ON OPTIMAL ASYMPTOTIC TESTS OF COMPOSITE STATISTICAL HYPOTHESES[1]

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0. Summary. A locally asymptotically most powerful test for a composite hypothesis $H: \xi = \xi_0$ has been developed for the case where the observable random variables $\{X_{nk}, k = 1, 2, \cdots, n\}$ are independently but not necessarily identically distributed. However, their distributions depend on $s + 1$ parameters, one being ξ under test and the other being a vector $\theta = (\theta_1, \cdots, \theta_s)$ of nuisance parameters. The theory is illustrated with an example from the field of astronomy.

1. Introduction and specification of the problem. This paper extends Neyman's [1] theory of optimal asymptotic tests of composite statistical hypotheses to the case where the observable random variables are independent but not necessarily identically distributed. Whenever possible, we shall use the terminology, notation, and definitions of Neyman's paper [1]. Also, we shall confine ourselves to the case where all estimates $\hat{\theta}_{jn}, j = 1, 2, \cdots, s$, form a weakly root n consistent system. Proofs of the theorems proceed in ways very similar to those in [1] and therefore will be omitted.

Let Ξ be an interval containing zero on the real line and Θ an arbitrary open set in the s-dimensional Euclidean space and let $\Omega = \Xi \times \Theta$. For each $\xi \varepsilon \Xi$, and $\theta = (\theta_1, \theta_2, \cdots, \theta_s) \varepsilon \Theta$ we consider a double sequence of independent random variables, $X_{nk}(\xi, \theta)$, $k = 1, 2, \cdots, n$; and $n = 1, 2, \cdots$. The sample space of $X_{nk}(\xi, \theta)$, denoted by W_{nk}, is assumed to be independent of $(\xi, \theta) \varepsilon \Omega$. Each variable $X_{nk}(\xi, \theta)$ is assumed to possess a probability density $p_{nk}(x; \xi, \theta)$ with respect to some σ-finite measure γ_{nk} which is independent of Ω. In addition we shall be concerned with vector random variables $X_n(\xi, \theta) = [X_{n1}(\xi, \theta), X_{n2}(\xi, \theta), \cdots, X_{nn}(\xi, \theta)]$. The corresponding sample space will be denoted by $W_{n.} = W_{n1} \times W_{n2} \times \cdots \times W_{nn}$. Integrals extending over the whole sample space, either W_{nk} or $W_{n.}$, will be written without specifying the region of integration.

The subject of this paper is an asymptotic optimal test of the hypotheses H_0 asserting that $\xi = \xi_0 \varepsilon \Xi$. In order to simplify the notation, it will frequently be assumed that $\xi_0 = 0$. If ξ_0 is an end point of a closed (or half open) interval Ξ then all derivatives with respect to ξ at $\xi = \xi_0$ are assumed to be the appropriate right hand or left hand derivatives. We shall also assume that for each $x \varepsilon W_{nk}$ and for all (ξ, θ), each of the density functions $p_{nk}(x; \xi, \theta)$ is at least twice differentiable with respect to all the $(s + 1)$ parameters, and that these differentiations are permissible under the sign of the integral extending over W_{nk}. The

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symbols $\varphi_{nk(\xi)}(x, \theta)$ and $\varphi_{nk(j)}(x, \theta)$ will denote the partial derivatives of $\log p_{nk}(x; \xi, \theta)$ with respect to ξ and θ_j, $j = 1, 2, \cdots, s$, all evaluated at $\xi = \xi_0$ and at an arbitrary point $\theta \varepsilon \Theta$. If $p_{nk} = 0$, define $\varphi_{nk(\xi)} = \varphi_{nk(j)} = 0$, $j = 1, 2, \cdots, s$. Assume that $E[\varphi_{nk}^2\{X_{nk}(\xi_0, \theta), \theta\}] < \infty$ for every n and k. Our additional assumption will be that, whatever be $\theta \varepsilon \Theta$, whatever be n and k, the quantities $\varphi_{nk(\xi)}[X_{nk}(\xi_0, \theta), \theta]$ and $\varphi_{nk(j)}[X_{nk}(\xi_0, \theta)\theta]. j = 1, 2, \cdots, s$, are linearly independent with positive probability.

