On Order-Reducible Sinc Discretizations and Block-Diagonal Preconditioning Methods for Linear Third-Order Ordinary Differential Equations *

Zhong-Zhi Bai

State Key Laboratory of Scientific/Engineering Computing
Institute of Computational Mathematics and Scientific/Engineering Computing
Academy of Mathematics and Systems Science
Chinese Academy of Sciences, P.O. Box 2719, Beijing 100190, P.R. China
Email: bzz@lsec.cc.ac.cn

Raymond H. Chan

Department of Mathematics
The Chinese University of Hong Kong
Shatin, Hong Kong
Email: rchan@math.cuhk.edu.hk

Zhi-Ru Ren

State Key Laboratory of Scientific/Engineering Computing
Institute of Computational Mathematics and Scientific/Engineering Computing
Academy of Mathematics and Systems Science
Chinese Academy of Sciences, P.O. Box 2719, Beijing 100190, P.R. China
Email: renzr@lsec.cc.ac.cn

February 3, 2012

Abstract

By introducing a variable substitution we transform the two-point boundary value problem of a third-order ordinary differential equation into a system of two second-order ordinary differential equations. We discretize this order-reduced system of ordinary differential equations by both sinc-collocation and sinc-Galerkin methods, and average these two discretized linear systems to obtain the target system of linear equations. We prove that the discrete solution resulting from the linear system converges exponentially to the true solution of the order-reduced system of ordinary differential equations. The coefficient matrix of the linear system is of block two-by-two structure and each of its blocks is a combination of Toeplitz

*Supported by The Hundred Talent Project of Chinese Academy of Sciences, and The National Basic Research Program (No. 2011CB309703), P.R. China; and by Hong Kong Research Grants Council Grant (No. 400510) and CUHK DAG 2060408.

and diagonal matrices. Because of its algebraic properties and matrix structures, the linear system can be effectively solved by Krylov subspace iteration methods such as GMRES preconditioned by block-diagonal matrices. We demonstrate that the eigenvalues of certain approximation to the preconditioned matrix are uniformly bounded within a rectangle on the complex plane independent of the size of the discretized linear system, and we use numerical examples to illustrate the feasibility and effectiveness of this new approach.

Keywords: third-order ordinary differential equation; order reduction; sinc-collocation discretization; sinc-Galerkin discretization; convergence analysis; preconditioning; eigenvalue estimate.

AMS(MOS) Subject Classifications: 65F08, 65F10, 65F15, 65L10, 65T10; CR: G1.3.

1 Introduction

We consider numerical solution for the following two-point boundary value problem of linear third-order *ordinary differential equation* (**ODE**):

$$\begin{cases} Ly_1(x) := y_1'''(x) + \mu_2(x)y_1''(x) + \mu_1(x)y_1'(x) + \mu_0(x)y_1(x) = \sigma(x), \quad a < x < b, \\ y_1(a) = 0, \quad y_1(b) = 0, \quad y_1'(a) = 0, \end{cases} \tag{1.1}$$

where $\mu_j(x)$ $(j = 0, 1, 2)$ and $\sigma(x)$ are known bounded functions, and a and b are given real numbers. This class of problems arise from many different applications where point/contact transformations and numerical linearizations are used. Examples are (i) the Korteweg-de Vries (KdV) equations in hydrodynamics modeling the propagation of solitary wave on water surfaces [34, 35], (ii) the non-conservative systems of realistic physical situations such as the Falkner-Skan and the Nosé equations modeling boundary layers in fluid dynamics and the interaction of a particle with a heat-bath, respectively [32, 11, 28], (iii) the draining or coating fluid-flow problems involving surface tension force [33], (iv) the flow of thin films of viscous fluid with a free surface having surface tension effects [9, 12, 14], and many others, see [30, 20, 21, 15, 23, 13, 10]. The main issue in solving (1.1) is that its highest term is of order three, which makes the coefficient matrix of the discretized linear system strongly nonsymmetric and highly ill-conditioned.

Recently, Bai, Chan and Ren [3] discretized the third-order ODE (1.1) by sinc methods and proved that the discrete solution converges to the true solution exponentially. The coefficient matrix of the resulted linear system is a combination of Toeplitz and diagonal matrices. In [3] the authors constructed a banded preconditioning matrix and employed the corresponding preconditioned Krylov subspace methods to iteratively solve such a discretized linear system. However, as the highest term of the ODE (1.1) is of order three, the discretized linear system is highly ill-conditioned as its coefficient matrix has a strongly dominant skew-symmetric part. As a result, in actual implementations there is considerable difficulty in iteratively computing the discrete solution.

In order to overcome the shortcoming of the direct discretization method in [3], in this paper we introduce a variable substitution that first transforms the third-order ODE (1.1) into a system of two second-order ODEs. Then we apply the approach similar to the one used in [3] to solve the resulting system. More precisely, we apply the sinc-collocation and sinc-Galerkin methods to discretize the system of second-order ODEs. Then we take the average of the two resulting systems to obtain a system $\mathbf{Aw} = \mathbf{p}$ that we need to solve. Because the highest order of both ODEs are 2, the coefficient matrix $\mathbf{A}$ will have strongly dominant symmetric part and diagonal blocks, as well as smaller-order off-diagonal blocks. Moreover, its diagonal blocks are positive definite. Thus $\mathbf{A}$ has better algebraic properties and matrix structures so that many numerical difficulties in solving the third-order ODE (1.1) caused by its third-order term can be alleviated. We will prove that the discrete solution of $\mathbf{Aw} = \mathbf{p}$ converges exponentially to the true solution of the order-reduced system of ODEs when the discretization step-size h tends to zero.

The matrix $\mathbf{A}$ is of block two-by-two structure with each of its blocks being a combination of Toeplitz and diagonal matrices. Unfortunately, direct methods such as the Gaussian elimination or the fast Toeplitz algorithms are not applicable to effectively solve this class of Toeplitz-plus-diagonal linear systems due to considerably high computational complexity; see [16, 17, 18, 25]. However, note that for $\mathbf{A}$ of size n-by-n, the matrix-vector product $\mathbf{Aq}$ can be computed in $\mathcal{O}(n \log n)$ operations for any n-vector $\mathbf{q}$. We can therefore employ Krylov subspace iteration methods such as GMRES to iteratively solve $\mathbf{Aw} = \mathbf{p}$. To guarantee fast convergence, one needs an efficient preconditioner; see [2, 6, 7, 8, 26]. Since the diagonal blocks of $\mathbf{A}$ are dominant over the off-diagonal blocks, we can construct a block-diagonal preconditioning matrix $\mathbf{P}$, with banded diagonal blocks, for $\mathbf{A}$, and make use of the Toeplitz structure and positive-definite property in $\mathbf{A}$. By denoting $\widehat{\mathbf{A}}$ the diagonal-block matrix of $\mathbf{A}$, we prove that $\|\widehat{\mathbf{A}} - \mathbf{A}\|_2 \leq \mathcal{O}(h)$ and the eigenvalues of the matrix $\mathbf{P}^{-1}\widehat{\mathbf{A}}$ are uniformly bounded within a rectangle on the complex plane independent of the size of the linear system. It follows that $\mathbf{P}$ will be a desirable preconditioner for $\mathbf{A}$, especially when the discretization step-size h is small. Numerical results show that $\mathbf{P}$ is effective in accelerating the convergence of the Krylov subspace iteration methods and the approximated solution is accurate.

The outline of this paper is as follows. In Section 2, we introduce the variable substitution that gives the order-reduced ODE system from the third-order ODE (1.1). In Section 3, we discretize the order-reduced ODE system by both sinc-collocation and sinc-Galerkin methods; and an averaging combination of these two sinc discretizations leads to the target linear system $\mathbf{Aw} = \mathbf{p}$. In Section 4, we derive an error bound of exponentially decreasing rate for the discrete solution of the order-reduced ODE system. A block-diagonal preconditioner $\mathbf{P}$ for the matrix $\mathbf{A}$ is constructed and the eigen-properties of the preconditioned matrix $\mathbf{P}^{-1}\mathbf{A}$ are discussed in Section 5. In Section 6, numerical examples are given to show the feasibility and effectiveness of our approach. Finally, in Section 7, we end the paper with a few concluding remarks.

2 The Order-Reduced ODE System

In this section, we give the substitution that transforms (1.1) into a system of two second-order ODEs. To this end, we define a function $y_2(x)$ implicitly by the following ODE:

$$y_1''(x) = p(x)y_2'(x) + q(x)y_2(x), \quad p(x) \neq 0, \tag{2.1}$$

where $p(x)$ and $q(x)$ are continuously differentiable functions to be specified later on. Differentiating (2.1) we have

$$y_1'''(x) = p(x)y_2''(x) + (p'(x) + q(x))y_2'(x) + q'(x)y_2(x). \tag{2.2}$$

Substituting (2.1) and (2.2) into (1.1), we obtain

$$\mu_1(x)y_1'(x) + \mu_0(x)y_1(x) + p(x)y_2''(x) + \nu_1(x)y_2'(x) + \nu_0(x)y_2(x) = \sigma(x), \tag{2.3}$$

where

$$\nu_0(x) = q^{'}(x) + \mu_2(x)q(x) \quad \text{and} \quad \nu_1(x) = p^{'}(x) + \mu_2(x)p(x) + q(x).$$

In order to derive the boundary conditions for $y_2(x)$, we define

$$r(x) = e^{-\int \frac{q(x)}{p(x)}\mathrm{d}x} \quad \text{and} \quad s(x) = \frac{1}{p(x)} e^{\int \frac{q(x)}{p(x)}\mathrm{d}x}.$$

Then by solving $y_2(x)$ from (2.1) we obtain

$$y_2(x) = r(x) \int s(x) y_1^{''}(x)\mathrm{d}x.$$

For any $x \in [a, b]$, it follows from setting the integral interval to be $[a, x]$ that $y_2(a) = 0$. This shows that the boundary condition for $y_2(x)$ defined through the ODE (2.1) is $y_2(a) = 0$.

Let $y(x) = (y_1(x), y_2(x))^T$. Then a combination of (2.1) and (2.3) immediately leads to the system of second-order ODEs:

$$\left\{ \begin{array}{lcl} L_1 y(x) & := & y_1^{''}(x) - p(x)y_2^{'}(x) - q(x)y_2(x) = 0, \\ L_2 y(x) & := & \mu_1(x)y_1^{'}(x) + \mu_0(x)y_1(x) + p(x)y_2^{''}(x) \\ & & +\nu_1(x)y_2^{'}(x) + \nu_0(x)y_2(x) = \sigma(x), \\ & & y_1(a) = 0, \quad y_1(b) = 0, \quad y_1^{'}(a) = 0 \quad \text{and} \quad y_2(a) = 0, \end{array} \right. \quad a < x < b. \tag{2.4}$$

Obviously, (2.4) is mathematically equivalent to (1.1). Alternatively, we may reformulate (2.4) into matrix-vector form as follows:

$$\left\{ \begin{array}{l} \Upsilon_2(x)y^{''}(x) + \Upsilon_1(x)y^{'}(x) + \Upsilon_0(x)y(x) = F(x), \\ y(a) = 0, \quad y_1^{'}(a) = 0 \quad \text{and} \quad y_1(b) = 0, \end{array} \right. \tag{2.5}$$

where

$$\Upsilon_0(x) = \begin{bmatrix} 0 & -q(x) \\ \mu_0(x) & \nu_0(x) \end{bmatrix}, \quad \Upsilon_1(x) = \begin{bmatrix} 0 & -p(x) \\ \mu_1(x) & \nu_1(x) \end{bmatrix}, \quad \Upsilon_2(x) = \begin{bmatrix} 1 & 0 \\ 0 & p(x) \end{bmatrix} \tag{2.6}$$

and

$$F(x) = \begin{bmatrix} 0 \\ \sigma(x) \end{bmatrix}. \tag{2.7}$$

We now specify the choices of $p(x)$ and $q(x)$, with the aim that the discretization matrix of (2.5)–(2.7) will have special properties and structures, so we can construct efficient preconditioners for the corresponding Krylov subspace iteration methods.

Case (i): $\mu_1(x) > 0, \forall x \in [a, b]$. For this case, we take $p(x) = \mu_1(x)$ and $q(x) = -\mu_0(x)$ such that the coefficient of the second-order term $\Upsilon_2(x)$ of (2.5) is symmetric and positive, and the off-diagonal coefficients of the first-order and the zero-order terms $\Upsilon_1(x)$ and $\Upsilon_0(x)$ are skew-symmetric and symmetric, respectively. As a result, the coefficient matrix of the correspondingly discretized linear system may have positive-definite diagonal blocks, and its off-diagonal blocks may be symmetric as far as possible.

Case (ii): $\mu_1(x) < 0, \forall x \in [a,b]$. For this case, we take $p(x) = -\mu_1(x)$ and $q(x) = \mu_0(x)$ such that $\Upsilon_2(x)$ is symmetric and positive while the off-diagonal parts of $\Upsilon_1(x)$ and $\Upsilon_0(x)$ are symmetric and skew-symmetric, respectively. As a result, the coefficient matrix of the resulting system may have positive-definite diagonal blocks, and its off-diagonal blocks may be skew-symmetric as far as possible.

The above two cases can be unified as $p(x) = |\mu_1(x)|$ and $q(x) = -\text{sign}(\mu_1(x))\mu_0(x)$, where

$$\text{sign}(\mu_1(x)) = \begin{cases} 1, & \text{for} \quad \mu_1(x) > 0, \\ 0, & \text{for} \quad \mu_1(x) = 0, \\ -1, & \text{for} \quad \mu_1(x) < 0. \end{cases}$$

Clearly, there could be other possible choices for $p(x)$ and $q(x)$, which may lead to different second-order ODE systems (2.4) with other matrix structures.

3 Sinc Discretization Methods

Let $\mathcal{D}$ be a simply-connected domain on the complex plane having boundary $\partial\mathcal{D}$. Let a and b denote two distinct points of $\partial\mathcal{D}$, and $t = \phi(z)$ denote a conformal mapping of $\mathcal{D}$ onto a strip region $\mathcal{D}_d$ such that $\phi(a) = -\infty$ and $\phi(b) = \infty$, where $\mathcal{D}_d := \{t \in \mathbb{C} : |\text{Im}(t)| < d\}$. Conversely, $z = \psi(t) := \phi^{-1}(t)$ maps $\mathcal{D}_d$ onto $\mathcal{D}$ with the boundary $\partial\mathcal{D}$ on which the points a and b lie. Here and in the following, we use $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ to denote the real and the imaginary parts of a complex number, and $(\cdot)^T$ to denote the transpose of either a vector or a matrix. We write a function $f(x)$ briefly as f if no confusion arises. In addition, for two vector functions $f = (f_1, f_2)^T$ and $g = (g_1, g_2)^T$, $f \odot g \equiv (f_1 g_1, f_2 g_2)^T$.

In this section, we discretize the order-reduced ODE system (2.4) by both sinc-collocation and sinc-Galerkin methods. The sinc function used is

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}, \quad -\infty < t < \infty,$$

and the set of basis functions adopted are

$$S(j,h)(t) := \frac{\sin[\pi(t - jh)/h]}{\pi(t - jh)/h}, \quad -\infty < t < \infty, \quad j \in \mathbb{Z}, \tag{3.1}$$

where h is the step-size and $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ denotes the set of all integers [31]. The points $t_j = jh$, $j \in \mathbb{Z}$, are called the sinc-grid points.

