ON ORDERED GROUPS

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1. **Introduction.** We base our discussion upon the concept of an ordered group, that is, the generalization of the l-group studied by G. Birkhoff [1](1) in which the lattice property is replaced by the weaker "Moore-Smith" or directed set axiom. An ordered group is embeddable in a complete ordered group if and only if it is integrally closed. We prove that if the commutator group of an ordered group is in its center, then integral closure of the group implies commutativity. Thus the conjecture of Birkhoff (see Problem 2, loc. cit.) is proved, although negative examples are given showing the falsity of Problems 1 and 2. Problem 3 is left open; the authors hope to settle it in a later paper. The linear group of functions $ax+A$, a, A real, $a>0$, admits no integrally closed order under composition.

Every ordered group is embeddable in a group which is sequence-complete in the sense of o-convergence. The results of [5] are thus extended to the non-commutative case.

Properties of the group of monotone continuous functions on (0, 1) to itself under composition are studied, and various orders in the free group with two generators are used to establish a curious property of the function group.

2. **Completion of ordered groups.** A group G is called an *ordered* or *o-group* in case G is (1) *partially ordered* by a relation $\geqq$ ($a\geqq a$; $a\geqq b$ and $b\geqq a$ imply $a=b$; $a\geqq b$ and $b\geqq c$ imply $a\geqq c$), (2) a *directed set*: for every a, b there is a $c\geqq a$, b, (3) *homogeneous*: $a\geqq b$ implies $c+a+d\geqq c+b+d$.

G is called an *l-group* in case the order is a lattice order, that is, every two elements a, b possess a l.u.b. $a\vee b$ and a g.l.b. $a\wedge b$ [1].

G is said to be *conditionally complete* in case every set of elements a_α bounded above has a l.u.b. $\vee a_\alpha$ (and hence also a g.l.b. $\wedge a_\alpha$ when bounded below).

G is *integrally closed* in case $na=a+\cdots+a\leqq b$, $n=1, 2, \cdots$, implies $a\leqq 0$ [4], and *archimedean* if $na\leqq b$, $n=0, \pm 1, \pm 2, \cdots$, implies $a=0$ [1].

LEMMA 1. *If G is an o-group, integral closure implies archimedean order.*

For if $na\leqq b$, $n=0, \pm 1, \pm 2, \cdots$, then $na\leqq b$ and $n(-a)\leqq b$, $n=1, 2, \cdots$. By integral closure, $a\leqq 0$ and $-a\leqq 0$, $a=0$.

We shall see later (Theorem 12) that the converse is false.

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1 Numbers in brackets refer to the Bibliography at the end of the paper.

LEMMA 2 [1, p. 313]. *If G is an l-group, integral closure and archimedean order are equivalent.*

Let G be archimedean, and $na \leqq b$, $n=1, 2, \cdots$. Then $n(a \vee 0) = na \vee (n-1)a \vee \cdots \vee a \vee 0 \leqq b \vee 0$, and $-n(a \vee 0) = n(-(a \vee 0)) \leqq 0 \leqq b \vee 0$. Hence $n(a \vee 0) \leqq b \vee 0$, $n=0, \pm 1, \cdots$, and $a \vee 0 = 0 \geqq a$.

A. H. Clifford [4] has proved that a commutative o-group is embeddable in a conditionally complete group if and only if it is integrally closed. We here extend this result to noncommutative groups.

THEOREM 1. *An o-group G is embeddable in a conditionally complete o-group with preservation of order,* g.l.b., *and* l.u.b. *if and only if G is integrally closed.*

If G is so embeddable, and $na \leqq b$, $n=1, 2, \cdots$, let $u = \bigvee(na; \text{all } n)$; then $u-a \geqq na$, $n=1, 2, \cdots$, $u-a \geqq u$, $a \leqq 0$ (cf. [1, p. 322]).