We adopt the definitions of asymptotic tests and the optimality criterion as given by Neyman [1]. Definition 1 given below is an extension of Definition 3 of Neyman.

DEFINITION 1. We shall say that $\{f_{nk}(x, \theta)\}$ is a Cramér sequence, if it satisfies the following conditions.

(i) For each n, for $k = 1, 2, \cdots, n$, and for all ξ and θ, the integral

$$\mu_{nk}(\xi, \theta) = \int f_{nk}(x, \theta) p_{nk}(x; \xi, \theta)\, d\gamma_{nk}(x) = E\{f_{nk}[X_{nk}(\xi, \theta), \theta]\} \tag{1}$$

exists and, at $\xi = \xi_0$ and arbitrary θ, is differentiable under its sign at least twice with respect to ξ and at least once with respect to $\theta_1, \theta_2, \cdots, \theta_s$.

(ii) For every θ, and for $j = 1, 2, \cdots, s$,

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} E\{f_{nk(j)}[X_{nk}(\xi, \theta), \theta]\} \tag{2}$$

exists and is finite for all ξ in some neighborhood V_{ξ_0} of $\xi = \xi_0$. Also for $j = 1, 2, \cdots, s$, and for all θ, as $n \to \infty$.

$$n^{-1} \sum_{k=1}^{n} E\{f_{nk(j)}[X_{nk}(\xi_0 + un^{-\frac{1}{2}}, \theta), \theta]\} \to \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} E\{f_{nk(j)}[X_{nk}(\xi_0, \theta), \theta]\} \tag{3}$$

and for a positive $\delta \leqq 1$,

$$(n^{1+\delta})^{-1} \sum_{k=1}^{n} E|f_{nk(j)}(X_{nk}(\xi_0 + un^{-\frac{1}{2}}, \theta), \theta) - E[f_{nk(j)}(X_{nk}(\xi_0 + un^{-\frac{1}{2}}, \theta), \theta]|^{1+\delta} \to 0 \tag{4}$$

both uniformly for all u such that $u^2 < U$ for some finite $U > 0$.

(iii) Whatever be $\vartheta \varepsilon \Theta$, there exists a sequence of functions $\{G_{nk\vartheta}(x)\}$, nonnegative for every n and $k = 1, 2, \cdots, n$, defined on $\{W_{nk}\}$ such that, for all $\xi \varepsilon \Xi$ and for every u and k

$$\int G_{nk\vartheta}(x) p_{nk}(x; \xi, \vartheta)\, d\gamma_{nk}(x) \tag{5}$$

exists, and that

$$n^{-1} \sum_{k=1}^{n} \int G_{nk\vartheta}(x) p_{nk}(x; \xi_0 + un^{-\frac{1}{2}}, \vartheta)\, d\gamma_{nk}(x) \tag{6}$$

converges to a finite limit as $n \to \infty$ uniformly for all u such that $u^2 < U$ for some finite $U > 0$, and such that for all θ in a neighborhood of ϑ,

$$|f_{nk(ij)}(x, \theta)| < G_{nk\vartheta}(x) \tag{7}$$

for $i, j = 1, 2, \cdots, s$ and $k = 1, 2, \cdots, n$.