We approximate the exact solution $y(x) = (y_1(x), y_2(x))^T$ of (2.4) by the function $y_N(x) = (y_{1N}(x), y_{2N}(x))^T$:

$$\begin{cases} y_{1N}(x) = \rho_1(x) \displaystyle\sum_{j=-N}^{N} u_j S(j,h) \circ \phi_1(x), \\ y_{2N}(x) = \rho_2(x) \displaystyle\sum_{j=-N}^{N} v_j S(j,h) \circ \phi_2(x), \end{cases} \tag{3.2}$$

where $\rho_i(x)$ $(i = 1, 2)$ are known bounded functions, $\phi_i(x)$ $(i = 1, 2)$ are conformal mappings from $\mathcal{D}$ to $\mathcal{D}_d$, $\{S(j,h)\}_{j \in \mathbb{Z}_N}$ are the sinc-basis functions in (3.1), and $\{u_j\}_{j \in \mathbb{Z}_N}$ and $\{v_j\}_{j \in \mathbb{Z}_N}$

are the unknown coefficients. Here, we have used the notation $\mathbb{Z}_N = \{-N, -N+1, \ldots, N\}$. Moreover, by denoting $\rho(x) = (\rho_1(x), \rho_2(x))^T$, $\varpi_j = (u_j, v_j)^T$ and $S(k,h) \circ \phi(x) = (S(k,h) \circ \phi_1(x), S(k,h) \circ \phi_2(x))^T$, we can briefly express the approximate solution $y_N(x)$ defined in (3.2) in vector form as

$$y_N(x) = \rho(x) \odot \sum_{j=-N}^{N} \varpi_j \odot S(j,h) \circ \phi(x).$$

For the sinc-collocation method [19], the unknown coefficients $\{u_j\}_{j=-N}^N$ and $\{v_j\}_{j=-N}^N$ of the functions $y_{1N}(x)$ and $y_{2N}(x)$ in (3.2) are determined by imposing the collocating conditions

$$\begin{cases} L_1 y_N(x_k) = 0, \\ L_2 y_N(x_k) = \sigma(x_k), \quad k \in \mathbb{Z}_N, \end{cases} \tag{3.3}$$

on the sinc grid-points $x_k = \psi_i(kh) = \phi_i^{-1}(kh)$ $(i = 1,2)$, with L_1 and L_2 being the ODE operators defined in (2.4). After substituting $y_N(x)$ in (3.2) into (3.3) and multiplying $h^2/[(\phi_1^{'})^2\rho_1]$ and $h^2/[(\phi_2^{'})^2\rho_2]$ through both sides of the two equations, respectively, we obtain a system of linear equations $\mathbf{A}_C\mathbf{w} = \mathbf{p}$ with respect to $\{u_j\}_{j=-N}^N$ and $\{v_j\}_{j=-N}^N$, where

$$\mathbf{A}_C = \begin{bmatrix} \mathbf{T}^{(2)} + \mathbf{D}_C^{(1)}\mathbf{T}^{(1)} + \mathbf{D}_C^{(5)} & \mathbf{D}_C^{(2)}\mathbf{T}^{(1)} + \mathbf{D}_C^{(6)} \\ \mathbf{D}_C^{(3)}\mathbf{T}^{(1)} + \mathbf{D}_C^{(7)} & \mathbf{D}_C^{(0)}\mathbf{T}^{(2)} + \mathbf{D}_C^{(4)}\mathbf{T}^{(1)} + \mathbf{D}_C^{(8)} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{3.4}$$

and

$$\begin{cases} \mathbf{w} = (u_{-N}, u_{-N+1}, \ldots, u_N, v_{-N}, v_{-N+1}, \ldots, v_N)^T \in \mathbb{R}^{2n}, \\ \mathbf{p} = -h^2\left(0, 0, \ldots, 0, \dfrac{\sigma}{(\phi_2^{'})^2\rho_2}(x_{-N}), \dfrac{\sigma}{(\phi_2^{'})^2\rho_2}(x_{-N+1}), \ldots, \dfrac{\sigma}{(\phi_2^{'})^2\rho_2}(x_N)\right)^T \in \mathbb{R}^{2n}, \end{cases} \tag{3.5}$$

with $n = 2N+1$. In addition, $\mathbf{T}^{(1)} = (\delta_{jk}^{(1)})$ and $\mathbf{T}^{(2)} = (-\delta_{jk}^{(2)})$ are the n-by-n Toeplitz matrices given by

$$\mathbf{T}^{(1)} = \begin{bmatrix} 0 & -1 & \frac{1}{2} & \cdots & \frac{(-1)^{n-1}}{n-1} \\ 1 & 0 & \ddots & \ddots & \vdots \\ -\frac{1}{2} & 1 & \ddots & -1 & \frac{1}{2} \\ \vdots & \ddots & \ddots & 0 & -1 \\ -\frac{(-1)^{n-1}}{n-1} & \cdots & -\frac{1}{2} & 1 & 0 \end{bmatrix} \tag{3.6}$$

and

$$\mathbf{T}^{(2)} = \begin{bmatrix} \frac{\pi^2}{3} & -2 & \frac{2}{2^2} & \cdots & \frac{2(-1)^{n-1}}{(n-1)^2} \\ -2 & \frac{\pi^2}{3} & \ddots & \ddots & \vdots \\ \frac{2}{2^2} & -2 & \ddots & -2 & \frac{2}{2^2} \\ \vdots & \ddots & \ddots & \frac{\pi^2}{3} & -2 \\ \frac{2(-1)^{n-1}}{(n-1)^2} & \cdots & \frac{2}{2^2} & -2 & \frac{\pi^2}{3} \end{bmatrix}, \tag{3.7}$$

and

$$\mathbf{D}_C^{(i)} := \operatorname{diag}(g_C^{(i)}(x_{-N}), g_C^{(i)}(x_{-N+1}), \dots, g_C^{(i)}(x_N)), \quad i = 0, 1, \dots, 8,$$

are diagonal matrices, with $g_C^{(0)} = p$ and

$$g_C^{(1)} = h\,\frac{\phi_1'\rho_1' + (\phi_1'\rho_1)'}{(\phi_1')^2\rho_1}, \qquad g_C^{(2)} = -h\,\frac{p\phi_2'\rho_2}{(\phi_1')^2\rho_1},$$

$$g_C^{(3)} = h\,\frac{\mu_1\phi_1'\rho_1}{(\phi_2')^2\rho_2}, \qquad g_C^{(4)} = h\,\frac{p(\phi_2'\rho_2' + (\phi_2'\rho_2)') + \nu_1\phi_2'\rho_2}{(\phi_2')^2\rho_2},$$

$$g_C^{(5)} = -h^2\,\frac{\rho_1''}{(\phi_1')^2\rho_1}, \qquad g_C^{(6)} = h^2\,\frac{p\rho_2' + q\rho_2}{(\phi_1')^2\rho_1},$$

$$g_C^{(7)} = -h^2\,\frac{\mu_1\rho_1' + \mu_0\rho_1}{(\phi_2')^2\rho_2}, \qquad g_C^{(8)} = -h^2\,\frac{p\rho_2'' + \nu_1\rho_2' + \nu_0\rho_2}{(\phi_2')^2\rho_2}.$$

For the sinc-Galerkin method [19], the unknown coefficients $\{u_j\}_{j=-N}^N$ and $\{v_j\}_{j=-N}^N$ of the functions $y_{1N}(x)$ and $y_{2N}(x)$ in (3.2) are determined by orthogonalizing the residual elements $L_1 y_N(x)$ and $L_2 y_N(x) - \sigma(x)$ with the functions $\{S(k,h)\circ\phi(x)\}_{k=-N}^N$, where L_1 and L_2 are the operators defined in (2.4). This yields the discretized system

$$\begin{cases} \langle L_1 y_N(x),\, S(k,h)\circ\phi(x)\rangle = 0, \\ \langle L_2 y_N(x) - \sigma(x),\, S(k,h)\circ\phi(x)\rangle = 0, \quad k \in \mathbb{Z}_N, \end{cases} \tag{3.8}$$

where $\langle\cdot,\cdot\rangle$ represents the inner product defined by

$$\langle f(x),\, g(x)\rangle = \int_a^b \omega_1(x) f_1(x) g_1(x)\mathrm{d}x + \int_a^b \omega_2(x) f_2(x) g_2(x)\mathrm{d}x,$$

with $f(x) = (f_1(x), f_2(x))^T$, $g(x) = (g_1(x), g_2(x))^T$, and $\omega_1(x)$ and $\omega_2(x)$ being weighting functions. After integrating (3.8) by part and using Corollary 4.2.15 in [31], we obtain a system of linear equations $\mathbf{A}_G\,\mathbf{w} = \mathbf{p}$ where

$$\mathbf{A}_G = \begin{bmatrix} \mathbf{T}^{(2)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(1)} + \mathbf{D}_G^{(5)} & \mathbf{T}^{(1)}\mathbf{D}_G^{(2)} + \mathbf{D}_G^{(6)} \\ \mathbf{T}^{(1)}\mathbf{D}_G^{(3)} + \mathbf{D}_G^{(7)} & \mathbf{T}^{(2)}\mathbf{D}_G^{(0)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(4)} + \mathbf{D}_G^{(8)} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{3.9}$$

$\mathbf{T}^{(1)} = (\delta_{jk}^{(1)})$ and $\mathbf{T}^{(2)} = (-\delta_{jk}^{(2)})$ are the Toeplitz matrices defined in (3.6)–(3.7), and $\mathbf{w}$ and $\mathbf{p}$ are the unknown and the right-hand side vectors defined in (3.5), with $n = 2N+1$. In addition,

$$\mathbf{D}_G^{(i)} := \operatorname{diag}(g_G^{(i)}(x_{-N}), g_G^{(i)}(x_{-N+1}), \dots, g_G^{(i)}(x_N)), \quad i = 0, 1, \dots, 8,$$

are diagonal matrices, with $g_G^{(0)} = p$ and

$$g_G^{(1)} = -h\,\frac{\phi_1'\omega_1' + (\phi_1'\omega_1)'}{(\phi_1')^2\omega_1}, \qquad g_G^{(2)} = -h\,\frac{p\rho_2\omega_2}{\phi_1'\rho_1\omega_1},$$

$$g_G^{(3)} = h\,\frac{\mu_1\rho_1\omega_1}{\phi_2'\rho_2\omega_2}, \qquad g_G^{(4)} = -h\,\frac{(p\omega_2)'\phi_2' + (p\omega_2\phi_2')' - \nu_1\omega_2\phi_2'}{(\phi_2')^2\omega_2},$$

$$g_G^{(5)} = -h^2\,\frac{\omega_1''}{(\phi_1')^2\omega_1}, \qquad g_G^{(6)} = -h^2\,\frac{((p\omega_2)' - q\omega_2)\rho_2}{\phi_1'\phi_2'\rho_1\omega_1},$$

$$g_G^{(7)} = h^2\,\frac{((\mu_1\omega_1)' - \mu_0\omega_1)\rho_1}{\phi_1'\phi_2'\rho_2\omega_2}, \qquad g_G^{(8)} = -h^2\,\frac{(p\omega_2)'' - (\nu_1\omega_2)' + \nu_0\omega_2}{(\phi_2')^2\omega_2}.$$

In order to construct a discretized linear system for (2.4) such that its coefficient matrix is as symmetric/skew-symmetric or as positive-definite[1] as possible [5, 4], we follow the strategy in [3] and average the sinc-collocation matrix $\mathbf{A}_C$ in (3.4) and the sinc-Galerkin matrix $\mathbf{A}_G$ in (3.9), and obtain the system of linear equations

$$\mathbf{A}\mathbf{w} = \mathbf{p}, \tag{3.10}$$

where

$$\mathbf{A} = \frac{1}{2}(\mathbf{A}_C + \mathbf{A}_G) := \begin{bmatrix} \mathbf{B} & \mathbf{E} \\ \mathbf{F} & \mathbf{C} \end{bmatrix}, \tag{3.11}$$

and $\mathbf{w}$ and $\mathbf{p}$ being defined as in (3.5), with

$$\begin{aligned}
\mathbf{B} &= \mathbf{T}^{(2)} + \frac{1}{2}(\mathbf{D}_C^{(1)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(1)}) + \mathbf{D}^{(5)},\\
\mathbf{E} &= \frac{1}{2}(\mathbf{D}_C^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(2)}) + \mathbf{D}^{(6)},\\
\mathbf{F} &= \frac{1}{2}(\mathbf{D}_C^{(3)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(3)}) + \mathbf{D}^{(7)},\\
\mathbf{C} &= \frac{1}{2}(\mathbf{D}^{(0)}\mathbf{T}^{(2)} + \mathbf{T}^{(2)}\mathbf{D}^{(0)}) + \frac{1}{2}(\mathbf{D}_C^{(4)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(4)}) + \mathbf{D}^{(8)},\\
\mathbf{D}^{(0)} &= \mathbf{D}_C^{(0)} = \mathbf{D}_G^{(0)},\\
\mathbf{D}^{(i)} &= \frac{1}{2}(\mathbf{D}_C^{(i)} + \mathbf{D}_G^{(i)}), \quad i = 5, 6, 7, 8.
\end{aligned} \tag{3.12}$$

Evidently, each block of the matrix $\mathbf{A}$ is more symmetrically or skew-symmetrically structured than either $\mathbf{A}_C$ or $\mathbf{A}_G$ in (3.4) and (3.9), respectively.

Let the functions $\rho_i(x)$, $\phi_i(x)$ and $\omega_i(x)$ $(i = 1, 2)$ satisfy $\rho_1(x) = \rho_2(x) \equiv 1$, $\phi_1(x) = \phi_2(x) \equiv \phi(x)$ and $\omega_1(x) = \omega_2(x) \equiv 1/\phi^{'}(x)$. Then we can obtain the actual expressions for the matrix $\mathbf{A}$ corresponding to the two special cases described at the end of Section 2.

Case (i) : For $\mu_1(x) > 0, \forall x \in [a, b]$, $p(x) = \mu_1(x)$, and $q(x) = -\mu_0(x)$, it holds that

$$g_C^{(0)} = g_G^{(0)} = \mu_1, \quad g_C^{(1)} = g_G^{(1)} = -h\left(\frac{1}{\phi'}\right)', \quad g_C^{(2)} = g_G^{(2)} = -h\frac{\mu_1}{\phi'}, \quad g_C^{(3)} = g_G^{(3)} = h\frac{\mu_1}{\phi'},$$

[1]A complex system of linear equations is called positive-definite if the Hermitian part of its coefficient matrix is positive-definite; see, e.g., [5, 4].

and

$$g_C^{(4)} = -h\left(\mu_1\left(\frac{1}{\phi'}\right)' - \frac{\nu_1}{\phi'}\right), \qquad g_G^{(4)} = -h\left(2\frac{\mu_1'}{\phi'} + \mu_1\left(\frac{1}{\phi'}\right)' - \frac{\nu_1}{\phi'}\right),$$

$$g_C^{(5)} = 0, \qquad g_G^{(5)} = -h^2\frac{1}{\phi'}\left(\frac{1}{\phi'}\right)'',$$

$$g_C^{(6)} = -h^2\frac{\mu_0}{(\phi')^2}, \qquad g_G^{(6)} = -h^2\left(\frac{1}{\phi'}\left(\frac{\mu_1}{\phi'}\right)' + \frac{\mu_0}{(\phi')^2}\right),$$

$$g_C^{(7)} = -h^2\frac{\mu_0}{(\phi')^2}, \qquad g_G^{(7)} = h^2\left(\frac{1}{\phi'}\left(\frac{\mu_1}{\phi'}\right)' - \frac{\mu_0}{(\phi')^2}\right),$$

$$g_C^{(8)} = -h^2\frac{\nu_0}{(\phi')^2}, \qquad g_G^{(8)} = -h^2\left(\frac{1}{\phi'}\left(\frac{\mu_1}{\phi'}\right)'' - \frac{1}{\phi'}\left(\frac{\nu_1}{\phi'}\right)' + \frac{\nu_0}{(\phi')^2}\right).$$

It then follows that

$$\mathbf{D}_C^{(1)} = \mathbf{D}_G^{(1)} := \mathbf{D}^{(1)}, \quad \mathbf{D}_C^{(2)} = \mathbf{D}_G^{(2)} = -\mathbf{D}_C^{(3)} = -\mathbf{D}_G^{(3)} := \mathbf{D}^{(2)} \quad \text{and} \quad \mathbf{D}_C^{(6)} = \mathbf{D}_C^{(7)}.$$

