

On Orthogonal and Symplectic Matrix Ensembles

Craig A. Tracy¹, Harold Widom²

¹ Department of Mathematics, and Institute of Theoretical Dynamics, University of California, Davis, CA 95616, USA.

E-mail: tracy@itd.ucdavis.edu

² Department of Mathematics, University of California, Santa Cruz, CA 95064, USA.

E-mail: widom@cats.ucsc.edu

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Abstract: The focus of this paper is on the probability, $E_\beta(0; J)$, that a set J consisting of a finite union of intervals contains no eigenvalues for the finite N Gaussian Orthogonal ($\beta = 1$) and Gaussian Symplectic ($\beta = 4$) Ensembles and their respective scaling limits both in the bulk and at the edge of the spectrum. We show how these probabilities can be expressed in terms of quantities arising in the corresponding unitary ($\beta = 2$) ensembles. Our most explicit new results concern the distribution of the largest eigenvalue in each of these ensembles. In the edge scaling limit we show that these largest eigenvalue distributions are given in terms of a particular Painlevé II function.

I. Introduction

In the standard random matrix models of $N \times N$ Hermitian or symmetric matrices the probability density that the eigenvalues lie in infinitesimal intervals about the points x_1, \dots, x_N is given by

$$P_\beta(x_1, \dots, x_N) = C_{N\beta} e^{-\beta \sum V(x_i)} \prod_{j < k} |x_j - x_k|^\beta,$$

where the constant $C_{N\beta}$ is such that the integral of the right side equals 1. In the Gaussian ensembles the potential $V(x)$ equals $x^2/2$ and the cases $\beta = 1, 2$ and 4 correspond to the orthogonal, unitary, and symplectic ensembles, respectively, since the underlying probability distributions are invariant under these groups.

When $\beta = 2$ the polynomials orthogonal with respect to the weight function $e^{-2V(x)}$ play an important role. If $\varphi_i(x)$ ($i = 0, 1, \dots$) is the family of functions obtained by orthonormalizing the sequence $x^i e^{-V(x)}$, then

$$P_2(x_1, \dots, x_N) = \det(K_N(x_i, x_j)) \quad (i, j = 1, \dots, N),$$

where

$$K_N(x, y) := \sum_{i=0}^{N-1} \varphi_i(x) \varphi_i(y). \quad (1)$$

It follows from this that the so-called “ n -level correlation function” is given by

$$R_{n2}(x_1, \dots, x_n) = \det(K_N(x_i, x_j)) \quad (i, j = 1, \dots, n), \tag{2}$$

and the “ n -level cluster function” by

$$T_{n2}(x_1, \dots, x_n) = \sum K_N(x_{\sigma 1}, x_{\sigma 2}) \cdots K_N(x_{\sigma(n-1)}, x_{\sigma n}), \tag{3}$$

where the sum is taken over all cyclic permutations σ of the integers $1, \dots, n$, in some order. (See [9], (5.1.2, 3) and (5.2.14, 15).) The probability $E_2(0; J)$ that no eigenvalues lie in the set J is equal to the Fredholm determinant of the integral operator on J (more precisely, on functions on J) with kernel $K_N(x, y)$. There are analogues of this for scaled limits of these ensembles. If one takes a scaled limit in “the bulk” of the spectrum for the Gaussian unitary ensemble then (2) and (3) become

$$R_{n2}(x_1, \dots, x_n) = \det(S(x_i, x_j)) \quad (i, j = 1, \dots, n), \tag{4}$$

$$T_{n2}(x_1, \dots, x_n) = \sum S(x_{\sigma 1}, x_{\sigma 2}) \cdots S(x_{\sigma(n-1)}, x_{\sigma n}), \tag{5}$$

where

$$S(x, y) := \frac{1}{\pi} \frac{\sin(x - y)}{x - y}.$$

Now $E_2(0, J)$ is the Fredholm determinant of the operator on J with kernel $S(x, y)$. If one scales the same ensemble at “the edge” of the spectrum this is replaced by the “Airy kernel”

$$\frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y},$$

where $\text{Ai}(x)$ is the Airy function, and if one scales the Laguerre ensemble (which corresponds to the choice of potential $V(x) = \frac{1}{2}x - \frac{1}{2}\alpha \log x$) at the edge one obtains the “Bessel kernel”

$$\frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y}, \quad \varphi(x) = J_\alpha(\sqrt{x}), \quad \psi(x) = x\varphi'(x).$$