It is well known that if $L(X)$, $U(X)$ denote the sets of all lower and of all upper bounds, respectively, for all the elements of a subset $X \subset G$, then the operation $X^* = L(U(X))$ has the closure properties: $X^* \supset X$; $X^{**} = X^*$; $X \supset Y$ implies $X^* \supset Y^*$; and that the class $\mathfrak{C}$ of all "closed" sets $C = C^*$ is a conditionally complete lattice under set inclusion, with set intersection effective as g.l.b., and closure of set union as l.u.b. [2, p. 25; 7]. The correspondence $a \rightarrow (a)^*$ embeds G in $\mathfrak{C}$ with preservation of order, g.l.b. and l.u.b. Moreover one has the important property: $\bigvee(x_\alpha) = x$ in G if and only if $(x_\alpha; \alpha)^* = (x)^*$.

Defining $X+Y = (\text{all } x+y;\ x \in X,\ y \in Y)^*$, X, $Y \in \mathfrak{C}$, one readily verifies $X+(Y+Z) = (X+Y)+Z$, $0+X = 0 = X+0$, where $0 = (0)^*$, and $A \supset B$ implies $X+A+Y \supset X+B+Y$. Since $(a+b)^* = (a)^* + (b)^*$, G is an o-subgroup of the lattice semigroup $\mathfrak{C}$. In general, $\mathfrak{C}$ is not a group.

LEMMA 3. *For $X \in \mathfrak{C}$, there exists $Y \in \mathfrak{C}$ such that $Y+X=0$ if and only if $0 = \bigwedge(-x+u;\ x \in X,\ u \in U(X))$; similarly $X+Y=0$ if and only if $\bigwedge(u-x) = 0$. There is a $Y \in \mathfrak{C}$ for which $Y+X=0=X+Y$ if and only if $\bigwedge(-x+u)=0 = \bigwedge(u-x)$ and when such Y exists $Y = L(\text{all } -x;\ x \in X)$.*

If $\bigwedge(-x+u) = 0 = \bigvee(-u+x) = \bigvee(l+x;\ l \in L(-x),\ x \in X)$, then $(l+x)^* = (0)^*$ and $L(-x)+X = 0$. Conversely, if $Y+X=0$, $(y+x)^* = (0)^*$, $\bigvee(y+x) = 0$, $y+x \leqq 0$, $y \leqq -x$, $Y \subset L(-x)$. Hence $(y+x) \subset (l+x; l \in L(-x))$, $0 = Y+X \subset L(-x)+X$. But $L(-x)+X \subset 0 = Y+X$, thus $0 = L(-x)+X$. Hence $0 = \bigvee(l+x) = \bigvee(-u+x) = \bigwedge(-x+u)$. Similarly for the second implication.

If both intersections are zero, $L(-x)+X = 0 = X+L(-x)$, and if $Y+X = 0 = X+Y$, both intersections are zero, $L(-x)+X=0$, $L(-x)+X+Y = Y = L(-x)$.

LEMMA 4. *If G is integrally closed, every $X \in \mathfrak{C}$ has an inverse.*

Let $X \in \mathfrak{C}$, $y \leqq u-x$, all $u \in U(X)$, $x \in X$. Then $x \leqq -y+u$, $-y+u \in U(X)$ for all $u \in U(X)$. Let u_0 be any element of $U(X)$. Then $-y+u_0 = u_1$, $-y+u_1$

$=u_2, \cdots$ where $u_i \in U(X)$, and $u_0 = ny + u_n$. Thus $ny = u_0 - u_n \leqq u_0 - x_0$, x_0 any fixed element of X. By integral closure, $y \leqq 0$. Hence $\bigwedge(u-x)=0$. Similarly $\bigwedge(-x+u)=0$.

3. **Commutativity in integrally closed groups.** G. Birkhoff has raised the question whether archimedean order in l-groups does not imply commutativity [1, p. 329]. In this connection he quotes the following theorem [3].

THEOREM 2 (H. CARTAN). *If G is an archimedean ordered o-group, then linear order implies commutativity.*

He also conjectures the truth of the following theorem.