Hence, the matrix $\mathbf{A}$ is of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{T}^{(2)} + \frac{1}{2}(\mathbf{D}^{(1)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(1)}) + \mathbf{D}^{(5)} & \frac{1}{2}(\mathbf{D}^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(2)}) + \mathbf{D}^{(6)} \\ -\frac{1}{2}(\mathbf{D}^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(2)}) + \mathbf{D}^{(7)} & \mathbf{C} \end{bmatrix},$$

with $\mathbf{C}$ being defined as in (3.12). Since the elements of the diagonal matrices $\mathbf{D}^{(5)}$, $\mathbf{D}^{(6)}$, $\mathbf{D}^{(7)}$ and $\mathbf{D}^{(8)}$ are all of the order $\mathcal{O}(h^2)$, we see that the off-diagonal blocks of $\mathbf{A}$ are almost symmetric. If, in particular, $\mathbf{D}_C^{(4)} = \mathbf{D}_G^{(4)}$, then the symmetric part of $\mathbf{A}$ is given by

$$\begin{aligned} \mathcal{H}(\mathbf{A}) &= \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \\ &= \begin{bmatrix} \mathbf{T}^{(2)} + \mathbf{D}^{(5)} & \frac{1}{2}(\mathbf{D}^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(2)}) + \frac{1}{2}(\mathbf{D}^{(6)} + \mathbf{D}^{(7)}) \\ -\frac{1}{2}(\mathbf{D}^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(2)}) + \frac{1}{2}(\mathbf{D}^{(6)} + \mathbf{D}^{(7)}) & \frac{1}{2}(\mathbf{D}^{(0)}\mathbf{T}^{(2)} + \mathbf{T}^{(2)}\mathbf{D}^{(0)}) + \mathbf{D}^{(8)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{T}^{(2)} & \frac{1}{2}(\mathbf{D}^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(2)}) \\ -\frac{1}{2}(\mathbf{D}^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(2)}) & \frac{1}{2}(\mathbf{D}^{(0)}\mathbf{T}^{(2)} + \mathbf{T}^{(2)}\mathbf{D}^{(0)}) \end{bmatrix} + \mathcal{O}(h^2). \end{aligned}$$

Case (ii) : For $\mu_1(x) < 0, \forall x \in [a,b]$, $p(x) = -\mu_1(x)$, and $q(x) = \mu_0(x)$, it holds that

$$g_C^{(0)} = g_G^{(0)} = -\mu_1, \quad g_C^{(1)} = g_G^{(1)} = -h\left(\frac{1}{\phi'}\right)', \quad g_C^{(2)} = g_G^{(2)} = h\frac{\mu_1}{\phi'}, \quad g_C^{(3)} = g_G^{(3)} = h\frac{\mu_1}{\phi'},$$

and

$$g_C^{(4)} = h\left(\mu_1\left(\frac{1}{\phi'}\right)' + \frac{\nu_1}{\phi'}\right), \qquad g_G^{(4)} = h\left(2\frac{\mu_1'}{\phi'} + \mu_1\left(\frac{1}{\phi'}\right)' + \frac{\nu_1}{\phi'}\right),$$

$$g_C^{(5)} = 0, \qquad g_G^{(5)} = -h^2\frac{1}{\phi'}\left(\frac{1}{\phi'}\right)'',$$

$$g_C^{(6)} = h^2\frac{\mu_0}{(\phi')^2}, \qquad g_G^{(6)} = h^2\left(\frac{1}{\phi'}\left(\frac{\mu_1}{\phi'}\right)' + \frac{\mu_0}{(\phi')^2}\right),$$

$$g_C^{(7)} = -h^2\frac{\mu_0}{(\phi')^2}, \qquad g_G^{(7)} = h^2\left(\frac{1}{\phi'}\left(\frac{\mu_1}{\phi'}\right)' - \frac{\mu_0}{(\phi')^2}\right),$$

$$g_C^{(8)} = -h^2\frac{\nu_0}{(\phi')^2}, \qquad g_G^{(8)} = h^2\left(\frac{1}{\phi'}\left(\frac{\mu_1}{\phi'}\right)'' + \frac{1}{\phi'}\left(\frac{\nu_1}{\phi'}\right)' - \frac{\nu_0}{(\phi')^2}\right).$$

It then follows that

$$\mathbf{D}_C^{(1)} = \mathbf{D}_G^{(1)} := \mathbf{D}^{(1)}, \quad \mathbf{D}_C^{(2)} = \mathbf{D}_G^{(2)} = \mathbf{D}_C^{(3)} = \mathbf{D}_G^{(3)} := \mathbf{D}^{(2)} \quad \text{and} \quad \mathbf{D}_C^{(6)} = -\mathbf{D}_C^{(7)}.$$

Hence, $\mathbf{A}$ is of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{T}^{(2)} + \frac{1}{2}(\mathbf{D}^{(1)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(1)}) + \mathbf{D}^{(5)} & \frac{1}{2}(\mathbf{D}^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(2)}) + \mathbf{D}^{(6)} \\ \frac{1}{2}(\mathbf{D}^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(2)}) + \mathbf{D}^{(7)} & \mathbf{C} \end{bmatrix},$$

with $\mathbf{C}$ being defined as in (3.12). Again, since the elements of the diagonal matrices $\mathbf{D}^{(5)}$, $\mathbf{D}^{(6)}$, $\mathbf{D}^{(7)}$ and $\mathbf{D}^{(8)}$ are all of the order $\mathcal{O}(h^2)$, the off-diagonal blocks of $\mathbf{A}$ are almost skew-symmetric. If, in particular, $\mathbf{D}_C^{(4)} = \mathbf{D}_G^{(4)}$, then the symmetric part of $\mathbf{A}$ is given by

$$\begin{aligned} \mathcal{H}(\mathbf{A}) &= \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \\ &= \begin{bmatrix} \mathbf{T}^{(2)} + \mathbf{D}^{(5)} & \frac{1}{2}(\mathbf{D}^{(6)} + \mathbf{D}^{(7)}) \\ \frac{1}{2}(\mathbf{D}^{(6)} + \mathbf{D}^{(7)}) & \frac{1}{2}(\mathbf{D}^{(0)}\mathbf{T}^{(2)} + \mathbf{T}^{(2)}\mathbf{D}^{(0)}) + \mathbf{D}^{(8)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{T}^{(2)} & \mathbf{O} \\ \mathbf{O} & \frac{1}{2}(\mathbf{D}^{(0)}\mathbf{T}^{(2)} + \mathbf{T}^{(2)}\mathbf{D}^{(0)}) \end{bmatrix} + \mathcal{O}(h^2), \end{aligned}$$

where $\mathbf{O}$ denotes the zero matrix of suitable dimension.

We remark that in **Cases (i)** and **(ii)** if $\mathbf{D}^{(5)}$ is positive semidefinite, the symmetric part of $\mathbf{B}$ is positive definite. Thereby, $\mathbf{B}$ is positive definite and nonsingular. Moreover, if $\mathbf{D}^{(8)}$ is positive semidefinite, the symmetric part of $\mathbf{C}$ is positive definite. Thereby, $\mathbf{C}$ is positive definite and nonsingular, too. Evidently, when $\mathbf{B}$ is nonsingular, the matrix $\mathbf{A}$ is nonsingular if and only if its Schur complement $\mathbf{S} = \mathbf{C} - \mathbf{F}\mathbf{B}^{-1}\mathbf{E}$ is nonsingular; see [1, 2].

4 Convergence Analysis

In this section, we show that the approximate solution $y_N(x)$ given in (3.2) converges exponentially to the true solution $y(x)$ of (2.4) as N tends to infinity. To this end, similar to the treatments for the third-order and the fourth-order ODEs in [3, 29], firstly we estimate $\|\mathbf{A}\tilde{\mathbf{y}} - \mathbf{p}\|_2$, where $\tilde{\mathbf{y}}$ is a $2n$-dimensional real vector defined by

$$\tilde{\mathbf{y}} = (\tilde{y}_1(x_{-N}), \ldots, \tilde{y}_1(x_N), \tilde{y}_2(x_{-N+1}), \ldots, \tilde{y}_2(x_N))^T, \tag{4.1}$$

with $\tilde{y}_i(x) := y_i(x)/\rho_i(x)$ $(i = 1, 2)$, then we derive an upper bound for $\|\mathbf{A}^{-1}\|_2$, and finally we prove the boundedness of the error $\|y(x) - y_N(x)\|_\infty$. Here and in the following, we denote by $\|\cdot\|_2$ the Euclidean vector or matrix norm, and for $f(x) = (f_1(x), f_2(x))^T$, with $f_i(x)$ $(i = 1, 2)$ defined in $\mathcal{D}$, we define the norm $\|f(x)\|_\infty$ as

$$\|f(x)\|_\infty = \max\left\{\sup_{x\in\mathcal{D}} |f_1(x)|, \quad \sup_{x\in\mathcal{D}} |f_2(x)|\right\}.$$

We introduce two functional spaces $\mathbb{L}_\alpha(\mathcal{D})$ and $\mathbb{H}^\infty(\mathcal{D})$: $\mathbb{L}_\alpha(\mathcal{D})$ is the set of all analytic functions F in $\mathcal{D}$ such that

$$|F(z)| \le \frac{c|e^{\phi(z)}|^\alpha}{(1 + |e^{\phi(z)}|)^{2\alpha}}, \quad \forall z \in \mathcal{D},$$

where c and α are positive constants, and $\phi : \mathcal{D} \to \mathcal{D}_d$ is a conformal mapping; and $\mathbb{H}^\infty(\mathcal{D})$ is the space of all analytic functions in $\mathcal{D}$ equipped with the maximum norm.

The following Lemma gives an upper bound for $\|\mathbf{A}\tilde{\mathbf{y}} - \mathbf{p}\|_2$.

Lemma 4.1 *Assume that the second-order ODE system (2.4) has a unique solution $y(x)$. Let $\tilde{y}(x) = (\tilde{y}_1(x), \tilde{y}_2(x))^T \in \mathbb{L}_\alpha(\mathcal{D})$ with $\tilde{y}_i(x) := y_i(x)/\rho_i(x)$ $(i = 1, 2)$, and $\sigma/[(\phi_2')^2\rho_2] \in \mathbb{L}_\alpha(\mathcal{D})$. Let $\mathbf{A}_C$, $\mathbf{A}_G$, $\mathbf{A}$, $\tilde{\mathbf{y}}$ and $\mathbf{p}$ be defined as in (3.4), (3.9), (3.11), (4.1) and (3.5), respectively.*

(i) *If $\rho_1'/[\phi_1'\rho_1]$, $p\rho_2\phi_2'/[(\phi_1')^2\rho_1]$, $\mu_1\rho_1\phi_1'/[(\phi_2')^2\rho_2]$, $p\rho_2'/[\phi_2'\rho_2]$, $(1/\phi_1')'$, $(1/\phi_2')'$ and ν_1/ϕ_2' belong to $\mathbb{H}^\infty(\mathcal{D})$, then there exists a constant c_1, independent of N, such that*

$$\|\mathbf{A}_C\tilde{\mathbf{y}} - \mathbf{p}\|_2 \le c_1 N^{1/2} e^{-(\pi d\alpha N)^{1/2}}.$$

(ii) *If $\omega_1'/[\omega_1\phi_1']$, $p\rho_2\omega_2/[\rho_1\omega_1\phi_1']$, $\mu_1\rho_1\omega_1/[\rho_2\omega_2\phi_2']$, $(\mu_1\omega_2)'/[\omega_2\phi_2']$, $(1/\phi_1')'$, $(1/\phi_2')'p$ and ν_1/ϕ_2' belong to $\mathbb{H}^\infty(\mathcal{D})$, then there exists a constant c_1', independent of N, such that*

$$\|\mathbf{A}_G\tilde{\mathbf{y}} - \mathbf{p}\|_2 \le c_1' N^{1/2} e^{-(\pi d\alpha N)^{1/2}}.$$

It follows immediately from both (i) and (ii) that

$$\|\mathbf{A}\tilde{\mathbf{y}} - \mathbf{p}\|_2 \le \frac{1}{2}(c_1 + c_1')N^{1/2} e^{-(\pi d\alpha N)^{1/2}}. \tag{4.2}$$

Proof. We put the lengthy proof in Appendix. □

We now derive an upper bound for $\|\mathbf{A}^{-1}\|_2$.

Lemma 4.2 *Let $\mathbf{A}$ be defined as in (3.11). Assume that $\mathbf{D}^{(5)}$ and $\mathbf{D}^{(8)}$ are positive semidefinite matrices, $\mathbf{D}_C^{(1)} = \mathbf{D}_G^{(1)}$, $\mathbf{D}_C^{(4)} = \mathbf{D}_G^{(4)}$, and $p(x)$ is a positive constant with $p(x) \equiv d^{(0)}$. Then there exists a constant γ_0, with $0 < \gamma_0 < 1$, such that*

$$\|\mathbf{A}^{-1}\|_2 \le \max\left\{1, \frac{1}{d^{(0)}}\right\} \frac{\tau(N)}{4\sqrt{1-\gamma_0}}, \tag{4.3}$$

where $\tau(N) = 1/\sin^2\left(\frac{\pi}{4(N+1)}\right)$*. Note that there exists a constant* c_2 *such that*

$$\tau(N) \approx \frac{16N^2}{\pi^2}(1 + c_2 N^{-1}) = \mathcal{O}(N^2)$$

holds for a sufficiently large N*.*

Proof. Denote by $\delta_1(\cdot)$ the smallest singular value of the corresponding matrix. Then from [22] and the assumptions we have

$$\begin{aligned}\delta_1(\mathbf{B}) &\geq \min_{1\leq i\leq n}\left|\lambda_i\left(\frac{\mathbf{B}+\mathbf{B}^T}{2}\right)\right| = \min_{1\leq i\leq n}\left|\lambda_i(\mathbf{T}^{(2)} + \mathbf{D}^{(5)})\right| \\ &\geq \min_{1\leq i\leq n}\left|\lambda_i(\mathbf{T}^{(2)})\right| \geq 4\sin^2\left(\frac{\pi}{4(N+1)}\right) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\delta_1(\mathbf{C}) &\geq \min_{1\leq i\leq n}\left|\lambda_i\left(\frac{\mathbf{C}+\mathbf{C}^T}{2}\right)\right| \\ &= \min_{1\leq i\leq n}\left|\lambda_i\left(\frac{1}{2}(\mathbf{D}^{(0)}\mathbf{T}^{(2)} + \mathbf{T}^{(2)}\mathbf{D}^{(0)}) + \mathbf{D}^{(8)}\right)\right| \\ &\geq \frac{1}{2}\min_{1\leq i\leq n}\left|\lambda_i(\mathbf{D}^{(0)}\mathbf{T}^{(2)} + \mathbf{T}^{(2)}\mathbf{D}^{(0)})\right| \\ &= d^{(0)}\min_{1\leq i\leq n}\left|\lambda_i(\mathbf{T}^{(2)})\right| \\ &\geq 4d^{(0)}\sin^2\left(\frac{\pi}{4(N+1)}\right). \end{aligned} \tag{4.5}$$

Based on (4.4), (4.5) and Lemma A.1, we immediately obtain the estimate in (4.3). □

Now we are ready to derive a bound for the error function $y(x) - y_N(x)$.

