ON OPTIMAL CONTROL PROBLEMS WITH BOUNDED STATE VARIABLES AND CONTROL APPEARING LINEARLY

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Summary: Necessary conditions for the junction of interior arcs and boundary arcs of an optimal control are given. These conditions are based on necessary conditions for the switching function holding at junction points or contact points with the state boundary. The junction theorems obtained are similar to junction theorems in singular control problems.

1. Formulation of the problem

We consider the following control problem with control appearing linearly: determine the scalar, piecewise continuous control $u(t)$, $t \in [0,T]$, which minimizes the functional

(1) $$J(u) = G(x(T))$$

subject to

(2) $$\dot{x} = f_1(x) + f_2(x)u$$

(3) $$x(0) = x_0 \ , \quad \psi(x(T)) = 0$$

(4) $$|u(t)| \leq K(t) \ , \quad K(t) > 0 \ , \quad 0 \leq t \leq T$$

and the scalar state inequality constraint of order p

(5) $$S(x) \leq 0 \ .$$

The state x is an n-vector. The functions $G: \mathbb{R}^n \to \mathbb{R}$, $\psi: \mathbb{R}^n \to \mathbb{R}^k$, $k < n$, are differentiable and for simplicity the functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}^n$, $S : \mathbb{R}^n \to \mathbb{R}$ are assumed to be analytic in a suitable domain; $K(t)$ is analytic in $[0,T]$.

Along a trajectory $x(t)$ of (2) the i-th time derivative of $S(x(t))$ is denoted by S^i, $i \geq 0$. Then by definition of the order p of the state constraint (5) S^p is the first derivative containing the control u explicitly and we have

(6) $$S^i = S^i(x) \ , \quad i=0,..,p-1 \ , \qquad S^p = S^p(x,u) = a(x) + b(x)u \ .$$

A subarc of $x(t)$ with $S(x(t)) < 0$ is called an <u>interior arc</u> and a subarc of $x(t)$ with $S(x(t)) = 0$ for $t_1 \leq t \leq t_2$, $t_1 < t_2$, is called a <u>boundary arc</u>. Here t_1 and t_2 are called <u>entry</u>- and <u>exit-point</u> or simply <u>junction points</u> of the boundary arc. If $S(x(t_1)) = 0$ and $S(x(t)) < 0$ for $t \neq t_1$ in a neighborhood of t_1 then t_1 is called a <u>contact point</u> of $x(t)$ with the boundary.

The <u>boundary control</u> is determined by $S^p(x,u) = 0$ which gives

(7) $$u = u(x) = -a(x)/b(x) .$$

Let $u(t) = u(x(t))$ and $b(t) = b(x(t))$. It is assumed that along a boundary arc in $[t_1,t_2]$ the following condition holds:

(8) $$b(t) \neq 0 \text{ for } t_1 \leq t \leq t_2 , \quad |u(t)| < K(t) \text{ for } t_1 < t < t_2 .$$

2. Necessary conditions of the Minimum-Principle

The necessary conditions for an extremal arc of (1)-(5) are developed in [1],[2]. It can be shown that the function η^* of bounded variation in [2,Th.2.3] has a continuous derivative η on the interior of a boundary arc for p-th order state constraints. Define the Hamiltonian

(9) $$H(x,u,\lambda,\eta) = \lambda^T f_1(x) + \lambda^T f_2(x)u + \eta S(x)$$

where $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}$ and where the superscript T denotes the transpose.

1. There exists a scalar function $\eta(t) \geq 0$ which satisfies $\eta(t)S(x(t)) = 0$, $t \in [0,T]$, and which is continuous on the interior of a boundary arc. The adjoint variable $\lambda(t)$ satisfies

(10) $$\dot{\lambda}^T = - \lambda^T(f_1+f_2u)_x - \eta S_x , \quad \lambda^T(T) = G_x(x(T))+\sigma^T\psi_x(x(T)) , \quad \sigma \in \mathbb{R}^k .$$

2. The jump condition at a contact point or junction point t_1 is

(11) $$\lambda^T(t_1^+) = \lambda^T(t_1^-) - \nu_1 S_x(x(t_1)) , \quad \nu_1 \geq 0 .$$

3. The optimal control $u(t)$ minimizes $H(x(t),u,\lambda(t),\eta(t))$ over u with $|u| \leq K(t)$.

The coefficient of u in (9) is called the <u>switching function</u>

(12) $$\Phi(t) = \lambda^T(t)f_2(x(t)) .$$

Then the optimal control $u(t)$ is given on an interior arc by

$$u(t) = -K(t)\operatorname{sgn}\Phi(t) \tag{13}$$

where for simplicity $\Phi(t)$ is assumed to have only isolated zeros, i.e. $u(t)$ is a nonsingular control. On a boundary arc in $[t_1,t_2]$ the optimal control is the boundary control (7). The assumption (8) and the Minimum-Principle then imply

$$H_u(t) = \Phi(t) = 0 \quad \text{for} \quad t_1^+ \leq t \leq t_2^- \ . \tag{14}$$

Thus the boundary control behaves like a singular control in singular control problems. We can expect therefore necessary conditions for junctions between interior and boundary arcs which are similar to those in McDanell, Powers [3] .