THEOREM 3. *If G is an integrally closed o-group (or archimedean l-group), and if the commutator subgroup of G is in the center of G, then G is commutative.*

Birkhoff bases a tentative proof on previous conjectures which we shall later show false. The following proof however establishes Theorem 3.

Let $c=b+a-b-a$ and note $b+a=b+(a+b)-b$. Then $c+(a+b)=(b+a)$ $= b + (a + b) - b$; $2c+(a+b)=c+b+(a+b)-b=(c+b-c)+c+(a+b)-b$ $=(c+b-c)+b+(a+b)-2b$; $3c+(a+b)=(2c+b-2c)+(c+b-c)+b+(a+b)$ $-3b$; by induction, $nc+(a+b)=((n-1)c+b-(n-1)c)+ \cdots +b+(a+b)$ $-nb$. For a, $b \geqq 0$, and c in the center, we have $nc+(a+b)=nb+(a+b)-nb$ $\geqq 0$. By integral closure, $b+a-b-a \geqq 0$, $b+a \geqq a+b$. Similarly $a+b \geqq b+a$, and $a+b=b+a$, all a, $b \geqq 0$. But then $-b+a=a-b$, and every positive element commutes with every negative element. Since every $g=g\bigvee 0+g\bigwedge 0$ [1, p. 306] and $h=h\bigvee 0+h\bigwedge 0$ we have $g+h=h+g$, all h, g of G.

THEOREM 4. *The linear group of elements $ax+A$, a, A real, $a>0$, under composition, admits no integrally closed o-group order.*

One computes $(gx+G)^{-1}=(x-G)/g$ and $(gx+G)(fx+F)(gx+G)^{-1}=fx$ $+gF+G(1-f)$. The conjugates of $fx+F$ $(f\neq 1)$ consist of all $fx+R$, R real; for example, use $g=1$, $G=(R-F)/(1-f)$.

There exists $f_0x+F_0 \geqq x$ $(f_0 \neq 1)$, for since G is a directed set, let $fx+F \geqq x$, $x/2$. If $f=1$, $x+F \geqq x$, $x/2$. Substitution into $2x$ yields $2(x+F)=2x+2F$ $\geqq 2(x/2)=x$.

Now all conjugates of f_0x+F_0, namely all f_0x+R (R real), are greater than or equal to x. Thus $f_0x-nk \geqq x$, k real, $n=1, 2, \cdots$, and substitution into $(x+k)^n=x+nk$ yields $f_0x \geqq x+nk=(x+k)^n$. By integral closure, $x+k \leqq x$, all k. Thus $(x+1) \leqq x$ and $(x+1)^{-1}=x-1 \leqq x$; $x+1=x$, a contradiction.

COROLLARY 1. *The group of functions αx^A, α, A real and positive, under composition and the group [1, p. 303] of real pairs (x, y) under $(x, y)+(x', y')$ $=(x+x', e^{x'}y+y')$ are isomorphic to the group of Theorem 4, hence cannot be integrally closed o-groups under any order.*

For consider the correspondences $ax+A \rightarrow e^A x^a$, $ax+A \rightarrow (-\log a, -A/a)$.

4. **On o-convergence and sequence completion in l-groups.** The absolute $|a| = a \vee -a$ has proved of importance in the study of commutative l-groups, since its fundamental properties: $|a| = |-a| \geqq 0$; $|a| = 0$ if and only if $a = 0$; $|a \vee b - a' \vee b| \leqq |a - a'|$ and dually; and $|a+b| \leqq |a| + |b|$ serve to establish an intrinsic topology in G via o-convergent sequences [2; 5; 6]. All these properties except the last are valid in l-groups. We suggest the following as a generalization which seems adequate:

LEMMA 5. *In an l-group,* $|a+b| \leqq (|a| + |b|) \vee (|b| + |a|) \leqq |a| + |b| + |a|$.