F.J. Dyson [6] discovered that the introduction of so-called “quaternion determinants” allows one to write down $\beta = 1$ and $\beta = 4$ analogues of (4) and (5). We define $\varepsilon(x) := \frac{1}{2} \text{sgn } x$ and

$$\begin{aligned} S(x) &:= \frac{\sin x}{\pi x}, & DS(x) &:= S'(x), \\ IS(x) &:= \int_0^x S(y) dy, & JS(x) &:= IS(x) - \varepsilon(x), \end{aligned} \tag{6}$$

$$\sigma_1(x, y) := \begin{pmatrix} S(x - y) & DS(x - y) \\ JS(x - y) & S(x - y) \end{pmatrix}, \tag{7}$$

$$\sigma_4(x, y) := \begin{pmatrix} S(2(x - y)) & DS(2(x - y)) \\ IS(2(x - y)) & S(2(x - y)) \end{pmatrix}. \tag{8}$$

If these 2×2 matrices are thought of as quaternions then the analogues of (4) and (5) for $\beta = 1$ and $\beta = 4$ are obtained by replacing $S(x, y)$ by $\sigma_\beta(x, y)$, and by

interpreting the right side of (4) as a quaternion determinant. The right side of (5) is a scalar quaternion, the scalar being equal to $\frac{1}{2}$ times the trace of the right side when it is interpreted as a matrix. It follows from these facts, by an argument to be found in A.7 of [9], that $E_\beta(0; J)$ is equal to the square root of the Fredholm determinant of the integral operator on J with matrix kernel $\sigma_\beta(x, y)$. There are similar but more complicated matrix kernels for the finite N matrix ensembles ([9], Chaps. 6–7).

The focus of this paper is $E_\beta(0; J)$ for $\beta = 1$ and 4 when J is a finite union of intervals, for both finite N and scaled Gaussian ensembles. We shall show how these can all be expressed in terms of quantities arising in the corresponding $\beta = 2$ ensembles. The $E_\beta(0; J)$ are Fredholm determinants of certain matrix-valued kernels, and by manipulating these Fredholm determinants we are able to write them as ordinary scalar determinants whose order depends only on the ensemble and the number of intervals in J , and not on N if the ensemble is finite. The entries of this determinant contain integrals involving the resolvent kernel for the $\beta = 2$ kernel (these are the “quantities” alluded to above). For the scaled Gaussian ensembles our results for general J are given below in (19) and (21) and for the finite N Gaussian ensembles the results for general J are in (33) and (34).

The evaluation of the integrals appearing in our above quoted expressions is a separate matter. In the cases of greatest interest to us— J a finite interval for the scaled ensembles, J a semi-infinite interval for the finite N ensembles—there are systems of differential equations associated with those integrals. The equations are easily solved in the cases of the scaled ensembles and we recover known formulas for the probability in these models of the absence of eigenvalues in an interval. (These are found in [9] as formulas (6.5.19) and (10.7.5).) It then follows from [8] that all these probabilities are expressible in terms of a Painlevé function of fifth kind (see also [2,11]). For finite N , and J the semi-infinite interval (t, ∞) , $E_\beta(0; J)$ is the probability distribution function $F_{N\beta}(t)$ for the largest eigenvalue. In [14] we showed that when $\beta = 2$ this is expressible in terms of a Painlevé function (this time P_{IV}) and we hoped to be able to find representations in the cases $\beta = 1$ and $\beta = 4$ also, but we were unable to solve the associated system of differential equations. However we succeeded for their limits scaled at “the edge” because the equations scale to a system which we can solve. To explain our results, we recall that $F_{N\beta}(2\sigma\sqrt{N} + t)$ tends to the Heaviside function as $N \rightarrow \infty$, where σ is the standard deviation of the Gaussian distribution on the off-diagonal matrix elements [1]. (Our choice of $P_\beta(x_1, \dots, x_N)$ corresponds to a standard deviation $\sigma = 1/\sqrt{2}$ which is the usual choice [9]. We also recall that this result holds for the larger class of so-called Wigner matrices [1], but the results that follow are only known for the Gaussian ensembles.) This says, roughly, that the largest eigenvalue is within $o(1)$ of $2\sigma\sqrt{N}$, and so this is thought of as the right edge of the spectrum. In fact, the largest eigenvalue is within $O(N^{-1/6})$ of $2\sigma\sqrt{N}$ and we consider here the more refined limits