(iv) Let

$$S_n^2(\xi, \theta) = n^{-1} \sum_{k=1}^{n} \mathrm{Var}\,(f_{nk}[X_{nk}(\xi, \theta), \theta]). \tag{8}$$

For every $\theta \varepsilon \Theta$, and for all ξ in V_{ξ_0}, $\lim_{n\to\infty} S_n(\xi, \theta)$ exists and is strictly positive. Also for every θ, as $n \to \infty$,

$$S_n(\xi_0, \theta + vn^{-\frac{1}{2}})/S_n(\xi_0, \theta) \to 1 \tag{9}$$

uniformly for all $v = (v_1, v_2, \cdots, v_s)$ with $\sum v_i^2 < U$ and

$$S_n(\xi_0 + un^{-\frac{1}{2}}, \theta)/S_n(\xi_0, \theta) \to 1 \tag{10}$$

uniformly for all u with $u^2 < U$ for some finite $U > 0$.

(v) There exist two positive constants δ and U, such that

$$[n^{\frac{1}{2}}S_n(\xi_0 + un^{-\frac{1}{2}}, \theta)]^{-(2+\delta)} \sum_{k=1}^{n} E|f_{nk}[X_{nk}(\xi_0 + un^{-\frac{1}{2}}, \theta), \theta] - \mu_{nk}(\xi_0 + un^{-\frac{1}{2}}, \theta)|^{2+\delta} \to 0 \tag{11}$$

uniformly in u with $u^2 < U$ for some finite $U > 0$.

(vi) Let

$$m_n(\xi, \theta) = n^{-1} \sum_{k=1}^{n} \mu_{nk}(\xi, \theta). \tag{12}$$

We assume that $m_n(\xi_0 + un^{-\frac{1}{2}}, \theta)$ converges uniformly for all u with $u^2 < U$ for some finite $U > 0$, to a finite limit for every fixed θ. We assume that the 1st derivative of $m_n(\xi, \theta)$, at $\xi = \xi_0$ tends to a nonzero limit (in Neyman's terminology we say then that the sequence $\{f_{nk}(x, \theta)\}$ is of index 1); and that $d^2m_n(\xi, \theta)/d\xi^2$ exists and is uniformly bounded for all n in the vicinity of $\xi = \xi_0$ and for all $\theta \varepsilon \Theta$. Finally,

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \mathrm{cov}\,\{f_{nk}[X_{nk}(\xi_0, \theta), \theta], \varphi_{nk(\xi)}[X_{nk}(\xi_0, \theta), \theta]\} \tag{13}$$

exists and is finite.

We note that condition (v) implies that if $f_{nk}(X_{nk}(\xi_n, \theta), \theta)$ is expectation centered, then

$$Z_n(\xi_0, \theta) = [nS_n^2(\xi_0, \theta)]^{-\frac{1}{2}} \sum_{k=1}^{n} f_{nk}(X_{nk}(\xi_0, \theta), \theta) \tag{14}$$

has a limiting normal distribution with zero mean and unit variance.

2. A preliminary result. To compute the value of $Z_n(\xi_0, \theta)$ at some point, θ must be known or estimated. To determine when the limiting distribution of $Z_n(\xi_0, \hat{\theta}_n)$, where $\hat{\theta}_n$ is a suitable weakly root n consistent estimator of θ, is equivalent to that of $Z_n(\xi_0, \theta)$ the following theorem is needed.

THEOREM 1. *If at $\xi = \xi_0$, $\{f_{nk}(x, \theta)\}$ is an expectation centered Cramér sequence then in order that as $n \to \infty$, the differences $Z_n(\xi_0, \hat{\theta}_n) - Z_n(\xi_0, \theta)$ tend to zero in probability, it is necessary and sufficient that*

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \lambda_{nk(j)}(\theta) = 0 \tag{15}$$

identically in θ, for $j = 1, 2, \cdots, s$, where

$$\lambda_{nk(j)}(\theta) = \mathrm{cov}\,\{f_{nk}[X_{nk}(\xi_0, \theta)], \varphi_{nk(j)}[X_{nk}(\xi_0, \theta), \theta]\}. \tag{16}$$

There remains to define a family $C(\alpha)$ of asymptotic tests based on the function $Z_n(\xi_0, \theta)$, and to prove that it contains tests that are optimal with respect to a certain class Γ of sequences $\{\xi_n\}$ of points belonging to Ξ and converging to ξ_0 and to develop a method of their construction.