$|a| \geqq a, -a$, $|b| \geqq b, -b$ yields $|a| + |b| \geqq a+b$ and $|b| + |a| \geqq -b-a$. Hence the first inequality. But $|a| + |b| \leqq |a| + |b| + |a|$ and $|b| + |a| \leqq |a| + |b| + |a|$.

We write $x_n \uparrow x$ for $x_1 \leqq x_2 \leqq \cdots$ with $x = \vee x_n$ and $x_n \downarrow x$ dually. Define $x_n \to x$ (o-convergence) as usual [2, pp. 28, 112] to mean there exist sequences l_n, u_n such that $l_n \leqq x_n \leqq u_n$ where $l_n \uparrow x$ and $u_n \downarrow x$. One proves $x_n \to x$ if and only if for some $w_n \downarrow 0$, $|x_n - x| \leqq w_n$; also if and only if $|-x + x_n| \leqq w_n' \downarrow 0$.

THEOREM 5. *o-convergence is a Fréchet convergence for which $a_n \to a$, $b_n \to b$ implies $a_n + b_n \to a + b$, $-a_n \to -a$, $a_n \vee b_n \to a \vee b$, and dually.*

One easily verifies $a, a, a, \cdots \to a$; $a_{n_i} \to a$; and $a_n \to a$, $a_n \to b$ implies $a = b$. Moreover $|(a_n + b_n) - (a+b)| = |a_n + b_n - b - a| = |(a_n - a) + (a + b_n - b - a)| \leqq |a_n - a| + |a + (b_n - b) - a| + |a_n - a| \leqq w_n + (a + w_n' - a) + w_n \downarrow 0$.

Finally, $|a_n \vee b_n - a \vee b| = |a_n \vee b_n - a_n \vee b + a_n \vee b - a \vee b| \leqq |a_n \vee b_n - a_n \vee b| + |a_n \vee b - a \vee b| + |a_n \vee b_n - a_n \vee b| \leqq |b_n - b| + |a_n - a| + |b_n - b| \leqq w_n' + w_n + w_n' \downarrow 0$.

We say a sequence is *o-regular* in case, for some $w_n \downarrow 0$, $|-a_{n+p} + a_n| \vee |a_n - a_{n+p}| \leqq w_n$, all $n, p = 1, 2, \cdots$.

THEOREM 6. *Every o-convergent sequence is o-regular.*

For $|a_n - a_{n+p}| = |(a_n - a) + (a - a_{n+p})| \leqq |a_n - a| + |a - a_{n+p}| + |a_n - a| \leqq 3w_n$. Similarly, $|-a_{n+p} + a_n| \leqq 3w_n'$, by the remark preceding Theorem 5.

G is called *o-complete* in case every o-regular sequence o-converges. In a previous paper conditions were given for a commutative l-group to be o-complete. The conditions and proofs given there [5] extend readily to the noncommutative case. We merely state the following theorem.

THEOREM 7. *In an l-group G the following conditions are equivalent:*

(i) *G is o-complete.*

(ii) *Every o-regular monotone sequence $y_1 \geqq y_2 \geqq \cdots$ o-converges.*

(iii) *For every o-regular monotone sequence $y_1 \geqq y_2 \geqq \cdots$, $\wedge y_n$ exists.*

(iv) *For every o-regular sequence x_n, $\wedge x_n$ exists.*

COROLLARY 2. *Every conditionally complete l-group is o-complete.*

(Cf. Kantorovitch [6] for the commutative case.) For if a_n is o-regular, $|a_n-a|\leqq w_n$, $-w_n\leqq a_n-a$, $-w_1+a\leqq -w_n+a\leqq a_n$ (all n). Hence by completeness, $\wedge a_n$ exists and (iv) above is satisfied.

One naturally asks whether an arbitrary l-group may be embedded in an o-complete l-group. Let $\mathfrak{G}$ be the set of all elements X of $\mathfrak{C}$ (cf. §2) for which an inverse exists: $X+Y=0=Y+X$.