$$F_\beta(s) := \lim_{N \rightarrow \infty} F_{N\beta} \left(2\sigma\sqrt{N} + \frac{\sigma s}{N^{1/6}} \right).$$

(As the notation suggests, F_β is independent of σ .) In earlier work [12] we showed that when $\beta = 2$ this is given in terms of another Painlevé function (P_{II} —see (52) below). Now we shall find representations for $F_1(s)$ and $F_4(s)$ in terms of this same function (see (53) and (54) below). The probability densities $f_\beta(s) = dF_\beta/ds$

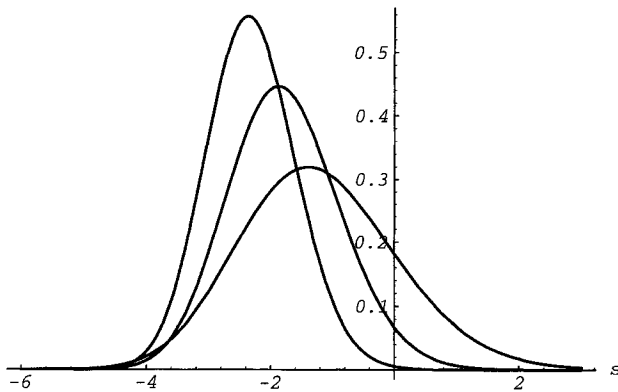


Fig. 1. The probability densities $f_\beta(s) = dF_\beta/ds$, $\beta = 1, 2$, and 4 for the position of the largest eigenvalue in the edge scaling limit. The probability density with the smaller value of β (“higher temperature”) has the larger variance.

are graphed in Fig. 1. Though these results strictly apply only in the limit as the size of the matrices tends to infinity, simulations of finite N GOE, GUE and GSE matrices show that the empirical probability density of the largest eigenvalue is well approximated by f_β , $\beta = 1, 2, 4$, for $N \geq 200$.

Although this paper treats exclusively the Gaussian ensembles, the methods appear quite general and should apply to other ensembles as well. In particular one might expect to be able to express the limiting distribution function for the smallest eigenvalue in the $\beta = 1$ and 4 Laguerre ensembles in terms of a P_{III} function, as is the case for $\beta = 2$ [13].

In Sects. II and III we derive our expressions for $E_\beta(0; J)$ for the bulk-scaled Gaussian ensembles when J is a finite union of intervals. In Sect. IV we specialize to the case of one interval and see how to recover the formulas cited above for the probability of absence of eigenvalues. Section V contains the analogous derivations, for general J , for the finite N ensembles. These are more complicated than in the scaled case, because the expressions for the analogous matrix functions $\sigma_{N\beta}(x, y)$ have “extra” terms. In Sect. VI we derive the differential equations associated with these ensembles when J is a semi-infinite interval, and in Sect. VII we derive the results on the limiting probability distribution for the largest eigenvalues in the orthogonal and symplectic ensembles.

To obtain our formulas for the Fredholm determinants we think of the operators with matrix kernel instead as matrices with operator kernels, and then manipulate their determinants (2×2 determinants with operator entries) in a way the reader might find suspect. These manipulations are, however, quite correct and in the last section we present in detail their justification for the bulk-scaled ensembles. For the basic definitions and properties of operator determinants we refer the reader to Chapter IV of [7], where everything we use will be found.

There is an alternative route to the results of Sect. VII, based on the fact that the limiting probability distributions are Fredholm determinants involving the scaled kernel, the Airy kernel. We did not choose this route because we would have had to give yet another derivation of a set of differential equations for the entries of a scalar determinant, and because we would have had to present yet another justification for the manipulation of the Fredholm determinants, in this case involving the Airy

kernel. This would be more delicate than the justification in Sect. VIII for the sine kernel because the Airy kernel is not as nicely behaved at $-\infty$. (We do use the Airy kernel in our derivation, but only on intervals semi-infinite on the right, where it is very well behaved.)