3. Family $C(\alpha)$ of asymptotic tests and their asymptotic powers.

Let $B(\alpha)$ be an arbitrary measurable set on the real line whose characteristic function is continuous almost everywhere and such that

$$(2\pi)^{-\frac{1}{2}} \int_{B(\alpha)} \exp\{-t^2/2\}\, dt = \alpha. \tag{17}$$

Let $\hat{\theta}_{1n}, \hat{\theta}_{2n}, \cdots, \hat{\theta}_{sn}$ form a weakly root n consistent system for parameters $\theta_1, \theta_2, \cdots, \theta_s$. Again let $\{f_{nk}(x, \theta)\}$ be an expectation centered Cramér sequence such that for $j = 1, 2, \cdots, s$

$$\lim_{n\to\infty} n^{-1} \textstyle\sum_{k=1}^{n} \operatorname{cov}\{f_{nk}[X_{nk}(\xi_0, \theta), \theta], \varphi_{nk(j)}[X_{nk}(\xi_0, \theta), \theta]\} = 0. \tag{18}$$

Thus $\{f_{nk}[X_{nk}(\xi_0, \theta), \theta]\}$ satisfies (15) of Theorem 1 and it follows that

$$Z_n(\xi_0, \hat{\theta}_n) = [n^{\frac{1}{2}} S_n(\xi_0, \hat{\theta}_n)]^{-1} \textstyle\sum_{k=1}^{n} f_{nk}[X_{nk}(\xi_0, \theta), \hat{\theta}_n] \tag{19}$$

has a limiting normal distribution with zero mean and unit variance. If ω_n denotes a subset of W_n. such that

$$[Z_n(\xi_0, \hat{\theta}_n) \,\varepsilon\, B(\alpha)] \Leftrightarrow [X_n \,\varepsilon\, \omega_n] \tag{20}$$

then $\{\omega_n\}$ would form a test of the hypothesis $H_0 : \xi = \xi_0$ with

$$\lim_{n\to\infty} P[X_n(\xi_0, \theta) \,\varepsilon\, \omega_n] = \alpha. \tag{21}$$

Definition 2. The family $C(\alpha)$ of asymptotic tests is defined to be composed of all sequences of critical regions of the type $\{\omega_n\}$ defined by (20).

Let Γ_1 be a subclass of Γ consisting of all sequences $\{\xi_n\} \,\varepsilon\, \Xi$ such that $\{n^{\frac{1}{2}}(\xi_n - \xi_0)\}$ remains bounded. We shall devote the rest of the paper to the problem of selecting members of the family $C(\alpha)$ which are asymptotically most powerful with reference to the subclass Γ_1.

Let $\{f_{nk}(x, \theta)\}$ be an arbitrary Cramér sequence which satisfies condition (15) of Theorem 1. Let $\xi^* = \{\xi_n\}$ be an arbitrary sequence of numbers belonging to Ξ which converges to ξ_0. In order to study the behavior of the probability $P\{Z_n(\xi_n, \hat{\theta}_n) \,\varepsilon\, B(\alpha)\}$, as $n \to \infty$, we notice that this in turn depends on the sequence of distribution functions of the form $P\{Z_n(\xi_n, \hat{\theta}_n) < x\}$ where

$$Z_n(\xi_n, \hat{\theta}_n) = [n^{\frac{1}{2}} S_n(\xi_0, \hat{\theta}_n)]^{-1} \textstyle\sum_{k=1}^{n} f_{nk}(\xi_n, \theta), \hat{\theta}_n]. \tag{22}$$

Let

$$Z_n(\xi_n, \theta) = [n^{\frac{1}{2}} S_n(\xi_0, \theta)]^{-1} \textstyle\sum_{k=1}^{n} f_{nk}[X_{nk}(\xi_n, \theta), \theta_n]. \tag{23}$$

