the non-negative orthant by time T, when started at $\mathfrak{Y}(0)=\mathbf{x}\in(0,\infty)^n$? How does $U(T,\mathbf{x})$ behave as $T\to\infty$?

How is the theory presented in this paper to be modified, if one insists at the outset on using long-only portfolios rather than general investment strategies in $\mathcal{H}$? How about replacing the quantity $\mathfrak{u}(T)$ of (6.1) by

$$\inf\big\{v>0 \mid \exists\; \pi(\cdot)\in\mathcal{H}\;\text{ s.t. } \mathbb{P}\big[\,V^{\,vX(0),\pi}(T)\ge X(T)\,\big]\ge 1-\eta\,\big\}\,,$$

for some given 'small' number $\eta\in(0,1)$? Can a theory be developed for this quantity along the lines of this paper, (at least) in a Markovian context?

13 Appendix

The transition probability density function

$$\mathbb{P}\big[\,X_1(T)\in dq_1,\cdots,X_n(T)\in dq_n \,|\, \mathfrak{X}(0)=\mathbf{x}\,\big] \;=\; \mathfrak{g}_T(q_1,\cdots,q_n \,|\, x_1,\cdots,x_n)\, dq_1\cdots dq_n$$

for the $(0,\infty)^n-$ valued process $\mathfrak{X}(\cdot)$ in (8.11) can be expressed in terms of the transition probability density function

$$\mathbb{P}\big[\,\mu_1(T)\in dm_1,\cdots,\mu_n(T)\in dm_n \,|\, \underline{\mu}(0)=\underline{\rho}\,\big] \;=\; \mathfrak{p}_T(m_1,\cdots,m_n \,|\, \rho_1,\cdots,\rho_n)\, dm_1\cdots dm_n$$

for the Δ^n_+-valued Jacobi diffusion in (10.8), as

$$\mathfrak{g}_T(q_1,\cdots,q_n \,|\, x_1,\cdots,x_n) \;=\; \frac{e^{-\delta^2 T/8}\,\big(\sum q_i\big)^{(\delta/2)-n+1}}{2(\sum x_i)^{\delta/2}}\;\mathfrak{p}_{\frac{T}{4}}\Big(\frac{q_1}{\sum q_i},\cdots,\frac{q_n}{\sum q_i}\,\Big|\,\frac{x_1}{\sum x_i},\cdots,\frac{x_n}{\sum x_i}\Big)\cdot$$

$$\cdot\int_0^\infty e^{-\sum(q_i+x_i)/(2t)}\,\Pi\left(\frac{T}{4}\,;\frac{1}{t}\Big(\sum q_i\cdot\sum x_i\Big)^{1/2}\right)\,\frac{I\big(\delta-1;t,T,\sqrt{\sum x_i}\big)}{I\big(\delta+1;t,T,\sqrt{\sum x_i}\big)}\,\frac{dt}{t}$$

for $\mathbf{q} = (q_1, \cdots, q_n)$, $\mathbf{x} = (x_1, \cdots, x_n)$ in $(0,\infty)^n$; $\mathrm{m} = (m_1, \cdots, m_n)$, $\underline{\rho} = (\rho_1, \cdots, \rho_n)$ in Δ_+^n .

Here $\delta = \big(n(1+\zeta)-1\big)/2$ is the constant appearing in (8.12),

$$\Pi(u;r) := \frac{re^{\pi^2/2u}}{\sqrt{2\pi^3 u}} \int_0^\infty e^{-y^2/(2u)}\, e^{-r\cosh y}\, \sinh y\, \sin\Big(\frac{\pi y}{u}\Big)\, dy$$

and

$$I(\gamma;t,T,\rho) = \int_0^\infty \Pi\big(T/4\,;\,(\xi\rho)/t\big)\, e^{-\xi^2/(2t)}\, \xi^\gamma\, d\xi\,.$$

For the details of this computation, and for an extensive study of the multivariate Jacobi process, see Goia (2008). Then, the function of (9.16) is given as

$$U(T,\mathbf{x}) = \frac{x_1\cdots x_n}{x_1+\cdots+x_n} \int_0^\infty \cdots \int_0^\infty \frac{q_1+\cdots+q_n}{q_1\cdots q_n}\, \mathfrak{g}_T(q_1,\cdots,q_n\,|\,x_1,\cdots,x_n)\, dq_1\cdots dq_n\,.$$

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