3. Relations for the switching function at contact or junction points

Let t_1 be a contact point or a junction point and let $u^{(r)}(t)$, $r \geq 0$, be the lowest order derivative of the control $u(t)$ which is discontinuous at t_1 . Furthermore let the integer q be the order of a singular arc, i.e. $\Phi^{(2q)}$ is the lowest order time derivative of Φ which contains the control u explicitly. Under the assumption $p \leq 2q+r$ one can show the following relations for the switching function by using the jump condition (11):

$$\Phi^{(i)}(t_1^+) = \Phi^{(i)}(t_1^-) \ , \quad i = 0,..,p-2 \ , \tag{15}$$

$$\Phi^{(p-1)}(t_1^+) = \Phi^{(p-1)}(t_1^-) - \nu_1(-1)^{p-1}b(t_1) \ . \tag{16}$$

The assumption $p \leq 2q+r$ always holds for $p \leq 2$. The relation (16) implies that $\nu_1 > 0$ is equivalent to the discontinuity of $\Phi^{(p-1)}(t)$ at t_1 . Now let t_1 be an entry-point of a boundary arc. Then we get $\Phi^{(i)}(t_1^+) = 0$ for $i \geq 0$ by virtue of (14) and hence (15),(16) yield

$$\Phi^{(i)}(t_1^-) = 0 \ , \quad i = 0,..,p-2 \ , \tag{17}$$

$$\nu_1 = (-1)^{p-1}\, \Phi^{(p-1)}(t_1^-)/b(t_1) \geq 0 \ . \tag{18}$$

The relations (17),(18) remain valid at an exit-point t_2 with t_1^- resp. ν_1 replaced by t_2^+ resp. $-\nu_2$.

4. Junction Theorems

Based on (17),(18) the following theorem can be proved using ideas similar to those in [3,Th.1]

Theorem 1: Let t_1 be a point where an interior nonsingular arc and a boundary arc of an optimal control u are joined and assume that u is piecewise analytic in a neighborhood of t_1. Let $u^{(r)}$, $r \geq 0$, be the lowest order derivative of u which is discontinuous at t_1 and let $p \leq 2q+r$. If $\nu_1 > 0$ then $p+r$ is an even integer.

Corollary 1: Under the assumptions of Theorem 1 the following statements are valid.

(i) If $p+r$ is odd then $\nu_1 = 0$ holds at a junction point t_1.

(ii) If $p+r$ is odd and if $\nu_1 > 0$ then t_1 cannot be a junction point but can only be a contact point with the boundary.

The next theorem treats the case $\nu_1 = 0$, i.e. $\Phi^{(p-1)}(t_1^-) = 0$, and is dual to a result for singular control problems [3,Th.2].

Theorem 2: Let t_1 be a point where an interior nonsingular arc and a boundary arc of an optimal control u are joined and assume that u is piecewise analytic in a neighborhood of t_1. Let $\Phi^{(p+m)}(t_1^-)$, $m \geq 0$, be the lowest order nonvanishing derivative of Φ and let $u^{(r)}$, $r \geq 0$, be the lowest order derivative of u which is discontinuous at t_1. If $p+m < 2q+r$ then $p+r+m$ is an odd integer.

In the 'normal case' $r = 0$ the preceding junction theorems allow a rough classification of the behaviour of the extremals with respect to the order p. For $p = 1$ numerical examples show that the extremals contain in general only boundary arcs. Hence $\nu_1 = 0$ at a junction point t_1 by Corollary 1 and the integer m in Theorem 2 is even. Usually we have $m = 0$ and thus $\Phi(t_1) = \Phi(t_2) = 0$, $\dot{\Phi}(t_1^-) \neq 0$, $\dot{\Phi}(t_2^+) \neq 0$ at an entry-point t_1 or exit-point t_2. If p is even then contact points and boundary arcs are possible for $\nu_1 > 0$. If p is odd and $p \geq 3$ then only contact points with the boundary are possible for $\nu_1 > 0$. A similar result holds for a regular Hamiltonian, cf. [1].

Proofs, further junction theorems and numerical examples will appear elsewhere [4]. The duality of control problems with bounded state variables and singular control problems is also displayed by similar numerical algorithms for both problems [5],[6].

References

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