Under the conditions of Definition 1 it can be shown that $Z_n(\xi_n, \theta)$ is asymptotically normally distributed with mean $m_n(\xi_n, \theta) n^{\frac{1}{2}}$ and unit variance where, in this case

$$m_n(\xi_n, \theta) = n^{-1} \textstyle\sum_{k=1}^{n} E\{f_{nk}[X_{nk}(\xi_n, \theta), \theta]/S_n(\xi_0, \theta)\}. \tag{24}$$

To show that $Z_n(\xi_n, \hat{\theta}_n)$ has the same asymptotic distribution as $Z_n(\xi_n, \theta)$ the following theorem is required.

THEOREM 2. *If the sequence* $\xi^* = \{\xi_n\} \varepsilon \Gamma_1$, *then* $Z_n(\xi_n, \hat{\theta}_n) - Z_n(\xi_n, \theta)$ *converges in probability to zero and hence, there exists for arbitrary* $\epsilon > 0$, *an integer* N_ϵ *such that for* $n > N_\epsilon$,

$$(25) \qquad |P\{Z_n(\xi_n, \hat{\theta}_n) < x\} - \Phi(x - m_n(\xi_n, \theta)n^{\frac{1}{2}})| < \epsilon$$

uniformly in x *where* $\Phi(t)$ *is the standard normal distribution function.*

If the first derivative of $m_n(\xi_n, \theta)$ tend to a non-zero limit and $(\xi_n - \xi_0)n^{\frac{1}{2}}$ remains bounded, it follows from (vi) of Definition 1 and the above theorem that $Z_n(\xi_n, \hat{\theta}_n)$ is asymptotically normal with mean $\mu_{1n}(\theta)(\xi_n - \xi_0)n^{\frac{1}{2}}$ and unit variance, where

$$(26) \qquad \mu_{1n}(\theta) = dm_n(\xi, \theta)/d\xi \,|_{\xi=\xi_0}.$$

This in turn yields the asymptotic power of a typical member $\{\omega_n\}$ of the family $C(\alpha)$.

4. Optimal test of class $C(\alpha)$. Without loss of generality we assume $\xi_0 = 0$ and consider the case where it is desired to test the hypothesis $H_0 : \xi = \xi_0 = 0$ against alternatives specifying values of ξ that are greater than zero. Other cases can be dealt with in a similar manner (see [1]). Let Γ_{1+} denote the subclass of sequences in Γ_1 such that $\xi_n > 0$.

Let $a_n^0(\theta) = (a_{1n}^0(\theta), a_{2n}^0(\theta), \cdots, a_{sn}^0(\theta))$ be a vector which minimizes the variance of

$$(27) \qquad \textstyle\sum_{k=1}^n \varphi_{nk(\xi)}[X_{nk}(\xi_0, \theta), \theta] - \sum_{j=1}^n a_{nk}(\theta) \sum_{k=1}^n \varphi_{nk(j)}[X_{nk}(\xi_0, \theta), \theta],$$

for each n and for fixed θ. The symbol $S_n^{*2}(\theta)$ will denote this minimum variance. Because of the assumption made in Section 1 to the effect that, for every $\theta \varepsilon \Theta$, for every n and $k = 1, 2 \cdots, n$, $\varphi_{nk(\xi)}[X_{nk}(\xi_0, \theta), \theta]$ and $\varphi_{nk(j)}[X_{nk}(\xi_0, \theta), \theta]$, $j = 1, 2, \cdots, s$, are linearly independent with non-zero probability, the values $a_{jn}^0(\theta)$ are always determined and $S_n^{*2}(\theta)$ is always positive. We note that $S_n^{*2}(\theta)$ is positive if and only if the index l (in the sense of Neyman [1]) of the sequence $\{f_{nk}^*(x, \theta)\}$ is equal to one.