THEOREM 8. *Every l-group G is embeddable with preservation of order*, g.l.b., l.u.b., *and o-convergence in the l-group $\mathfrak{G}$, which is o-complete.*

The correspondence $a\rightarrow(a)^*$ maps G into $\mathfrak{G}$ with preservation of order, g.l.b., l.u.b. (and hence o-convergence), by the discussion in §2. That $\mathfrak{G}$ is an o-group is obvious. We must show that the order is a lattice order. We need only the following lemma.

LEMMA 6. *If X, Y, are in $\mathfrak{G}$ then the $X\wedge Y$ (of $\mathfrak{C}$) is in $\mathfrak{G}$.*

The proof is given in [5].

Finally we verify Theorem 7 (iii) using the following lemma.

LEMMA 7. *If Y_p, $W_p\in\mathfrak{G}$, $W_p\downarrow 0$ in $\mathfrak{G}$, and $Y\in\mathfrak{C}$, where $Y_p-W_p\subset Y\subset Y_p$* and $-W_p+Y_p\subset Y\subset Y_p$, *all p, then $Y\in\mathfrak{G}$.*

The proof is given in [5].

Hence condition (iii) holds; for let Y_n be o-regular in $\mathfrak{G}$, $Y_1\supset Y_2\supset\cdots$, $|Y_n-Y_{n+p}|=Y_n-Y_{n+p}\subset W_n\downarrow 0$ in $\mathfrak{G}$. Then $-W_1+Y_1\subset Y_n$ (all n), and $Y=\wedge(Y_n)$ exists in $\mathfrak{C}$ with $-W_n+Y_n\subset Y\subset Y_n$ for $-W_n+Y_n\subset Y_{n+p}$. Similarly we fulfill the other condition of Lemma 7.

5. **The group of topological transformations of the line into itself.** Let T be the class of all functions $f(x)$ on $0\leqq x\leqq 1$, having the properties:

(1) $f(x)$ is continuous monotone increasing on $0\leqq x\leqq 1$,

(2) $f(0)=0$, $f(1)=1$.

For f, g in T, define $fg=f(g(x))$, and $f\geqq g$ to mean $f(x)\geqq g(x)$, $0\leqq x\leqq 1$.

THEOREM 9. *T is an l-group, non-integrally closed.*

Under composition, T is the well known [8] group of topological transformations of (0, 1) into itself, with identity $e(x)=x$. Verification of the order postulates is trivial. The function $u(x)=\max\,(f(x), e(x))$, $0\leqq x\leqq 1$, is in the class T and has the properties of a l.u.b. for $f(x)$ and $e(x)$. This is sufficient for lattice order. The function $b(x)$ in T defined by a broken line of four segments: $b(x)\equiv x$ on $(0, 1/4)$ and on $(3/4, 1)$ with $b(1/2)=5/8$ is obviously bounded: $b^n\leqq f$ for some f, all n, but b not less than or equal to e.

THEOREM 10. *In an l-group, the relation $(aba^{-1}b^{-1})^n<ab$, a, $b>e$, $n=1, 2, \cdots$, need not be true. Indeed in the l-group T, there are elements f, $g>e$ for which the elements $(fgf^{-1}g^{-1})^n$, $n=0, \pm 1, \pm 2, \cdots$, are unbounded by any element.*

For, any two polynomials $f, g > e$ have the property that either $fgf^{-1}g^{-1}$ or $(fgf^{-1}g^{-1})^{-1} = gfg^{-1}f^{-1}$ is greater than x on an open interval $(0, d)$, where d is the first fixed point of the commutator. The powers of this commutator $c(x)$ are then unbounded, since, if $\lim_n c^n(x) = L(x)$, $0 \leqq x \leqq d$, then $\lim_n c(c^n(x)) = c(L(x)) = L(x)$. But since $x \leqq d$, $c^n(x) \leqq c(d) = d$. Hence $L(x) \equiv d$ and no *continuous* function can serve as upper bound.

This settles Problem 2 [1, p. 329].