Theorem 4.1 *Let* $y(x)$ *be the exact solution of the second-order ODE system (2.4), and* $y_N(x)$ *be its sinc approximation given in (3.2). Under the assumptions of Lemmas 4.1 and 4.2, there exists a constant* c*, independent of* N*, such that*

$$\|y(x) - y_N(x)\|_\infty \leq cN^{5/2}e^{-(\pi d\alpha N)^{1/2}} \tag{4.6}$$

holds for a sufficiently large N*.*

Proof. Define the functions

$$\zeta_{iN}(x) := \rho_i(x)\sum_{j=-N}^{N}\frac{y_i(x_j)}{\rho_i(x_j)}S(j,h)\circ\phi_i(x), \quad i = 1, 2,$$

and denote by $\zeta_N(x) = (\zeta_{1N}(x), \zeta_{2N}(x))^T$. Then the function $\zeta_N(x)$ can be expressed as

$$\zeta_N(x) = \rho(x)\odot\sum_{j=-N}^{N}\tilde{y}(x_j)\odot S(j,h)\circ\phi(x),$$

where $\tilde{y}(x_j) = (\tilde{y}_1(x_j), \tilde{y}_2(x_j))^T$ with $\tilde{y}_i(x_j) := y_i(x_j)/\rho_i(x_j)$ $(i = 1, 2)$. By making use of the triangular inequality, we have

$$\|y(x) - y_N(x)\|_\infty \le \|y(x) - \zeta_N(x)\|_\infty + \|\zeta_N(x) - y_N(x)\|_\infty. \tag{4.7}$$

Since $\tilde{y}(x) \in \mathbb{L}_\alpha(\mathcal{D})$, from [31] we know that there exists a $c_3 > 0$, independent of N, such that

$$\|y(x) - \zeta_N(x)\|_\infty \le c_3 N^{1/2} e^{-(\pi d\alpha N)^{1/2}}. \tag{4.8}$$

The second term on the right-hand side of (4.7) satisfies

$$\begin{aligned}
\|\zeta_N(x) - y_N(x)\|_\infty &= \left\| \sum_{j=-N}^{N} [\tilde{y}(x_j) - \varpi_j] \odot [S(j,h) \circ \phi(x)] \odot \rho(x) \right\|_\infty \\
&\le \sum_{j=-N}^{N} \|\tilde{y}(x_j) - \varpi_j\|_\infty \cdot \|[S(j,h) \circ \phi(x)] \odot \rho(x)\|_\infty \\
&\le \left(\sum_{j=-N}^{N} \|\tilde{y}(x_j) - \varpi_j\|_\infty^2 \right)^{1/2} \left(\sum_{j=-N}^{N} \|[S(j,h) \circ \phi(x)] \odot \rho(x)\|_\infty^2 \right)^{1/2},
\end{aligned}$$

where $\varpi_j = (u_j, v_j)^T$. Because $\rho(x)$ is a bounded function and $x \in \phi^{-1}((-\infty, \infty))$, the summation $\sum_{j=-\infty}^{\infty} \|[S(j,h) \circ \phi(x)] \odot \rho(x)\|_\infty^2$ is bounded by a constant, say, $(c_3')^2$. Hence, we get

$$\|\zeta_N(x) - y_N(x)\|_\infty \le c_3' \left(\sum_{j=-N}^{N} \|\tilde{y}(x_j) - \varpi_j\|_\infty^2 \right)^{1/2} \le c_3' \|\tilde{\mathbf{y}} - \mathbf{w}\|_2,$$

with $\mathbf{w}$, defined in (3.5), being the exact solution of (3.10). By (4.2) and (4.3), we can obtain

$$\|\tilde{\mathbf{y}} - \mathbf{w}\|_2 = \|\mathbf{A}^{-1}(\mathbf{A}\tilde{\mathbf{y}} - \mathbf{p})\|_2 \le \|\mathbf{A}^{-1}\|_2 \|\mathbf{A}\tilde{\mathbf{y}} - \mathbf{p}\|_2 \le c_3'' N^{5/2} e^{-(\pi d\alpha N)^{1/2}}, \tag{4.9}$$

where c_3'' is a constant independent of N. Now the estimate (4.6) follows immediately by substituting (4.8) and (4.9) into (4.7). □

5 Block-Diagonal Preconditioning

In this section, we discuss how to construct an efficient preconditioner $\mathbf{P}$ for the coefficient matrix $\mathbf{A}$ defined in (3.11), so that the convergence speeds of the Krylov subspace methods such as GMRES for solving the system of linear equations (3.10) can be further accelerated. To this end, we propose a block-diagonal preconditioner $\mathbf{P}$ based on the special structure and actual properties of the matrix $\mathbf{A}$. Let $\widehat{\mathbf{A}}$ be the block-diagonal part of $\mathbf{A}$. Then we demonstrate that $\|\widehat{\mathbf{A}} - \mathbf{A}\|_2 \le \mathcal{O}(h)$. Moreover, we prove that the eigenvalues of the matrix $\mathbf{P}^{-1}\widehat{\mathbf{A}}$ are uniformly bounded within a rectangle on the complex plane independent of the size of the linear system.

5.1 Construction of the Preconditioners

As the block two-by-two matrix $\mathbf{A}$ is dominated by its diagonal blocks and each of its blocks is a combination of Toeplitz and diagonal matrices, following the approach in [24, 27] we construct the block-diagonal preconditioning matrix

$$\mathbf{P} = \begin{bmatrix} \widetilde{\mathbf{B}} & \mathbf{O} \\ \mathbf{O} & \widetilde{\mathbf{C}} \end{bmatrix}, \tag{5.1}$$

where

$$\widetilde{\mathbf{B}} = \mathbf{B}^{(2)} + \frac{1}{2}(\mathbf{D}_C^{(1)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}_G^{(1)}) + \mathbf{D}^{(5)}$$

and

$$\widetilde{\mathbf{C}} = \frac{1}{2}(\mathbf{D}^{(0)}\mathbf{B}^{(2)} + \mathbf{B}^{(2)}\mathbf{D}^{(0)}) + \frac{1}{2}(\mathbf{D}_C^{(4)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}_G^{(4)}) + \mathbf{D}^{(8)},$$

with

$$\mathbf{B}^{(1)} = \text{tridiag}\left[\frac{1}{2}, 0, -\frac{1}{2}\right] \quad \text{and} \quad \mathbf{B}^{(2)} = \text{tridiag}[-1, 2, -1]$$

being tridiagonal matrices approximating the Toeplitz matrices $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$, respectively. The preconditioner $\mathbf{P}$ is a tridiagonal matrix and, hence, the linear system with respect to it can be solved fast and economically.

Hereafter in this section, we use $\mathbf{I}$ to denote the identity matrix and focus on the special case that $\mathbf{D}^{(0)} = d^{(0)}\mathbf{I}$ with $d^{(0)} > 0$, $\mathbf{D}_C^{(1)} = \mathbf{D}_G^{(1)} := \mathbf{D}^{(1)}$ and $\mathbf{D}_C^{(4)} = \mathbf{D}_G^{(4)} := \mathbf{D}^{(4)}$. It turns out that the block-diagonal preconditioner $\mathbf{P}$ is positive definite.

Theorem 5.1 *Assume that $\mathbf{D}^{(5)}$ and $\mathbf{D}^{(8)}$ are positive semidefinite matrices. Then $\mathcal{H}(\mathbf{P})$ is symmetric positive definite. Hence, the matrix $\mathbf{P}$ is positive definite and, thus, nonsingular.*

Proof. Evidently, the symmetric and skew-symmetric parts of $\mathbf{P}$ are given by

$$\mathcal{H}(\mathbf{P}) = \frac{1}{2}(\mathbf{P} + \mathbf{P}^T) = \begin{bmatrix} \mathbf{B}^{(2)} + \mathbf{D}^{(5)} & \mathbf{O} \\ \mathbf{O} & \frac{1}{2}(\mathbf{D}^{(0)}\mathbf{B}^{(2)} + \mathbf{B}^{(2)}\mathbf{D}^{(0)}) + \mathbf{D}^{(8)} \end{bmatrix}$$

and

$$\mathcal{S}(\mathbf{P}) = \frac{1}{2}(\mathbf{P} - \mathbf{P}^T) = \begin{bmatrix} \frac{1}{2}(\mathbf{D}^{(1)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(1)}) & \mathbf{O} \\ \mathbf{O} & \frac{1}{2}(\mathbf{D}^{(4)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(4)}) \end{bmatrix},$$

respectively. Because $\mathbf{B}^{(2)}$ is symmetric positive definite, $\mathbf{D}^{(5)}$ and $\mathbf{D}^{(8)}$ are diagonal and positive semidefinite, and $\mathbf{D}^{(0)} = d^{(0)}\mathbf{I}$ with $d^{(0)} > 0$, we easily see that $\mathcal{H}(\mathbf{P})$ is positive definite. Therefore, the matrix $\mathbf{P}$ is positive definite and, thus, nonsingular. □

5.2 Analysis of the Preconditioned Matrix

Denote by

$$\widehat{\mathbf{A}} = \begin{bmatrix} \mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{C} \end{bmatrix} \tag{5.2}$$

the block-diagonal part of $\mathbf{A}$. Then we can prove the following result.

Theorem 5.2 *Let* $\mathbf{A}$ *and* $\widehat{\mathbf{A}}$ *be defined as in (3.11) and (5.2), respectively. Then*

$$\|\widehat{\mathbf{A}} - \mathbf{A}\|_2 \leq \mathcal{O}(h).$$

Proof. By straightforward computations we have

$$\|\widehat{\mathbf{A}} - \mathbf{A}\|_2 = \left\| \begin{bmatrix} \mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{C} \end{bmatrix} - \begin{bmatrix} \mathbf{B} & \mathbf{E} \\ \mathbf{F} & \mathbf{C} \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} \mathbf{O} & \mathbf{E} \\ \mathbf{F} & \mathbf{O} \end{bmatrix} \right\|_2 \leq \max\left\{\|\mathbf{E}\|_2,\, \|\mathbf{F}\|_2\right\}.$$

Because

$$\|\mathbf{E}\|_2 = \left\| \frac{1}{2}(\mathbf{D}_C^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(2)}) + \mathbf{D}^{(6)} \right\|_2 \leq \left\| \frac{1}{2}(\mathbf{D}_C^{(2)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(2)}) \right\|_2 + \|\mathbf{D}^{(6)}\|_2 = \mathcal{O}(h)$$

and

$$\|\mathbf{F}\|_2 = \left\| \frac{1}{2}(\mathbf{D}_C^{(3)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(3)}) + \mathbf{D}^{(7)} \right\|_2 \leq \left\| \frac{1}{2}(\mathbf{D}_C^{(3)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}_G^{(3)}) \right\|_2 + \|\mathbf{D}^{(7)}\|_2 = \mathcal{O}(h),$$

we immediately obtain the result. □

Theorem 5.2 clearly shows that the matrix $\widehat{\mathbf{A}}$ accurately approximates $\mathbf{A}$ when the step-size h is small. Under the assumption that both $\mathbf{D}^{(5)}$ and $\mathbf{D}^{(8)}$ are positive semidefinite matrices, analogously to Theorem 5.1 we can demonstrate that the matrix $\widehat{\mathbf{A}}$ is positive definite and, thus, nonsingular, too.

The generalized Bendixson theorem, established in [8], is an essential tool for deriving a rectangular domain that bounds the eigenvalues of the preconditioned matrix $\mathbf{P}^{-1}\widehat{\mathbf{A}}$. Based on Lemma 4.7 in [3], we can readily obtain a rectangular domain that bounds the eigenvalues of the preconditioned matrix $\mathbf{P}^{-1}\widehat{\mathbf{A}}$.

Theorem 5.3 *Let* $\mathbf{P}$ *and* $\widehat{\mathbf{A}}$ *be defined in (5.1) and (5.2), respectively. Let the diagonal matrices* $\mathbf{D}^{(5)}$ *and* $\mathbf{D}^{(8)}$ *be positive definite, and denote by*

$$\eta = \max\left\{ \frac{d^{(1)}\pi}{\sqrt{d^{(5)}(\pi^2 + d^{(5)})}}, \quad \frac{d^{(4)}\pi}{\sqrt{d^{(8)}(d^{(0)}\pi^2 + d^{(8)})}} \right\}$$

and

$$\xi = \max\left\{ \frac{d^{(1)}(\sqrt{4 + d^{(5)}} - \sqrt{d^{(5)}})}{\sqrt{d^{(5)}}}, \quad \frac{d^{(4)}(\sqrt{4d^{(0)} + d^{(8)}} - \sqrt{d^{(8)}})}{d^{(0)}\sqrt{d^{(8)}}} \right\},$$

with

$$d^{(1)} = \max_{1\leq\ell\leq n}\{|[\mathbf{D}^{(1)}]_{\ell\ell}|\}, \quad d^{(4)} = \max_{1\leq\ell\leq n}\{|[\mathbf{D}^{(4)}]_{\ell\ell}|\}$$

and

$$d^{(5)} = \min_{1\leq\ell\leq n}\{|[\mathbf{D}^{(5)}]_{\ell\ell}|\}, \quad d^{(8)} = \min_{1\leq\ell\leq n}\{|[\mathbf{D}^{(8)}]_{\ell\ell}|\}.$$

Then it holds that

$$\begin{cases} \dfrac{(1-\eta\xi)}{1+\xi^2} \leq \mathrm{Re}(\lambda(\mathbf{P}^{-1}\widehat{\mathbf{A}})) \leq \dfrac{\pi^2(1+\eta\xi)}{4}, \quad \text{for } \eta\xi < 1, \\ -\dfrac{\pi^2(\eta+\xi)}{4} \leq \mathrm{Im}(\lambda(\mathbf{P}^{-1}\widehat{\mathbf{A}})) \leq \dfrac{\pi^2(\eta+\xi)}{4}. \end{cases}$$

Proof. The symmetric and skew-symmetric parts of the matrices $\widehat{\mathbf{A}}$ and $\mathbf{P}$ are given by

$$\mathcal{H}(\widehat{\mathbf{A}}) = \frac{1}{2}(\widehat{\mathbf{A}} + \widehat{\mathbf{A}}^T) = \begin{bmatrix} \mathbf{T}^{(2)} + \mathbf{D}^{(5)} & \mathbf{O} \\ \mathbf{O} & d^{(0)}\mathbf{T}^{(2)} + \mathbf{D}^{(8)} \end{bmatrix},$$

$$\mathcal{S}(\widehat{\mathbf{A}}) = \frac{1}{2}(\widehat{\mathbf{A}} - \widehat{\mathbf{A}}^T) = \begin{bmatrix} \frac{1}{2}(\mathbf{D}^{(1)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(1)}) & \mathbf{O} \\ \mathbf{O} & \frac{1}{2}(\mathbf{D}^{(4)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(4)}) \end{bmatrix}$$

and

$$\mathcal{H}(\mathbf{P}) = \frac{1}{2}(\mathbf{P} + \mathbf{P}^T) = \begin{bmatrix} \mathbf{B}^{(2)} + \mathbf{D}^{(5)} & \mathbf{O} \\ \mathbf{O} & d^{(0)}\mathbf{B}^{(2)} + \mathbf{D}^{(8)} \end{bmatrix},$$

$$\mathcal{S}(\mathbf{P}) = \frac{1}{2}(\mathbf{P} - \mathbf{P}^T) = \begin{bmatrix} \frac{1}{2}(\mathbf{D}^{(1)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(1)}) & \mathbf{O} \\ \mathbf{O} & \frac{1}{2}(\mathbf{D}^{(4)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(4)}) \end{bmatrix}.$$

From [3, Lemma 4.7], for all $\mathbf{z} = (\mathbf{u}^T, \mathbf{v}^T)^T \in \mathbb{C}^{2n} \backslash \{0\}$, we have

$$\begin{aligned} h(\mathbf{z}) := \frac{\mathbf{z}^T \mathcal{H}(\widehat{\mathbf{A}})\mathbf{z}}{\mathbf{z}^T \mathcal{H}(\mathbf{P})\mathbf{z}} &= \frac{\mathbf{u}^T(\mathbf{T}^{(2)} + \mathbf{D}^{(5)})\mathbf{u} + \mathbf{v}^T(d^{(0)}\mathbf{T}^{(2)} + \mathbf{D}^{(8)})\mathbf{v}}{\mathbf{u}^T(\mathbf{B}^{(2)} + \mathbf{D}^{(5)})\mathbf{u} + \mathbf{v}^T(d^{(0)}\mathbf{B}^{(2)} + \mathbf{D}^{(8)})\mathbf{v}} \\ &\le \max\left\{ \frac{\mathbf{u}^T(\mathbf{T}^{(2)} + \mathbf{D}^{(5)})\mathbf{u}}{\mathbf{u}^T(\mathbf{B}^{(2)} + \mathbf{D}^{(5)})\mathbf{u}}, \quad \frac{\mathbf{v}^T(d^{(0)}\mathbf{T}^{(2)} + \mathbf{D}^{(5)})\mathbf{v}}{\mathbf{v}^T(d^{(0)}\mathbf{B}^{(2)} + \mathbf{D}^{(5)})\mathbf{v}} \right\} < \frac{\pi^2}{4}. \end{aligned} \tag{5.3}$$

Similarly, we can obtain

$$\frac{\mathbf{z}^T \mathcal{H}(\widehat{\mathbf{A}})\mathbf{z}}{\mathbf{z}^T \mathcal{H}(\mathbf{P})\mathbf{z}} > 1. \tag{5.4}$$