Let us assume that

$$(28) \qquad a^0 = (a_1^0, a_2^0, \cdots, a_s^0) = \lim_{n\to\infty} a_n^0$$

exists. We assume that the sequence $\{p_{nk}(x; \xi, \theta)\}$ is regular enough so that the sequence $\{f_{nk}^*(x, \theta)\}$ where

$$(29) \qquad f_{nk}^*(x, \theta) = \varphi_{nk(\xi)}(x, \theta) - \textstyle\sum_{j=1}^s a_j^0 \varphi_{nk(j)}(x, \theta)$$

form an expectation centered Cramér sequence which of course, satisfies (15) of Theorem 1.

Using Neyman's [1] optimality criterion the following theorem establishes the optimality of the test based on the Cramér sequence $\{f_{nk}^*(x, \theta)\}$.

THEOREM 3. *If the estimates* $\hat{\theta}_{nk}$ *of the parameters* θ_j, $j = 1, 2, \cdots, s$, *are all* *locally root n consistent then the test of the hypothesis* $H_0 : \xi = \xi_0 = 0$ *against the alternative* $H_1 : \Gamma_{1+}$ *based on the sequence* $\{\omega_n^0\}$, *where* ω_n^0 *is defined by*

$$(30) \qquad Z_n^*(\xi_0, \hat{\theta}_n) = [1/n^{\frac{1}{2}} S_n^*(\hat{\theta}_n)] \textstyle\sum_{k=1}^n f_{nk}^*[X_{nk}(\xi_0, \theta), \hat{\theta}_n] > \nu(\alpha)$$

is an optimal test of class $C(\alpha)$, *with reference to the sequences of the family* Γ_{1+}. *Furthermore, its asymptotic power is given by*

$$(31) \qquad 1 - \Phi\{\nu(\alpha) - S_n^*(\theta)\xi_n n^{\frac{1}{2}}\}$$

where $\nu(\alpha)$ *satisfies*

$$(32) \qquad \Phi(\nu(\alpha)) = 1 - \alpha.$$

Finally it should be remarked that for certain types of problems, it may not be possible to determine the value of $a^0 = \lim_{n\to\infty} a_n^0$. For such cases we suggest an asymptotically equivalent optimal $C(\alpha)$ test, which may be constructed by replacing a_j^0 by a_{jn}^0, $j = 1, 2, \cdots, s$. The test function so obtained has the same asymptotic properties as those of $Z_n^*(\xi_0, \hat{\theta}_n)$.

5. An example: Homogeneity of frequencies of Supernovae.

A problem in astronomy posed by Professor Neyman is to test whether galaxies (perhaps of a restricted class) are homogeneous with respect to the frequency of occurrence of supernovae. It is customary to assume that, if galaxies are homogeneous, the number X of supernovae has a Poisson distribution. If they are not, the number X_i of supernovae observed during controlled time t_i in a randomly selected ith galaxy would be distributed as a mixture of Poisson variables and its distribution may be described by

$$(33) \qquad P[X_i = k_i] = (\lambda_0 t_i)^{k_i}/k_i! \int_a^b \exp -[\lambda_0 t_i e^{x\xi^{\frac{1}{2}}} - k_i x \xi^{\frac{1}{2}}]\, dF(x)$$

$$k_i = 0, 1, 2, \cdots; \qquad i = 1, 2, \cdots, n$$

where

n = number of galaxies observed,
$\ln \lambda_0$ = average of the logarithms of frequencies of supernovae per unit time.
ξ = variance of the logarithms of frequencies of supernovae over all galaxies, with

$$(34) \qquad \int_a^b x\, dF(x) = \int_a^b x^3\, dF(x) = 0; \qquad \int_a^b x^2\, dF(x) = \int_a^b dF(x) = 1,$$

$a < b$ being two arbitrary but appropriate finite constants. Furthermore, the t_i's are such that $n^{-1} \sum_{i=1}^n t_i$ has a finite positive limit as $n \to \infty$ and that $\max t_i < C$ for some constant $C > 0$. Calculations yield $f_{nk}^*(x, \lambda_0^i) = f_k^*(x, \lambda_0) = [(x_k - \lambda_0 t_k)^2 - x_k]/2$.