THEOREM 11. *In an l-group, $a^m > b^m$ for some m, $a, b > e$ does not imply $a \geqq b$.*

In T, we exhibit the broken line $f(x)$ of two segments with vertex at $f(3/8) = 3/4$, and the broken line of four segments $g(x) \equiv x$ on $(0, 1/4)$ and on $(3/4, 1)$ with vertex at $g(5/16) = 11/16$. One verifies $f^2(x) > g^2(x)$ for all x on open $(0, 1)$, but $f(x)$ not greater than or equal to $g(x)$.

This settles Problem 1 [1, p. 329].

THEOREM 12. *The subgroup of algebraic functions of T is an archimedean ordered o-group, but is not integrally closed.*

If $f(x) \neq x$ and $g(x)$ are algebraic and $f^n(x) < g(x)$, $n = 0, \pm 1, \pm 2, \cdots$, then either $f(x)$ or $f^{-1}(x)$ must be greater than or equal to x on an interval $(0, d)$, hence its powers cannot be bounded by any continuous function (see the argument of Theorem 10). It is clear however that an algebraic function similar to the broken line of Theorem 9 can be defined, hence the group is not integrally closed.

6. **The free group, combinatorial order.** Let F be the free group with two generators a, b. We recall that an order may be established in a group G by defining a subsemi-group of "positive" elements K which (a) is closed under multiplication, (b) is closed under conjugation: $gKg^{-1} \subset K$, all $g \in G$, (c) contains e, and no other element along with its inverse. Then $a \geqq b$ is defined to mean $ab^{-1} \in K$ [1]. These conditions are equivalent to (1, 3) of §2. A first attempt to define an o-group on F consists in letting K_0 be the set of all elements expressible as products of conjugates of a and of b, together with the identity e.

THEOREM 13. *The order $x \geqq y$ meaning $xy^{-1} \in K_0$ defines a non-integrally closed o-group on F.*

Properties (a, b, c) are trivial, the latter because the sum of exponents of any K_0 element not e is positive. Moreover, F is a directed set, since $a^n \leqq a^{|n|}$, hence $a^m b^n \cdots \leqq a^{|m|} b^{|n|} \cdots$. Thus both $a^m b^n \cdots$ and $a^{m'} b^{n'} \cdots \leqq (a^{|m|} b^{|n|} \cdots)(a^{|m'|} b^{|n'|} \cdots)$. The remainder of the proof requires the following lemma.

LEMMA 8. *In any l-group, $a^n \geqq e$ for some n implies $a \geqq e$.*

Proof. $(a \wedge e)^n = a^n \wedge a^{n-1} \wedge \cdots \wedge a \wedge e = (a \wedge e)^{n-1}$, $a \wedge e = e \leqq a$.

However, in the group F one has $A=ab^{-2}a^2b^2a^{-1}ba^{-1}b$ not greater than or equal to e and $A^2=(ab^{-2}a^2b^2a^{-1})(ba^{-1}bab^{-1})(b^{-1}a(ab^2a^{-1}b)a^{-1}b)>e$.

Now if F were integrally closed, by Theorem 1, F would be embeddable in an l-group. Or one may argue directly that $A^{2n}>e$, $A^{2n+1}>A$, hence $A^n>B$ for any $B<e$, A (directed set property), and integral closure would imply $A\geqq e$.

7. **The free group, function order.** Again let F be the free group with generators a, b, and let T be the group of continuous monotone functions of §5. Denote by T^+ the functions $f(x)\geqq x$, $0\leqq x\leqq 1$, of T. We now introduce an order into F by defining a positive class F^+ consisting of e, and of all formal products $a^mb^n\cdots$ of F for which $f^mg^n\cdots(x)\in T^+$ for all f, g of T^+.

It is clear that F^+ is closed under multiplication and conjugation, inasmuch as T^+ is. Moreover, if a formal product $a^mb^n\cdots\neq e$ were in F^+ along with its inverse, we should have $f^mg^n\cdots(x)\equiv x$, $0\leqq x\leqq 1$, for all f, g of T^+. We show that this is impossible.