Again, by making use of [3, Lemma 4.7], with straightforward computations we have

$$\begin{aligned} f_{\widehat{\mathbf{A}}}(\mathbf{z}) := \left| \frac{1}{\imath} \frac{\mathbf{z}^T \mathcal{S}(\widehat{\mathbf{A}})\mathbf{z}}{\mathbf{z}^T \mathcal{H}(\widehat{\mathbf{A}})\mathbf{z}} \right| &= \left| \frac{\frac{1}{2}\mathbf{u}^T(\mathbf{D}^{(1)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(1)})\mathbf{u} + \frac{1}{2}\mathbf{v}^T(\mathbf{D}^{(4)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(4)})\mathbf{v}}{\mathbf{u}^T(\mathbf{T}^{(2)} + \mathbf{D}^{(5)})\mathbf{u} + \mathbf{v}^T(d^{(0)}\mathbf{T}^{(2)} + \mathbf{D}^{(8)})\mathbf{v}} \right| \\ &\le \frac{1}{2} \frac{\left|\mathbf{u}^T(\mathbf{D}^{(1)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(1)})\mathbf{u}\right| + \left|\mathbf{v}^T(\mathbf{D}^{(4)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(4)})\mathbf{v}\right|}{\mathbf{u}^T(\mathbf{T}^{(2)} + \mathbf{D}^{(5)})\mathbf{u} + \mathbf{v}^T(d^{(0)}\mathbf{T}^{(2)} + \mathbf{D}^{(8)})\mathbf{v}} \\ &\le \frac{1}{2} \max\left\{ \frac{\left|\mathbf{u}^T(\mathbf{D}^{(1)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(1)})\mathbf{u}\right|}{\mathbf{u}^T(\mathbf{T}^{(2)} + \mathbf{D}^{(5)})\mathbf{u}}, \quad \frac{\left|\mathbf{v}^T(\mathbf{D}^{(4)}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{D}^{(4)})\mathbf{v}\right|}{\mathbf{v}^T(d^{(0)}\mathbf{T}^{(2)} + \mathbf{D}^{(8)})\mathbf{v}} \right\} \\ &< \max\left\{ \frac{d^{(1)}\pi}{\sqrt{d^{(5)}(\pi^2 + d^{(5)})}}, \quad \frac{d^{(4)}\pi}{\sqrt{d^{(8)}(d^{(0)}\pi^2 + d^{(8)})}} \right\} = \eta \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} f_{\mathbf{P}}(\mathbf{z}) := \left| \frac{1}{\imath} \frac{\mathbf{z}^T \mathcal{S}(\mathbf{P})\mathbf{z}}{\mathbf{z}^T \mathcal{H}(\mathbf{P})\mathbf{z}} \right| &= \left| \frac{\frac{1}{2}\mathbf{u}^T(\mathbf{D}^{(1)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(1)})\mathbf{u} + \frac{1}{2}\mathbf{v}^T(\mathbf{D}^{(4)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(4)})\mathbf{v}}{\mathbf{u}^T(\mathbf{B}^{(2)} + \mathbf{D}^{(5)})\mathbf{u} + \mathbf{v}^T(d^{(0)}\mathbf{B}^{(2)} + \mathbf{D}^{(8)})\mathbf{v}} \right| \\ &\le \frac{1}{2} \frac{\left|\mathbf{u}^T(\mathbf{D}^{(1)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(1)})\mathbf{u}\right| + \left|\mathbf{v}^T(\mathbf{D}^{(4)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(4)})\mathbf{v}\right|}{\mathbf{u}^T(\mathbf{B}^{(2)} + \mathbf{D}^{(5)})\mathbf{u} + \mathbf{v}^T(d^{(0)}\mathbf{B}^{(2)} + \mathbf{D}^{(8)})\mathbf{v}} \\ &\le \frac{1}{2} \max\left\{ \frac{\left|\mathbf{u}^T(\mathbf{D}^{(1)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(1)})\mathbf{u}\right|}{\mathbf{u}^T(\mathbf{B}^{(2)} + \mathbf{D}^{(5)})\mathbf{u}}, \quad \frac{\left|\mathbf{v}^T(\mathbf{D}^{(4)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}^{(4)})\mathbf{v}\right|}{\mathbf{v}^T(d^{(0)}\mathbf{B}^{(2)} + \mathbf{D}^{(8)})\mathbf{v}} \right\} \\ &< \max\left\{ \frac{d^{(1)}(\sqrt{4 + d^{(5)}} - \sqrt{d^{(5)}})}{\sqrt{d^{(5)}}}, \quad \frac{d^{(4)}(\sqrt{4d^{(0)} + d^{(8)}} - \sqrt{d^{(8)}})}{d^{(0)}\sqrt{d^{(8)}}} \right\} = \xi. \end{aligned} \tag{5.6}$$

Table 5.1: Estimated and Computed eigenvalue bounds for $\mathbf{P}^{-1}\widehat{\mathbf{A}}$ of Example 6.1

N	ν	Estimated Eigenvalue Bounds	Computed Eigenvalue Bounds
8	2×10^3	$[0.5011, 3.5549]\times[-4.0313, 4.0313]$	$[1.0000, 1.0159]\times[-0.1595, 0.1595]$
16	4×10^4	$[0.5315, 3.4845]\times[-3.9288, 3.9288]$	$[1.0000, 1.0198]\times[-0.1945, 0.1945]$
32	2×10^6	$[0.4500, 3.6766]\times[-4.2128, 4.2128]$	$[1.0000, 1.0321]\times[-0.2702, 0.2702]$
64	6×10^8	$[0.5553, 3.4300]\times[-3.8433, 3.8433]$	$[1.0000, 1.0293]\times[-0.2832, 0.2832]$

By applying (5.3)–(5.6) to the generalized Bendixson theorem [8, Theorem 2.4] (see also [3, Theorem 4.3]), we immediately obtain the bounds in Theorem 5.3 for the eigenvalues of $\mathbf{P}^{-1}\widehat{\mathbf{A}}$. □

When using Theorem 5.3, we should suitably scale (2.4) and appropriately choose the conformal mapping $\phi(x)$ such that $\eta\xi < 1$, so that correct and accurate estimates about the eigenvalue bounds may be obtained. For example, if we take $\phi(x) = \nu^{-1}\ln(x/(1-x))$ in Example 6.1, with $\nu > 0$ a scaling factor, then corresponding to different mesh-sizes $h = \pi/\sqrt{2N}$ we can obtain the computed and estimated eigenvalue bounds about $\mathbf{P}^{-1}\widehat{\mathbf{A}}$ as shown in Table 5.1. Clearly, from Table 5.1 we observe that the estimated rectangles tightly contain the computed eigenvalues of $\mathbf{P}^{-1}\widehat{\mathbf{A}}$. Moreover, the rectangles bounding the computed eigenvalues are almost unchanged with respect to h.

6 Numerical Examples

In this section, we verify the feasibility of the order-reduction method, examine the accuracy of the sinc discretization, and test the effectiveness of the proposed block-diagonal preconditioner. To this end, we apply GMRES and BiCGSTAB, incorporated with the block-diagonal preconditioner $\mathbf{P}$ defined in (5.1), to the system of linear equations (3.10) obtained from the sinc discretization of the second-order ODE system (2.4).

The two examples of the ODEs used in our tests are given below.

Example 6.1 *The third-order ODE (1.1) is given by*

$$\begin{cases} y_1'''(x) - \dfrac{1}{x}y_1''(x) + y_1'(x) - \dfrac{1}{x}y_1(x) = \sigma(x), \quad 0 < x < 1, \\ y_1(0) = 0, \quad y_1(1) = 0, \quad y_1'(0) = 0, \end{cases}$$

where $\sigma(x) = 3x^3 - 4x^2 + 13x - 2/x$. *It can be transformed into the second-order ODE system (2.4) as follows:*

$$\begin{cases} y_1''(x) - y_2'(x) - \dfrac{1}{x}y_2(x) = 0, \\ y_1'(x) - \dfrac{1}{x}y_1(x) + y_2''(x) - \dfrac{2}{x^2}y_2(x) = \sigma(x), \quad 0 < x < 1, \\ y_1(0) = 0, \quad y_1(1) = 0, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \end{cases}$$

which has the exact solution $y_1(x) = x^2(1-x)^2$, $y_2(x) = 3x^3 - 4x^2 + x$.

Example 6.2 *The third-order ODE (1.1) is given by*

$$\begin{cases} y_1^{'''}(x) - y_1^{''}(x) - y_1^{'}(x) + \dfrac{1}{x} y_1(x) = \sigma(x), \quad 0 < x < 1, \\ y_1(0) = 0, \quad y_1(1) = 0, \quad y_1^{'}(0) = 0, \end{cases}$$

where $\sigma(x) = -(\pi^3 + \pi)\cos(\pi x) + (\pi^2 + 1/x)\sin(\pi x) - \pi x - 2\pi$. *It can be transformed into the second-order ODE system (2.4) as follows:*

$$\begin{cases} y_1^{''}(x) - y_2^{'}(x) - \dfrac{1}{x} y_2(x) = 0, \quad 0 < x < 1, \\ -y_1^{'}(x) + \dfrac{1}{x} y_1(x) + y_2^{''}(x) + (\dfrac{1}{x} - 1) y_2^{'}(x) - (\dfrac{1}{x} + \dfrac{1}{x^2}) y_2(x) = \sigma(x), \\ y_1(0) = 0, \quad y_1(1) = 0, \quad y_1^{'}(0) = 0 \quad \textit{and} \quad y_2(0) = 0, \end{cases}$$

which has the exact solution $y_1(x) = \sin(\pi x) + \pi(x^2 - x)$, $y_2(x) = \pi x + \pi\cos(\pi x) - \sin(\pi x)/x$.

In Examples 6.1 and 6.2, the conformal mappings are chosen as $\phi_1(x) = \phi_2(x) = \ln(x/(1-x))$, the weighting functions are chosen as $\rho_1(x) = \rho_2(x) = 1$ and $\omega_1(x) = \omega_2(x) = 1/\ln(x/(1-x))$, and the mesh-size is set to be the optimal one $h = \pi/\sqrt{2N}$. Both test examples are ODEs of homogeneous boundary values and with known solutions, so that we can easily verify the accuracy of both discrete and computed solutions.

In our tests, all codes are written in MATLAB 7.04 and run on a personal computer with 0.98G memory. In addition, the initial guess is taken to be zero and the iteration process is terminated once $\|\mathbf{A}\mathbf{w} - \mathbf{p}\|_2 \le 10^{-6} \times \|\mathbf{p}\|_2$ or once the number of iteration steps exceeds 1000.

In Table 6.1, we list the errors $E_s(h)$ between the approximated solutions $y_N(x)$ and the true solutions $y(x)$ at the sinc points. More precisely, the error $E_s(h)$ is defined as

$$\begin{aligned} E_s(h) &= \max_{-N \le j \le N} \|y(x_j) - y_N(x_j)\|_\infty \\ &\equiv \max_{-N \le j \le N} \{|y_1(x_j) - y_{1N}(x_j)|, |y_2(x_j) - y_{2N}(x_j)|\}, \end{aligned}$$

where the coefficients $\{w_j\}_{j=-N}^N$ in $y_N(x_j)$ are solved by the direct method $\mathbf{w} = \mathbf{A} \setminus \mathbf{p}$ with MATLAB. From this table we see that the error function $E_s(h)$ reduces exponentially for both examples when N is growing up.

Tables 6.2 and 6.3 list the numbers of iteration steps for solving the system of linear equations (3.10) by two different methods. In the tables, the "new method" denotes the order-reduced sinc-discretization, incorporated with the block-diagonal preconditioning, proposed in this paper; and the "method in [3]" represents the sinc-discretization directly applied to the third-order ODE (1.1), combined with the penta-diagonal preconditioning, presented in [3]. In these two tables, we use " $*$ " to indicate that the iteration method does not converge within 1000 iteration steps, "**I**" to represent the iteration method with no preconditioner, and "**P**" to denote the iteration method with either the block-diagonal preconditioner defined in (5.1) or the penta-diagonal preconditioner given in [3]. In addition, "$\mathbf{P}_{\text{iter}}$" and "$\mathbf{I}_{\text{iter}}$" stand for the numbers of iteration steps required for convergence corresponding to the preconditioning matrices **P** and **I**, respectively.

Table 6.1: Errors for Examples 6.1 and 6.2

N	$E_s(h)$	
	Example 6.1	Example 6.2
8	1.94e-03	5.99e-03
16	1.72e-04	5.38e-04
32	4.97e-06	1.56e-05
64	3.01e-08	9.46e-08
128	2.05e-11	6.45e-11
256	4.57e-15	1.24e-14

Table 6.2: Numerical Results for Example 6.1

N	New Method				Method in [3]			
	GMRES		BiCGSTAB		GMRES		BiCGSTAB	
	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$
8	31	9	23	5	17	15	34	13
16	50	9	35	5	33	20	206	18
32	82	8	61	4	65	25	$*$	27
64	92	7	97	4	129	32	$*$	44
128	125	7	129	4	257	42	$*$	75
256	168	7	182	5	513	55	$*$	117

We now discuss and analyze the numerical results in Tables 6.2 and 6.3. If no preconditioner is used, we see that the new method successfully results in an approximate solution for the second-order ODE system (2.4) within the required maximal number of iterations when both GMRES and BiCGSTAB are employed as the linear solvers, while the method in [3] only succeeds when GMRES is employed, but fails for almost all cases when BiCGSTAB is employed as the linear solver. For GMRES, the new method also requires considerably less iteration steps than the method in [3] when $N \geq 64$.

If the preconditioners are used, Tables 6.2 and 6.3 show that both methods can successfully and accurately produce approximate solutions for the second-order ODE system (2.4), and they require much less numbers of iteration steps than their counterparts without using preconditioners. Therefore, the preconditioners can greatly improve the numerical properties of both GMRES and BiCGSTAB. Evidently, the new method outperforms the method in [3] in terms of iteration steps. Moreover, when N is increasing, the iteration steps of both preconditioned GMRES and BiCGSTAB are nearly constants and even roughly decreasing for the new method, while they are growing up quickly for the method in [3]. Consequently, the new method shows h-independent convergence property, but the method in [3] does not. Hence, for both Examples 6.1 and 6.2 our order-reduction approach incorporated with the block-diagonal preconditioner can produce accurate approximation to the solution of the third-order ODE (1.1) and accelerate the convergence rates of GMRES and BiCGSTAB.

Figures 6.1–6.4 depict the distributions of the eigenvalues of the original matrix $\mathbf{A}$ and the preconditioned matrix $\mathbf{P}^{-1}\mathbf{A}$ for Examples 6.1 and 6.2. These figures clearly show that the original matrices are very ill-conditioned and, therefore, the corresponding GMRES and BiCGSTAB may converge very slowly. However, the preconditioned matrices have tightly clustered eigen-

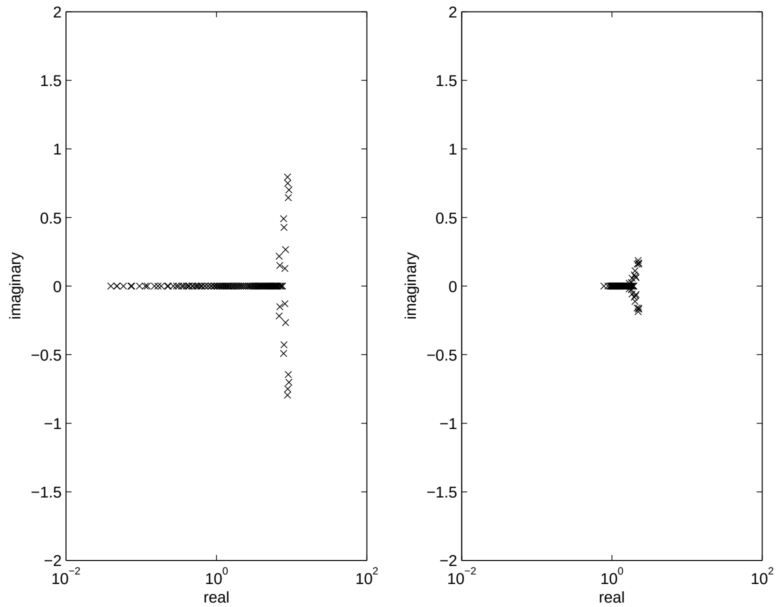

Figure 6.1: Spectra of $\mathbf{A}$ (left) and $\mathbf{P}^{-1}\mathbf{A}$ (right) for Example 6.1 with N=32.