It can be demonstrated that $\{f_{nk}^*(x, \lambda_0)\}$ is a Cramér sequence satisfying the conditions of Theorem 1.

Furthermore

$$(35) \qquad Z_n^*(\lambda_0) = [2\lambda_0^2 \textstyle\sum_{k=1}^n t_k^2]^{-\frac{1}{2}} \sum_{k=1}^n \{(x_k - \lambda_0 t_k)^2 - x_k\},$$

where the test criterion (30) will be to reject the hypothesis $H_0 : \xi = 0$. Whenever $Z_n^*(\lambda_0) > \nu(\alpha)$. The asymptotic power of the test is

$$(36) \qquad 1 - \Phi\{\nu(\alpha) - \xi\lambda_0(\textstyle\sum t_i^2/2)^{\frac{1}{2}}\}.$$

λ_0 in (35) may be replaced by any root n consistent estimate, for example

$$\{\textstyle\sum_{k=1}^n (t_k x_k)\}/\{\sum_{k=i}^n t_k\}.$$

The test criterion (35) deduced on the assumption that the times $\{t_k\}$ during which the galaxies are observed for supernovae are fixed constants, is closely related to the criterion deduced by Neyman [2] under the assumption that the values $\{t_k\}$ form a sample from some unspecified distribution. This test has certain interesting properties described in the following remarks.

Remark 1. One property is described by Neyman as "robustness of optimality". This concept of robustness is to be clearly distinguished from the usual concept of robustness, which may be labeled "robustness of distribution".

It will be noticed that in deducing the test criterion (35) the class of hypotheses alternative to the one tested is described by a distribution function $F(x)$ of which it is only assumed that (34) holds. With reference to the expected number λ of supernovae per unit time for a single galaxy, this is equivalent to asserting that, if $\lambda \neq \lambda_0 =$ constant, then it varies from galaxy to galaxy as $\ln \lambda = \ln \lambda_0 + X\xi^{\frac{1}{4}}$, where X is a random variable with distribution function $F(x)$ and ξ a positive constant. The interesting point is that (35) retains its optimal property independently of what $F(x)$ might be subject to (34). In particular, this includes the possibility that all the galaxies fall into two categories with expected numbers of supernovae $\lambda_1 < \lambda_2$. Also, there is the possibility that the frequency of galaxies with varying λ varies continuously from zero to some limit, etc. In all these cases, the optimal criterion for testing the hypothesis of homogeneity is the same, namely (35). Here, then, the optimality of the test criterion (35) is "robust".

Remark 2. The evaluation of the asymptotic power is essentially based on (36) which implies the following interesting conclusion regarding the design of a survey of galaxies for supernovae. It refers to the question of the relative advantage or disadvantage of, say, doubling the number n of galaxies to be observed and simultaneously of cutting by half the times t_k over which each galaxy will be observed.

As seen from (36) this operation will result in replacing the original noncentrality parameter

$$\xi\lambda_0(\textstyle\sum t_k^2/2)^{\frac{1}{2}} = \xi\lambda_0(n\bar{t}^2/2)^{\frac{1}{2}}$$

where $\bar{t}^2$ represents the mean square of the t_k, by $\xi\lambda_0[n\bar{t}^2/4]^{\frac{1}{2}}$ which is equivalent to dividing the number of observations by two, while the average of the t_k^2 remains unchanged. Thus it is clear that, granted that the number of galaxies observed is large, it is advantageous to diminish this number somewhat in order to be able to prolong the observations for all these galaxies that will be observed.

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