LEMMA 9. *If $a^mb^n\cdots\neq e$, there exist functions f, g of T^+ for which $f^mg^n\cdots(x)\not\equiv x$.*

Assume $P=a^mb^n\cdots$ completely reduced, that is, with no adjacent a, a^{-1} or b, b^{-1}. Define the first function (f, f^{-1}, g, or g^{-1}) from the right at $x=1/2$ by $f(1/2)=3/4$, or $g(1/2)=3/4$ if $P=a^m\cdots a$ or $P=a^m\cdots b$ respectively, and $f(1/4)=1/2$ or $g(1/4)=1/2$, that is, $f^{-1}(1/2)=1/4$, or $g^{-1}(1/2)=1/4$ in case $P=a^m\cdots a^{-1}$ or $P=a^m\cdots b^{-1}$. Now suppose the functions f, g have been defined on a finite set of points of (0, 1) so that

(1) f, g are monotone increasing and greater than or equal to x,

(2) the values $1/2$, $F_1(1/2)$, $\cdots$, $F_m(1/2)$ are all distinct, where $F_i(x)=(\cdots f\cdots g\cdots)$ is the product of the i right-most factors of P with f, g substituted for a, b.

Suppose $F_{m+1}(x)=f(F_m(x))$. We may define $F_{m+1}(1/2)$ distinct from $F_m(1/2)$, $\cdots$, $1/2$, and f monotone with the previous values, and $f(F_m(1/2))>F_m(1/2)$ provided only that $F_m(x)$ was not previously a point of definition for f. But previous definition of f occurred in only two ways. Either $F_k=f(F_{k-1})$, $k\leqq m$ (and this is impossible by (2)), or $F_k=f^{-1}(F_{k-1}), f(F_k)=F_{k-1}$. Again by (2), $k<m$ is impossible, and if $k=m$, we have $F_{m+1}=fF_m=ff^{-1}F_{m-1}$ contradicting the hypothesis on irreducibility of P.

Finally suppose $F_{m+1}(x)=f^{-1}(F_m(x))$. We can define f^{-1} (and hence f) and $F_{m+1}(1/2)$ as we wish provided only that $F(1/2)$ has not previously occurred as a point of definition of f^{-1}. If previously $F_k=f^{-1}F_{k-1}$, $k\leqq m$, we would contradict (2). If $F_k=fF_{k-1}$ and thus $f^{-1}(F_k)=F_{k-1}$ either $k<m$ is impossible by (2) or $k=m$ means $F_{m+1}=f^{-1}F_m=f^{-1}fF_{m-1}$ contradicting irreducibility of P. It follows that we have the theorem:

THEOREM 14. *The order $x\geqq y$ meaning $xy^{-1}\in F^+$ defines a non-integrally* *closed o-group on F, for which, however, $x^n \geqq e$ for some n implies $x \geqq e$. The functionally positive elements F^+ properly include the elements of K_0.*

Conditions (1, 3) of §2 are immediate. Since every element of K_0 is a product of conjugates of a and b, clearly $K_0 \subset F^+$. Hence function order defines a directed set since K_0 did. Hence also (2).

Suppose now $P = a^m b^n \cdots \in F$ and $P^\nu \in F^+$ for some ν. Then for every f, g in T^+, $(f^m g^n \cdots)^\nu \in T^+$, and since T is an l-group (Theorem 9), $(f^m g^n \cdots) \in T^+$, hence $P \in F^+$ (Lemma 8). Note that the element A of Theorem 12 (proof) is in F^+, not in K_0.

The fact that F^+-order is non-integrally closed seems deeper, and has curious consequences for function theory which we shall point out later.

COROLLARY 3. *The composite function $fg^{-2}f^2g^2f^{-1}gf^{-1}g(x)$ is in T^+ for all f, g of T^+.*

For $(ab^{-2}a^2b^2a^{-1}ba^{-1}b)^2 \in K_0 \subset F^+$.