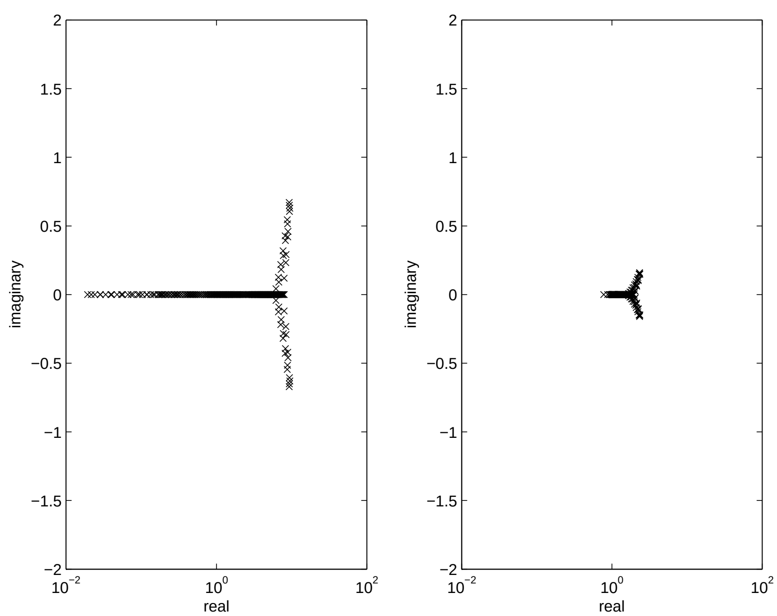

Figure 6.2: Spectra of $\mathbf{A}$ (left) and $\mathbf{P}^{-1}\mathbf{A}$ (right) for Example 6.1 with N=64.

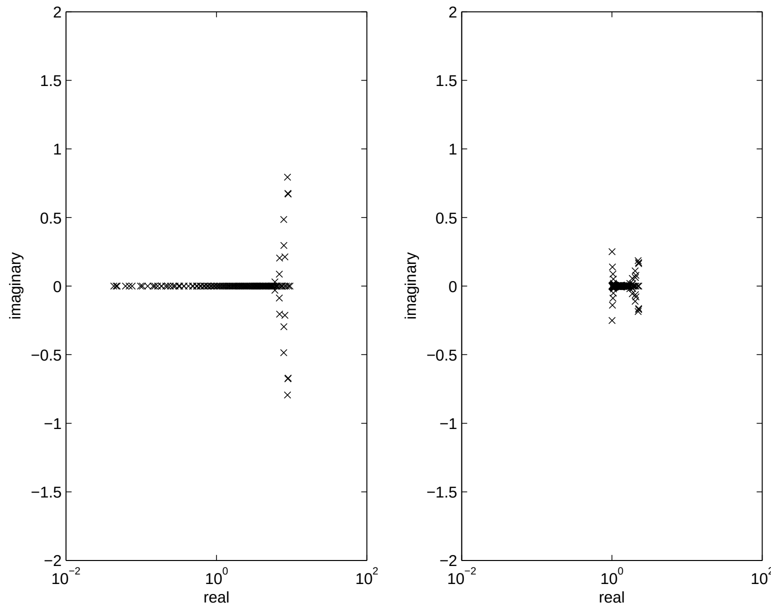

Figure 6.3: Spectra of $\mathbf{A}$ (left) and $\mathbf{P}^{-1}\mathbf{A}$ (right) for Example 6.2 with N=32.

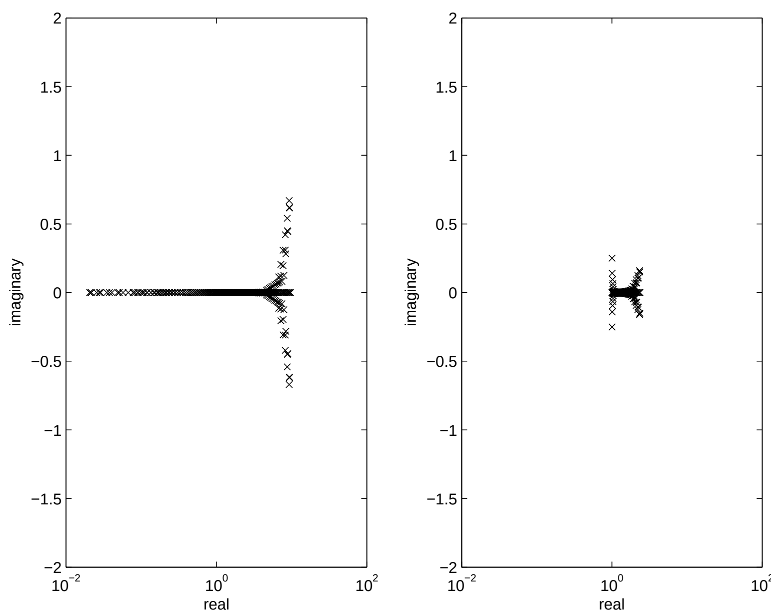

Figure 6.4: Spectra of $\mathbf{A}$ (left) and $\mathbf{P}^{-1}\mathbf{A}$ (right) for Example 6.2 with N=64.

Table 6.3: Numerical Results for Example 6.2

N	New Method				Method in [3]			
	GMRES		BiCGSTAB		GMRES		BiCGSTAB	
	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$
8	32	9	26	6	17	14	158	13
16	53	9	41	5	33	19	$*$	20
32	86	8	59	6	65	27	$*$	29
64	101	7	100	4	129	35	$*$	32
128	162	7	141	5	257	45	$*$	70
256	228	7	193	5	513	58	$*$	121

values and, thus, are well-conditioned; as a result, the corresponding preconditioned GMRES and BiCGSTAB converge very fast to the exact solution of the discretized system of (2.4).

7 Concluding Remarks

The order-reduction method transforms the third-order ODE (1.1) into the second-order ODE system (2.4). We prove that the sinc-discretization solution of the latter converges exponentially to its true solution. The coefficient matrix possesses more nice algebraic properties, such as structured, block-diagonally dominant, and positive definite, and is better conditioned than that from the direct sinc-discretization of the third-order ODE (1.1). Hence, with a suitable block-diagonal preconditioning, the system can be solved faster and more economically by the preconditioned Krylov subspace iteration methods such as GMRES and BiCGSTAB. We confirmed these advantages by both theoretical analysis and numerical experiments.

We emphasize that our order-reduction method is quite different from the common approach used frequently in the literature, where the third-order ODE (1.1) is equivalently transformed into an ODE system of a first-order and a second-order ODEs. That is to say, the reduced-order ODE system is of the form

$$\begin{cases} L_1 y(x) := y_2(x) - y_1'(x) = 0, \quad a < x < b, \\ L_2 y(x) := y_2''(x) + \mu_2(x) y_2'(x) + \mu_1(x) y_2(x) + \mu_0(x) y_1(x) = \sigma(x), \\ y_1(a) = 0, \quad y_1(b) = 0, \quad y_2(a) = 0. \end{cases}$$

Let $y(x) = (y_1(x), y_2(x))^T$ and the approximate solution $y_N(x)$ be defined by (3.2). Then, by combining both sinc-collocation and sinc-Galerkin discretizations and averaging the resulting discretized linear systems, we obtain the system $\mathbf{Aw} = \mathbf{p}$, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{T}^{(1)} + \mathbf{D}^{(1)} & \mathbf{D}^{(2)} \\ \mathbf{D}^{(3)} & \mathbf{T}^{(2)} + \frac{1}{2}(\mathbf{D}_C^{(4)} \mathbf{T}^{(1)} + \mathbf{T}^{(1)} \mathbf{D}_G^{(4)}) + \mathbf{D}^{(5)} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \tag{7.1}$$

$\mathbf{w}, \mathbf{p} \in \mathbb{R}^{2n}$ are given by (3.5), and

$$\mathbf{D}^{(i)} := \text{diag}(g^{(i)}(x_{-N}), g^{(i)}(x_{-N+1}), \ldots, g^{(i)}(x_N)), \quad i = 1, 2, \ldots, 5,$$

Table 7.1: Numerical Results about GMRES Method for Examples 6.1 and 6.2, with $\alpha = 0.01$

N	Example 6.1		Example 6.2	
	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$	$\mathbf{I}_{\text{iter}}$	$\mathbf{P}_{\text{iter}}$
8	34	16	34	16
16	66	20	66	21
32	129	25	130	25
64	251	30	253	29
128	481	39	489	37
256	945	50	945	51

with

$$g^{(1)} = h\left(\frac{\rho_2\omega_1'}{\rho_1\omega_1\phi_1'} - \frac{\rho_1'}{\rho_1\phi_1'}\right), \qquad g^{(2)} = h\left(\frac{\mu_0\rho_2}{\rho_1\phi_1'} + \frac{\rho_2\omega_2}{\rho_1\omega_1\phi_1'}\right),$$

$$g^{(3)} = -h^2\left(\frac{\mu_0\rho_1}{(\phi_2')^2\rho_2} + \frac{\mu_0\rho_1\omega_1}{\phi_1'\phi_2'\rho_2\omega_2}\right), \qquad g_C^{(4)} = h\left(\frac{\rho_2'\phi_2' + (\rho_2\phi_2')' + \mu_2\rho_2\phi_2'}{(\phi_2')^2\rho_2}\right),$$

$$g_G^{(4)} = -h\left(\frac{\omega_2'\phi_2' + (\omega_2\phi_2')' - \mu_2\omega_2\phi_2'}{(\phi_2')^2\omega_2}\right), \quad g^{(5)} = -h^2\left(\frac{\rho_2'' + \mu_2\rho_2' + \mu_1\rho_2}{(\phi_2')^2\rho_2} + \frac{\omega_2'' - (\mu_2\omega_2)' + \mu_1\omega_2}{(\phi_2')^2\omega_2}\right).$$

Because $\mathbf{T}^{(1)} + \mathbf{D}^{(1)}$ is nearly singular, the matrix $\mathbf{A}$ is almost singular, especially when the stepsize h is small. Moreover, $\mathbf{A}$ is likely indefinite and not block-diagonally dominant. Therefore, this approach can not effectively reduce the ill-conditioning in $\mathbf{A}$. Even if we precondition $\mathbf{A}$ in (7.1) by the block-diagonal matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{B}^{(1)} + \mathbf{D}^{(1)} + \alpha\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{B}^{(2)} + \frac{1}{2}(\mathbf{D}_C^{(4)}\mathbf{B}^{(1)} + \mathbf{B}^{(1)}\mathbf{D}_G^{(4)}) + \mathbf{D}^{(5)} \end{bmatrix}, \quad \text{with} \quad \alpha > 0,$$

and adopt an experimentally found optimal parameter α, the preconditioned GMRES method can not exhibit satisfactory numerical performance for both Examples 6.1 and 6.2; see Table 7.1.

A Appendix

A.1 Proof of Lemma 4.1

Part (i): To prove Lemma 4.1 (i), we need an error expression for the cardinal expansion of $\tilde{y}(x) = (\tilde{y}_1(x), \tilde{y}_2(x))^T$ with $\tilde{y}_i(x) := y_i(x)/\rho_i(x)$ $(i = 1, 2)$. For $i = 1, 2$, $m = 0, 1, 2$ and $j \in \mathbb{Z}_N$, define $K_{i,m}(x, z)$ and $\omega_{i,m,j}(x)$ as

$$K_{i,m}(x, z) = \frac{[\rho_i(x)]^{m-1}}{2\pi\imath}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\frac{\rho_i(x)\sin[\pi\phi_i(x)/h]}{\phi_i(z) - \phi_i(x)}\right],$$

$$\omega_{i,m,j}(x) = [\rho_i(x)]^{m-1}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\rho_i(x)S(j, h) \circ \phi_i(x)\right].$$

Then it follows from $\tilde{y}(x) \in \mathbb{L}_\alpha(\mathcal{D})$ that $\tilde{y}(x)\phi'(x) \in \mathbb{H}^1(\mathcal{D})$, where $\mathbb{H}^1(\mathcal{D})$ is the space of all analytic functions in $\mathcal{D}$ equipped with the 1-norm. Hence, by making use of [19, Theorem 3.2], we know that the cardinal series expansion of $\tilde{y}_i(x)$ has an error term

$$\tilde{y}_i(x) - \sum_{j=-\infty}^{\infty} \omega_{i,0,j}(x)\tilde{y}_i(x_j) = \int_{\partial\mathcal{D}} \frac{K_{i,0}(x,z)\tilde{y}_i(z)\phi_i'(z)}{\sin[\pi\phi_i(z)/h]}\mathrm{d}z, \quad i = 1, 2.$$

So, in general, we have

$$\frac{\mathrm{d}^m y_i(x)}{\mathrm{d}x^m} - \sum_{j=-\infty}^{\infty} \frac{\omega_{i,m,j}(x)\tilde{y}_i(x_j)}{[\rho_i(x)]^{m-1}} = \int_{\partial\mathcal{D}} \frac{K_{i,m}(x,z)\tilde{y}_i(z)\phi_i'(z)}{[\rho_i(x)]^{m-1}\sin[\pi\phi_i(z)/h]}\mathrm{d}z.$$

Let r_{1k} and r_{2k} denote the kth and $(k+n)$th components of the vector $\mathbf{A}_C\tilde{\mathbf{y}} - \mathbf{p}$, respectively. Then it holds that

$$r_{1k} = r_{1k}^{(1)} + r_{1k}^{(2)} \quad \text{and} \quad r_{2k} = r_{2k}^{(1)} + r_{2k}^{(2)},$$

where

$$\begin{aligned} r_{1k}^{(1)} &= \frac{h^2}{(\phi_1')^2\rho_1}\left(y_1'' - py_2'\right)(x_k) - \frac{h^2}{(\phi_1')^2\rho_1}\sum_{j=-\infty}^{\infty}\left[\frac{\omega_{1,2,j}}{\rho_1}(x_k)\tilde{y}_1(x_j) - p\omega_{2,1,j}(x_k)\tilde{y}_2(x_j)\right] \\ &= h^2\int_{\partial\mathcal{D}}\left[\frac{K_{1,2}(x_k,z)}{(\phi_1')^2\rho_1^2}\cdot\frac{\phi_1'(z)\tilde{y}_1(z)}{\sin[\pi\phi_1(z)/h]} - \frac{pK_{2,1}(x_k,z)}{(\phi_1')^2\rho_1}\cdot\frac{\phi_2'(z)\tilde{y}_2(z)}{\sin[\pi\phi_2(z)/h]}\right]\mathrm{d}z, \\ r_{1k}^{(2)} &= \sum_{|j|>N}\left[\frac{\omega_{1,2,j}}{\rho_1}(x_k)\tilde{y}_1(x_j) - p\omega_{2,1,j}(x_k)\tilde{y}_2(x_j)\right], \end{aligned}$$

and

$$\begin{aligned} r_{2k}^{(1)} &= \frac{h^2}{(\phi_2')^2\rho_2}\left(\mu_1 y_1' + py_2'' + \nu_1 y_2'\right)(x_k) \\ &\quad - \frac{h^2}{(\phi_2')^2\rho_2}\sum_{j=-\infty}^{\infty}\left[\mu_1\omega_{1,1,j}(x_k)\tilde{y}_1(x_j) + \frac{p\omega_{2,2,j}}{\rho_2}(x_k)\tilde{y}_2(x_j) + \nu_1\omega_{2,1,j}(x_k)\tilde{y}_2(x_j)\right] \\ &= h^2\int_{\partial\mathcal{D}}\left[\frac{\mu_1 K_{1,1}(x_k,z)}{(\phi_2')^2\rho_2}\cdot\frac{\phi_1'(z)\tilde{y}_1(z)}{\sin[\pi\phi_1(z)/h]}\right. \\ &\quad \left. + \left(\frac{pK_{2,2}(x_k,z)}{(\phi_2')^2\rho_2^2} + \frac{\nu_1 K_{2,1}(x_k,z)}{(\phi_2')^2\rho_2}\right)\cdot\frac{\phi_2'(z)\tilde{y}_2(z)}{\sin[\pi\phi_2(z)/h]}\right]\mathrm{d}z, \\ r_{2k}^{(2)} &= \sum_{|j|>N}\left[\mu_1\omega_{1,1,j}(x_k)\tilde{y}_1(x_j) + \frac{p\omega_{2,2,j}}{\rho_2}(x_k)\tilde{y}_2(x_j) + \nu_1\omega_{2,1,j}(x_k)\tilde{y}_2(x_j)\right]. \end{aligned}$$

In the above expressions, $K_{i,m}(x_k, z)$ ($i = 1, 2$, $m = 0, 1, 2$) have the following explicit forms:

$$K_{i,0}(x_k,z) = 0, \quad K_{i,1}(x_k,z) = \frac{(-1)^k}{2\imath h[\phi_i(z) - kh]}\,\rho_i\phi_i'(x_k),$$

and

$$K_{i,2}(x_k,z)=\frac{(-1)^k\rho_i}{2\imath h[\phi_i(z)-kh]^2}\left[2\rho_i(\phi_i^{'})^2+(\phi_i(z)-kh)(2\rho_i^{'}\phi_i^{'}+\rho_i\phi_i^{''})(x_k)\right].$$

Since $|\mathrm{Im}(t)|=d$ and $|t-kh|\geq d$ hold on $\partial\mathcal{D}_d$, we have $|\mathrm{Im}(\phi_i(z))|=d$ and $|\phi_i(z)-kh|\geq d$ on $\partial\mathcal{D}$. Using these facts, as well as the assumptions on the coefficients of the second-order ODE system (2.4) and on the mapping ϕ_i, we obtain

$$\begin{aligned}
h^2\left|\frac{K_{1,2}(x_k,z)}{(\phi_1^{'})^2\rho_1^2}\right| &\leq \frac{c_4^{'}h}{[(\mathrm{Re}(\phi_1(z))-kh)^2+d^2]^{1/2}},\\
h^2\left|\frac{pK_{2,1}(x_k,z)}{(\phi_1^{'})^2\rho_1}\right| &\leq \frac{c_4^{''}h}{[(\mathrm{Re}(\phi_2(z))-kh)^2+d^2]^{1/2}},\\
h^2\left|\frac{\mu_1K_{1,1}(x_k,z)}{(\phi_2^{'})^2\rho_2}\right| &\leq \frac{c_5^{'}h}{[(\mathrm{Re}(\phi_1(z))-kh)^2+d^2]^{1/2}},\\
h^2\left|\frac{pK_{2,2}(x_k,z)}{(\phi_2^{'})^2\rho_2^2}+\frac{\nu_1K_{2,1}(x_k,z)}{(\phi_2^{'})^2\rho_2}\right| &\leq \frac{c_5^{''}h}{[(\mathrm{Re}(\phi_2(z))-kh)^2+d^2]^{1/2}},
\end{aligned}$$

where $c_4^{'}$, $c_4^{''}$, $c_5^{'}$ and $c_5^{''}$ are positive constants depending on the bounds for the coefficients of the second-order ODE system (2.4), on the bounds for derivatives of the inverses of the mappings ϕ_i, and on the half-band-width d of the strip region $\mathcal{D}_d$. Therefore, it holds that

$$\begin{aligned}
\|\mathbf{A}_C\tilde{\mathbf{y}}-\mathbf{p}\|_2 &= \left(\sum_{k=-N}^{N}|r_{1k}|^2+\sum_{k=-N}^{N}|r_{2k}|^2\right)^{1/2} \qquad (A.1)\\
&\leq \left(\sum_{k=-N}^{N}|r_{1k}^{(1)}|^2\right)^{1/2}+\left(\sum_{k=-N}^{N}|r_{1k}^{(2)}|^2\right)^{1/2}+\left(\sum_{k=-N}^{N}|r_{2k}^{(1)}|^2\right)^{1/2}+\left(\sum_{k=-N}^{N}|r_{2k}^{(2)}|^2\right)^{1/2}.
\end{aligned}$$

The first term in the right-hand side of (A.1) satisfies

$$\begin{aligned}
\sum_{k=-N}^{N}|r_{1k}^{(1)}|^2 &\leq \sum_{k=-\infty}^{\infty}\left[\left|\int_{\partial\mathcal{D}}\frac{c_4^{'}h}{[(\mathrm{Re}(\phi_1(z))-kh)^2+d^2]^{1/2}}\frac{|\phi_1^{'}(z)\tilde{y}_1(z)|}{|\sin[\pi\phi_1(z)/h]|}|\mathrm{d}z|\right|\right.\\
&\qquad\left.+\left|\int_{\partial\mathcal{D}}\frac{c_4^{''}h}{[(\mathrm{Re}(\phi_2(z))-kh)^2+d^2]^{1/2}}\frac{|\phi_2^{'}(z)\tilde{y}_2(z)|}{|\sin[\pi\phi_2(z)/h]|}|\mathrm{d}z|\right|\right]^2\\
&\leq \sum_{k=-\infty}^{\infty}\frac{h^2}{k^2h^2+d^2}\left[c_4^{'}\int_{\partial\mathcal{D}}\frac{|\phi_1^{'}(z)\tilde{y}_1(z)\mathrm{d}z|}{|\sin[\pi\phi_1(z)/h]|}+c_4^{''}\int_{\partial\mathcal{D}}\frac{|\phi_2^{'}(z)\tilde{y}_2(z)\mathrm{d}z|}{|\sin[\pi\phi_2(z)/h]|}\right]^2\\
&\leq \frac{c_4}{[\sinh(\pi d/h)]^2}. \qquad (A.2)
\end{aligned}$$

We remark that the second inequality in (A.2) comes from the fact that there exists a $k_0\in\mathbb{Z}$ such that $k_0h\leq\mathrm{Re}(\phi_i(z))-kh\leq(k_0+1)h$, and the last inequality in (A.2) comes from the bounds $\sin[\pi\phi_i(z)/h]\geq\sinh[\pi d/h]$ on $\partial\mathcal{D}$ and from the existence of the integrals about both $\phi_1^{'}\tilde{y}_1$ and $\phi_2^{'}\tilde{y}_2$.

Analogous to the derivation of (A.2), we know that the third term in the right-hand side of (A.1) also satisfies

$$\sum_{k=-N}^{N} |r_{2k}^{(1)}|^2 \le \frac{c_5}{[\sinh(\pi d/h)]^2}.$$

For the second term in the right-hand side of (A.1), using the assumptions on the mappings ϕ_1, ϕ_2 and on the coefficients of (2.4), and utilizing the expressions for $\{\delta_{jk}^{(m)}\}_{j,k=-N}^{N}$ $(m=1,2)$, we have

$$\begin{aligned}
\sum_{k=-N}^{N} |r_{1k}^{(2)}|^2 &= \sum_{k=-N}^{N} \left| \sum_{|j|>N} \left[(\delta_{jk}^{(2)} + g_C^{(1)} \delta_{jk}^{(1)}) \tilde{y}_1(x_j) + g_C^{(2)} \delta_{jk}^{(1)} \tilde{y}_2(x_j) \right] \right|^2 \\
&\le c_6^{''} \sum_{k=-N}^{N} \left| \sum_{|j|>N} \gamma_{jk} e^{-\alpha|j|h} \right|^2 \\
&\le c_6^{''} \sum_{|j|>N} \sum_{|\ell|>N} \sum_{k=-\infty}^{\infty} \gamma_{jk} \gamma_{\ell k} e^{-\alpha|j|h} e^{-\alpha|\ell|h} \\
&\le \frac{c_6}{h^2} e^{-2\alpha N h}, \qquad \text{(A.3)}
\end{aligned}$$

where γ_{jk} is the maximum of $|\delta_{jk}^{(m)}|$ $(m=1,2)$. We remark that the first inequality in (A.3) results from the fact that $|\tilde{y}_1(x_j)|$ and $|\tilde{y}_2(x_j)|$ are bounded by exponentially decaying factors.

Analogous to the derivation of (A.3), we know that the fourth term in the right-hand side of (A.1) also satisfies

$$\sum_{k=-N}^{N} |r_{2k}^{(2)}|^2 \le \frac{c_6^{'}}{h^2} e^{-2\alpha N h}.$$

Finally, by replacing h with its optimal choice $[\pi d/(\alpha N)]^{1/2}$, through substituting the bounds with respect to $r_{1k}^{(1)}$, $r_{1k}^{(2)}$, $r_{2k}^{(1)}$ and $r_{2k}^{(2)}$ into (A.1), and after computing and re-arranging the terms in the estimate we straightforwardly obtain

$$\|\mathbf{A}_C \tilde{\mathbf{y}} - \mathbf{p}\|_2 \le c_1 N^{1/2} e^{-(\pi d \alpha N)^{1/2}}.$$

Part (ii): We select an arbitrary integer in the range $[-N, N]$, and simply write $S(k,h) \circ \phi(x)$ as $S(x)$ and $S(k,h) \circ \phi_i(x)$ as $S_i(x)$ $(i=1,2)$. Then it holds that

$$\begin{aligned}
0 &= \frac{h}{\rho_1 \omega_1 \phi_1^{'}} \langle L_1 y(x), S(x) \rangle \\
&= \frac{h}{\rho_1 \omega_1 \phi_1^{'}} \int_a^b \left\{ (S_1 \omega_1)^{''}(x) y_1(x) + \left[(p S_2 \omega_2)^{'} - q S_2 \omega_2 \right](x) y_2(x) \right\} \mathrm{d}x \\
&= r_{1k}^{(1)} + r_{1k}^{(2)} + r_{1k}^{(3)}
\end{aligned}$$

and

$$
\begin{aligned}
0 &= \frac{h}{\rho_2\omega_2\phi_2'}\langle L_2 y(x)-\sigma(x), S(x)\rangle \\
&= \frac{h}{\rho_2\omega_2\phi_2'}\int_a^b \Big\{\Big[-(\mu_1 S_1\omega_1)' + \mu_0 S_1\omega_1\Big](x)y_1(x) \\
&\quad + \Big[(pS_2\omega_2)'' - (\nu_1 S_2\omega_2)' + \nu_0 S_2\omega_2\Big](x)y_2(x) - \sigma(x)S_2\omega_2(x)\Big\}\,\mathrm{d}x \\
&= r_{2k}^{(1)} + r_{2k}^{(2)} + r_{2k}^{(3)},
\end{aligned}
$$

where $r_{1k}^{(1)}$ and $r_{2k}^{(1)}$ denote the kth and $(k+n)$th components of the vector $\mathbf{A}_G\mathbf{y}-\mathbf{p}$, respectively,

$$
\begin{aligned}
r_{1k}^{(2)} &= \sum_{|j|>N}\Big[(\delta_{kj}^{(2)} - g_G^{(1)}\delta_{kj}^{(1)})\tilde{y}_1(x_j) - g_G^{(2)}\delta_{kj}^{(1)}\tilde{y}_2(x_j)\Big], \\
r_{2k}^{(2)} &= \sum_{|j|>N}\Big[-g_G^{(1)}\delta_{kj}^{(1)}\tilde{y}_1(x_j) + (g_G^{(0)}\delta_{kj}^{(2)} - g_G^{(4)}\delta_{kj}^{(1)})\tilde{y}_2(x_j)\Big],
\end{aligned}
$$

and $r_{1k}^{(3)}$ and $r_{2k}^{(3)}$ represent the errors of infinite-point quadratures, which can be explicitly expressed by means of Theorem 4.2.1 in [31] as follows:

$$
\begin{aligned}
r_{1k}^{(3)} &= \frac{\imath h}{2\rho_1\omega_1\phi_1'}\left\{\int_{\partial\mathcal{D}} \frac{\kappa_2(z,h)\Big[(pS_2\omega_2)' - qS_2\omega_2\Big](z)y_2(z)}{\sin[\pi\phi_2(z)/h]}\mathrm{d}z\right. \\
&\quad \left. + \int_{\partial\mathcal{D}} \frac{\kappa_1(z,h)(S_1\omega_1)''(z)y_1(z)}{\sin[\pi\phi_1(z)/h]}\mathrm{d}z\right\}, \qquad \text{(A.4)} \\
r_{2k}^{(3)} &= \frac{\imath h}{2\rho_2\omega_2\phi_2'}\left\{\int_{\partial\mathcal{D}} \frac{\kappa_2(z,h)\Big[(pS_2\omega_2)'' - (\nu_1 S_2\omega_2)' + \nu_0 S_2\omega_2\Big](z)y_2(z) - \sigma S_2\omega_2(z)}{\sin[\pi\phi_2(z)/h]}\mathrm{d}z\right. \\
&\quad \left. + \int_{\partial\mathcal{D}} \frac{\kappa_1(z,h)\Big[(-\mu_1 S_1\omega_1)' + \mu_0 S_1\omega_1\Big](z)y_1(z)}{\sin[\pi\phi_1(z)/h]}\mathrm{d}z\right\}, \qquad \text{(A.5)}
\end{aligned}
$$

with

$$
\kappa_i(z,h) = \exp\{(\imath\pi\phi_i(z)/h)\,\mathrm{sign}(\mathrm{Im}(\phi_i(z)))\}, \quad i=1,2,
$$

such that $|\kappa_i(z,h)| = e^{-\pi d/h}$ holds for $z\in\partial\mathcal{D}$, where $\mathrm{sign}(x)$ denotes the sign function; see Section 2. Recall that if $z\in\partial\mathcal{D}$, then $|\phi_i(z)-kh|\geq d$. Therefore, under the assumptions in (ii), with the explicit expressions of the numerators in (A.4) and (A.5), we know that there exist positive constants c_7 and c_7', independent of h, such that

$$
\begin{aligned}
\left|r_{1k}^{(3)}\right| &\leq c_7 e^{-\pi d/h}\int_{\partial\mathcal{D}} \frac{|\mathrm{d}z|}{[(\mathrm{Re}(\phi_1(z))-kh)^2+d^2]^{1/2}}, \\
\left|r_{2k}^{(3)}\right| &\leq c_7' e^{-\pi d/h}\int_{\partial\mathcal{D}} \frac{|\mathrm{d}z|}{[(\mathrm{Re}(\phi_2(z))-kh)^2+d^2]^{1/2}}.
\end{aligned}
$$

Analogous to (A.2), we obtain

$$\left(\sum_{k=-N}^{N}|r_{1k}^{(3)}|^2\right)^{1/2} \le c_8 h^{-1/2}e^{-\pi d/h} \quad \text{and} \quad \left(\sum_{k=-N}^{N}|r_{2k}^{(3)}|^2\right)^{1/2} \le c_8' h^{-1/2}e^{-\pi d/h}.$$

Also, similar to (A.3), we have

$$\left(\sum_{k=-N}^{N}|r_{1k}^{(2)}|^2\right)^{1/2} \le c_9 h^{-1}e^{-\alpha N h} \quad \text{and} \quad \left(\sum_{k=-N}^{N}|r_{2k}^{(2)}|^2\right)^{1/2} \le c_9' h^{-1}e^{-\alpha N h}.$$

It then follows that

$$\|\mathbf{A}_G\tilde{\mathbf{y}} - \mathbf{p}\|_2 \le c_1' N^{1/2} e^{-(\pi d\alpha N)^{1/2}}.$$

□

A.2 Upper Bound for Inverse of Block Two-by-Two Matrix

Lemma A.1 *Let the matrix* $\mathbf{A}$ *be nonsingular and of the block two-by-two structure*

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{E} \\ \mathbf{F} & \mathbf{C} \end{bmatrix},$$

where $\mathbf{B} \in \mathbb{R}^{n\times n}$, $\mathbf{C} \in \mathbb{R}^{m\times m}$, $\mathbf{E} \in \mathbb{R}^{n\times m}$ *and* $\mathbf{F} \in \mathbb{R}^{m\times n}$. *Then there exists a constant* $\gamma_0 \in (0,1)$ *such that*

$$\|\mathbf{A}^{-1}\|_2 \le \frac{1}{\sqrt{1-\gamma_0}} \max\left\{\frac{1}{\delta_1(\mathbf{B})}, \quad \frac{1}{\delta_1(\mathbf{C})}\right\},$$

where $\delta_1(\cdot)$ *denotes the smallest singular value of the corresponding matrix.*