LEMMA 10. *In F, let $A = ab^{-2}a^2b^2a^{-1}ba^{-1}b$, and $B = ab^{-2}ab^2a^{-1}$. Under function order, $A^2 > B^2$, but A, B are incomparable.*

For $A^2 \geqq B^2$, see the computation of §6. Indeed, $B^{-2}A^2 \in K_0 \subset F^+$. One easily constructs broken lines f, g of T^+ for which $P = B^{-1}A = ab^{-2}ab^2a^{-1}ba^{-1}b$ under substitution yields a function $P(f, g) = fg^{-2}fg^2f^{-1}gf^{-1}g$ which has $P(f, g)(1/2) > 1/2$, and other broken lines for which $P(f, g)(1/2) < 1/2$. (This is most easily accomplished graphically by a point by point construction of the functions and their inverses from the right end of P, in the first case always assigning f, g values as great as is possible, consistent with monotonicity, in the second case, as small.)

LEMMA 11. *If for some f_0, g_0 of T^+ and x_0 on* (0, 1) *one has $P(f_0, g_0)(x_0) = f_0g_0^{-2}f_0g_0^2f_0^{-1}g_0f_0^{-1}g_0(x_0) < x_0$, then $P(f_0, g_0)(x_1) > x_1$, and $P(f_0, g_0)(x_2) > x_2$, where $x_1 = B^{-1}(f_0, g_0)(x_0) < x_0 < x_2 = A(f_0, g_0)(x_0)$.*

Let $P(x_0) = B^{-1}A(x_0) < x_0$ (where throughout we understand that f_0, g_0 are substituted for a, b respectively). Then $B^{-1}A(A(x_0)) \geqq B(x_0) > A(x_0) > x_0$. Similarly, $B^{-1}A(B^{-1}(x_0)) \geqq B^{-1}A^{-1}B(x_0) > B^{-1}(x_0)$.

LEMMA 12. *In the group F with function order F^+, one has $P^\nu B > e$ for all ν, with P not greater than or equal to e.*

For all $\nu = 1, 2, \cdots$, and all x on (0, 1), all f, g of T^+, one has $P^\nu B(f, g)(x) \geqq x$. For if not, then there are functions f_0, g_0 of T^+, such that $P^\nu B(f_0, g_0)(x) \geqq x$, all x on (0, 1), $\nu = 1, 2, \cdots, N-1$, but for $\nu = N$ there exists an x_0 on (0, 1) for which $P^N B(f_0, g_0)(x_0) < x_0 \leqq P^{N-1}B(f_0, g_0)(x_0)$. Hence $PB(f_0, g_0)(x_0) < B(f_0, g_0)(x_0)$. By Lemma 11, $PB^{-1}B(f_0, g_0)(x_0) = P(f_0, g_0)(x_0) > B^{-1}B(f_0, g_0)(x_0) = x_0 > P^N B(f_0, g_0)(x_0)$. Hence $x_0 > P^{N-1}B(f_0, g_0)(x_0)$.

This concludes the proof of Theorem 13.

COROLLARY 4. *Given functions f, g of T^+ for which some $f^m g^n \cdots (x) \leqq x$ on certain subintervals of* $(0, 1)$, *there do not in general exist functions f', g' of T^+ arbitrarily close to $e(x) = x$ for which $f'^m g'^n \cdots (x) \leqq x$ on these subintervals.*

For if this were true, we could prove F integrally closed under function order. Either $S \geqq e$, or if not, for some f_0, g_0 of T^+ one has $S(f_0, g_0)(x_0) < x_0$. One would then have $S(f_0', g_0')(x_0) < x_0$ with f_0', g_0' arbitrarily close to $e(x) = x$. But then S could not be bounded; indeed if $S^\nu T(f_0', g_0')(x) \geqq x$, we should have $S^\nu(f_0', g_0')(x_0) > T^{-1}(f_0', g_0')(x_0)$. Since T is fixed, this would be a contradiction.

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