Proof. From the definition of the Euclidean norm of a matrix, we know that

$$\|\mathbf{A}^{-1}\|_2 = \max_{\mathbf{z}\neq 0}\frac{\|\mathbf{z}\|_2}{\|\mathbf{A}\mathbf{z}\|_2} = \max_{\|\mathbf{z}\|_2=1}\frac{1}{\sqrt{\mathbf{z}^T\mathbf{A}^T\mathbf{A}\mathbf{z}}} = \frac{1}{\sqrt{\tilde{\mathbf{z}}^T\mathbf{A}^T\mathbf{A}\tilde{\mathbf{z}}}},$$

where $\tilde{\mathbf{z}} \in \mathbb{R}^{n+m}$ is the vector such that $1/\sqrt{\mathbf{z}^T\mathbf{A}^T\mathbf{A}\mathbf{z}}$ attains the maximum at $\tilde{\mathbf{z}}$ on the region $\{\mathbf{z} : \|\mathbf{z}\|_2 = 1\}$. Because

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \mathbf{B}^T\mathbf{B} + \mathbf{F}^T\mathbf{F} & \mathbf{B}^T\mathbf{E} + \mathbf{F}^T\mathbf{C} \\ \mathbf{E}^T\mathbf{B} + \mathbf{C}^T\mathbf{F} & \mathbf{E}^T\mathbf{E} + \mathbf{C}^T\mathbf{C} \end{bmatrix},$$

for $\tilde{\mathbf{z}} = (\tilde{\mathbf{u}}^T, \tilde{\mathbf{v}}^T)^T \in \mathbb{R}^{n+m}$, with $\tilde{\mathbf{u}} \in \mathbb{R}^n$ and $\tilde{\mathbf{v}} \in \mathbb{R}^m$, we can obtain

$$\begin{aligned}
\tilde{\mathbf{z}}^T\mathbf{A}^T\mathbf{A}\tilde{\mathbf{z}} &= \tilde{\mathbf{u}}^T(\mathbf{B}^T\mathbf{B} + \mathbf{F}^T\mathbf{F})\tilde{\mathbf{u}} + \tilde{\mathbf{u}}^T(\mathbf{B}^T\mathbf{E} + \mathbf{F}^T\mathbf{C})\tilde{\mathbf{v}} \\
&\quad + \tilde{\mathbf{v}}^T(\mathbf{E}^T\mathbf{B} + \mathbf{C}^T\mathbf{F})\tilde{\mathbf{u}} + \tilde{\mathbf{v}}^T(\mathbf{E}^T\mathbf{E} + \mathbf{C}^T\mathbf{C})\tilde{\mathbf{v}} \\
&\ge \tilde{\mathbf{u}}^T(\mathbf{B}^T\mathbf{B} + \mathbf{F}^T\mathbf{F})\tilde{\mathbf{u}} + \tilde{\mathbf{v}}^T(\mathbf{E}^T\mathbf{E} + \mathbf{C}^T\mathbf{C})\tilde{\mathbf{v}} - 2|\tilde{\mathbf{u}}^T(\mathbf{B}^T\mathbf{E} + \mathbf{F}^T\mathbf{C})\tilde{\mathbf{v}}| \\
&\ge \tilde{\mathbf{u}}^T(\mathbf{B}^T\mathbf{B} + \mathbf{F}^T\mathbf{F})\tilde{\mathbf{u}} + \tilde{\mathbf{v}}^T(\mathbf{E}^T\mathbf{E} + \mathbf{C}^T\mathbf{C})\tilde{\mathbf{v}} - 2|\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{E}\tilde{\mathbf{v}}| - 2|\tilde{\mathbf{u}}^T\mathbf{F}^T\mathbf{C}\tilde{\mathbf{v}}|. \qquad (A.6)
\end{aligned}$$

Now, we further estimate a lower bound for $\tilde{\mathbf{z}}^T\mathbf{A}^T\mathbf{A}\tilde{\mathbf{z}}$ in three cases.

Case (**a**) If $\tilde{\mathbf{u}} \neq 0$, $\tilde{\mathbf{v}} = 0$ and $\|\tilde{\mathbf{u}}\|_2 = 1$, from (A.6) we have

$$\tilde{\mathbf{z}}^T\mathbf{A}^T\mathbf{A}\tilde{\mathbf{z}} \geq \tilde{\mathbf{u}}^T(\mathbf{B}^T\mathbf{B} + \mathbf{F}^T\mathbf{F})\tilde{\mathbf{u}} \geq \tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{B}\tilde{\mathbf{u}} \geq [\delta_1(\mathbf{B})]^2.$$

Therefore, it holds that $\|\mathbf{A}^{-1}\|_2 \leq 1/\delta_1(\mathbf{B})$.

Case (**b**) If $\tilde{\mathbf{u}} = 0$, $\tilde{\mathbf{v}} \neq 0$ and $\|\tilde{\mathbf{v}}\|_2 = 1$, from (A.6) we have

$$\tilde{\mathbf{z}}^T\mathbf{A}^T\mathbf{A}\tilde{\mathbf{z}} \geq \tilde{\mathbf{v}}^T(\mathbf{E}^T\mathbf{E} + \mathbf{C}^T\mathbf{C})\tilde{\mathbf{v}} \geq \tilde{\mathbf{v}}^T\mathbf{C}^T\mathbf{C}\tilde{\mathbf{v}} \geq [\delta_1(\mathbf{C})]^2.$$

Therefore, it holds that $\|\mathbf{A}^{-1}\|_2 \leq 1/\delta_1(\mathbf{C})$.

Case (**c**) If $\tilde{\mathbf{u}} \neq 0$, $\tilde{\mathbf{v}} \neq 0$ and $\|\tilde{\mathbf{u}}\|_2^2 + \|\tilde{\mathbf{v}}\|_2^2 = 1$, by making use of the Cauchy-Schwarz inequality [1] we have

$$|\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{E}\tilde{\mathbf{v}}| \leq \sqrt{\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{B}\tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}^T\mathbf{E}^T\mathbf{E}\tilde{\mathbf{v}}}.$$

This inequality becomes equality if and only if $\mathbf{B}\tilde{\mathbf{u}} = \beta\mathbf{E}\tilde{\mathbf{v}}$, with $\beta \neq 0$ a constant. Therefore, there must exist a constant $\gamma_1 \in (0, 1]$ such that

$$|\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{E}\tilde{\mathbf{v}}| \leq \gamma_1\sqrt{\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{B}\tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}^T\mathbf{E}^T\mathbf{E}\tilde{\mathbf{v}}} \leq \frac{\gamma_1}{2}(\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{B}\tilde{\mathbf{u}} + \tilde{\mathbf{v}}^T\mathbf{E}^T\mathbf{E}\tilde{\mathbf{v}}). \tag{A.7}$$

Similarly, there must exist a constant $\gamma_2 \in (0, 1]$ such that

$$|\tilde{\mathbf{u}}^T\mathbf{F}^T\mathbf{C}\tilde{\mathbf{v}}| \leq \gamma_2\sqrt{\tilde{\mathbf{u}}^T\mathbf{F}^T\mathbf{F}\tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}^T\mathbf{C}^T\mathbf{C}\tilde{\mathbf{v}}} \leq \frac{\gamma_2}{2}(\tilde{\mathbf{u}}^T\mathbf{F}^T\mathbf{F}\tilde{\mathbf{u}} + \tilde{\mathbf{v}}^T\mathbf{C}^T\mathbf{C}\tilde{\mathbf{v}}). \tag{A.8}$$

By substituting (A.7) and (A.8) into (A.6) we immediately obtain

$$\begin{aligned}\tilde{\mathbf{z}}^T\mathbf{A}^T\mathbf{A}\tilde{\mathbf{z}} &\geq (1-\gamma_1)(\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{B}\tilde{\mathbf{u}} + \tilde{\mathbf{v}}^T\mathbf{E}^T\mathbf{E}\tilde{\mathbf{v}}) + (1-\gamma_2)(\tilde{\mathbf{v}}^T\mathbf{C}^T\mathbf{C}\tilde{\mathbf{v}} + \tilde{\mathbf{u}}^T\mathbf{F}^T\mathbf{F}\tilde{\mathbf{u}}) \\ &\geq (1-\gamma_1)\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{B}\tilde{\mathbf{u}} + (1-\gamma_2)\tilde{\mathbf{v}}^T\mathbf{C}^T\mathbf{C}\tilde{\mathbf{v}}.\end{aligned}$$

Note that both γ_1 and γ_2 can not be simultaneously equal to 1 if the matrix $\mathbf{A}$ is nonsingular. Let $\gamma_0 = \max\{\gamma_1, \gamma_2\}$ for $\gamma_1 < 1$ and $\gamma_2 < 1$, $\gamma_0 = \gamma_1$ for $\gamma_1 < 1$ and $\gamma_2 = 1$, and $\gamma_0 = \gamma_2$ for $\gamma_1 = 1$ and $\gamma_2 < 1$. Then it follows that $\gamma_0 \in (0, 1)$ and

$$\begin{aligned}\|\mathbf{A}^{-1}\|_2 &\leq \frac{1}{\sqrt{(1-\gamma_1)\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{B}\tilde{\mathbf{u}} + (1-\gamma_2)\tilde{\mathbf{v}}^T\mathbf{C}^T\mathbf{C}\tilde{\mathbf{v}}}} \\ &\leq \frac{1}{\sqrt{1-\gamma_0}}\max\left\{\sqrt{\frac{\tilde{\mathbf{u}}^T\tilde{\mathbf{u}}}{\tilde{\mathbf{u}}^T\mathbf{B}^T\mathbf{B}\tilde{\mathbf{u}}}},\quad \sqrt{\frac{\tilde{\mathbf{v}}^T\tilde{\mathbf{v}}}{\tilde{\mathbf{v}}^T\mathbf{C}^T\mathbf{C}\tilde{\mathbf{v}}}}\right\} \\ &\leq \frac{1}{\sqrt{1-\gamma_0}}\max\left\{\frac{1}{\delta_1(\mathbf{B})},\quad \frac{1}{\delta_1(\mathbf{C})}\right\}.\end{aligned}$$

In summary, we have demonstrated the upper bound for $\|\mathbf{A}^{-1}\|_2$. □

References

[1] O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, Cambridge, 1996.

[2] Z.-Z. Bai, *Structured preconditioners for nonsingular matrices of block two-by-two structures*, Math. Comput., 75 (2006), 791–815.

[3] Z.-Z. Bai, R.H. Chan and Z.-R. Ren, *On sinc discretization and banded preconditioning for linear third-order ordinary differential equations*, Numer. Linear Algebra Appl., 18 (2011), 471–497.

[4] Z.-Z. Bai, G.H. Golub, L.-Z. Lu and J.-F. Yin, *Block triangular and skew-Hermitian splitting methods for positive-definite linear systems*, SIAM J. Sci. Comput., 26 (2005), 844–863.

[5] Z.-Z. Bai, G.H. Golub and M.K. Ng, *Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems*, SIAM J. Matrix Anal. Appl., 24 (2003), 603–626.

[6] Z.-Z. Bai, Y.-M. Huang and M.K. Ng, *On preconditioned iterative methods for Burgers equations*, SIAM J. Sci. Comput., 29 (2007), 415–439.

[7] Z.-Z. Bai, Y.-M. Huang and M.K. Ng, *On preconditioned iterative methods for certain time-dependent partial differential equations*, SIAM J. Numer. Anal., 47 (2009), 1019–1037.

[8] Z.-Z. Bai and M.K. Ng, *Preconditioners for nonsymmetric block Toeplitz-like-plus-diagonal linear systems*, Numer. Math., 96 (2003), 197–220.

[9] B.R. Duffy and S.K. Wilson, *A third-order differential equation arising in thin-film flows and relevant to tanner's law*, Appl. Math. Lett., 10 (1997), 63–68.

[10] N. Euler, T. Wolf, P.G.L. Leach and M. Euler, *Linearisable third-order ordinary differential equations and generalised Sundman transformations: the case* $X''' = 0$, Acta Appl. Math., 76 (2003), 89–115.

[11] V.M. Falkner and S.W. Skan, *Solutions of the boundary layer equations*, Philos. Mag., 7 (1931), 865–896.

[12] W.F. Ford, *A third-order differential equation*, SIAM Rev., 34 (1992), 121–122.

[13] G. Grebot, *The characterization of third order ordinary differential equations admitting a transitive fiber-preserving point symmetry group*, J. Math. Anal. Appl., 206 (1997), 364–388.

[14] F.A. Howes, *The asymptotic solution of a class of third-order boundary-value problems arising in the theory of thin film flows*, SIAM J. Appl. Math., 43 (1983), 993–1004.

[15] N.H. Ibragimov and S.V. Meleshko, *Linearization of third-order ordinary differential equations by point and contact transformations*, J. Math. Anal. Appl., 308 (2005), 266–289.

[16] X.-Q. Jin, *A note on preconditioned block Toeplitz matrices*, SIAM J. Sci. Comput., 16 (1995), 951–955.

[17] X.-Q. Jin, *Band Toeplitz preconditioners for block Toeplitz systems*, J. Comput. Appl. Math., 70 (1996), 225–230.

[18] X.-Q. Jin, *Developments and Applications of Block Toeplitz Iterative Solvers*, Kluwer Academic Publishers, Dordrecht; Science Press, Beijing, 2002.

[19] J. Lund and K. Bowers, *Sinc Methods for Quadrature and Differential Equations*, SIAM, Philadelphia, 1992.

[20] H.-P. Ma and W.-W. Sun, *A Legendre-Petrov-Galerkin method and Chebyshev collocation method for the third-order differential equations*, SIAM J. Numer. Anal., 38 (2000), 1425–1438.

[21] H.-P. Ma and W.-W. Sun, *Optimal error estimates of the Legendre-Petrov-Galerkin method for the Korteweg-de Vries equation*, SIAM J. Numer. Anal., 39 (2001), 1380–1394.

[22] A.W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, New York, 1979.

[23] S.V. Meleshko, *On linearization of third-order ordinary differential equations*, J. Phys. A: Math. Gen., 39 (2006), 15135–15145.

[24] M.K. Ng, *Fast iterative methods for symmetric sinc-Galerkin systems*, IMA J. Numer. Anal., 19 (1999), 357–373.

[25] M.K. Ng, *Iterative Methods for Toeplitz Systems*, Oxford University Press, Oxford, 2004.

[26] M.K. Ng and Z.-Z. Bai, *A hybrid preconditioner of banded matrix approximation and alternating direction implicit iteration for symmetric sinc-Galerkin linear systems*, Linear Algebra Appl., 366 (2003), 317–335.

[27] M.K. Ng and D. Potts, *Fast iterative methods for sinc systems*, SIAM J. Matrix Anal. Appl., 24 (2002), 581–598.

[28] S. Nosé, *A unified formulation of the constant temperature molecular-dynamics methods*, J. Chem. Phys., 81 (1984), 511–519.

[29] A. Nurmuhammad, M. Muhammad, M. Mori and M. Sugihara, *Double exponential transformation in the sinc-collocation method for a boundary value problem with fourth-order ordinary differential equation*, J. Comput. Appl. Math., 182 (2005), 32–50.

[30] J. Shen, *A new dual-Petrov-Galerkin method for third and higher odd-order differential equations: application to the KdV equation*, SIAM J. Numer. Anal., 41 (2003), 1595–1619.

[31] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Springer Ser. Comput. Math., Springer-Verlag, New York, 1993.

[32] P. Swinnerton-Dyer and T. Wagenknecht, *Some third-order ordinary differential equations*, Bull. Lond. Math. Soc., 40 (2008), 725–748.

[33] E.O. Tuck and L.W. Schwartz, *A numerical and asymptotic study of some third-order ordinary differential equations relevant to draining and coating flows*, SIAM Rev., 32 (1990), 453–469.

[34] C.-X. Zheng, *Numerical simulation of a modified KdV equation on the whole real axis*, Numer. Math., 105 (2006), 315–335.

[35] C.-X. Zheng, X. Wen and H.-D. Han, *Numerical solution to a linearized KdV equation on unbounded domain*, Numer. Methods Partial Differential Equations, 24 (2008), 383–399.