II. The Scaled Gaussian Orthogonal Ensemble

First, we introduce some terminology. If $K(x, y)$ is the kernel of an integral operator K then we shall speak interchangeably of the determinant *for* $K(x, y)$ or K , and the determinant *of* the operator $I - K$. We denote by S and ε the integral operators on the entire real line \mathbf{R} with kernels $S(x - y)$ and $\varepsilon(x - y)$, respectively (see (6)), and write D for d/dx . Notice that DS has kernel $DS(x - y)$ (fortunately!) and it follows from the evenness of $S(x)$ that $IS(x - y)$ and $JS(x - y)$ are the kernels of the operators εS and $\varepsilon S - \varepsilon$, respectively. We denote by χ the operator of multiplication by $\chi_J(x)$, the characteristic function of J .

$E_1(0; J)^2$ equals the determinant for the kernel $\sigma_1(x, y)$ on J , or, what is the same thing, the determinant for the operator with kernel

$$\chi_J(x) \begin{pmatrix} S(x - y) & DS(x - y) \\ JS(x - y) & S(x - y) \end{pmatrix} \chi_J(y)$$

on \mathbf{R} . This can be represented as the 2×2 operator matrix

$$\begin{pmatrix} \chi S \chi & \chi D S \chi \\ \chi(\varepsilon S - \varepsilon)\chi & \chi S \chi \end{pmatrix}. \tag{9}$$

Since $D\varepsilon = I$, the identity operator, this can be factored as

$$\begin{pmatrix} \chi D & 0 \\ 0 & \chi \end{pmatrix} \begin{pmatrix} \varepsilon S \chi & S \chi \\ (\varepsilon S - \varepsilon)\chi & S \chi \end{pmatrix}. \tag{10}$$

It is a general fact that for operators A and B the determinants for AB and BA are equal. So we can take the factor on the left above and bring it around to the right, combine the two factors χ into one, and find that the above can be replaced by

$$\begin{pmatrix} \varepsilon S \chi D & S \chi \\ (\varepsilon S - \varepsilon)\chi D & S \chi \end{pmatrix}. \tag{11}$$

Subtracting row 1 from row 2 and then adding column 2 to column 1 we see that the determinant for this is the same as that for

$$\begin{pmatrix} \varepsilon S \chi D + S \chi & S \chi \\ -\varepsilon \chi D & 0 \end{pmatrix}, \tag{12}$$

and so equals the determinant of the operator

$$\begin{pmatrix} I - \varepsilon S \chi D - S \chi & -S \chi \\ \varepsilon \chi D & I \end{pmatrix}.$$

Next we subtract column 2, right-multiplied by $\varepsilon \chi D$, from column 1 and then add row 2, left-multiplied by $S \chi$, to row 1. (Column operations are always associated

with right-multiplication and row operations with left.) The result is

$$\begin{pmatrix} I - \varepsilon S\chi D - S\chi + S\chi\varepsilon\chi D & 0 \\ 0 & I \end{pmatrix},$$

and the determinant of this equals

$$\det(I - \varepsilon S\chi D - S\chi + S\chi\varepsilon\chi D). \tag{13}$$

So we have shown that $E_1(0; J)^2$ equals the determinant of the operator with scalar kernel

$$I - \varepsilon S\chi D - S\chi + S\chi\varepsilon\chi D = I - S\chi - S(I - \chi)\varepsilon\chi D,$$

where we used here the fact that ε and S commute. Now $I - S\chi$ is precisely the operator which arises in the bulk-scaled Gaussian unitary ensemble. (In particular its determinant, which is the same as the determinant of $I - \chi S\chi$, is exactly $E_2(0; J)$.) Because J is a finite union of intervals the last summand will turn out to be a finite rank operator, so if we factor out $I - S\chi$, whose determinant we know, then we obtain an operator of the form $I -$ finite rank operator, whose determinant is just a numerical determinant. Of course this factoring out requires introduction of the inverse $(I - S\chi)^{-1}$ which is why the resolvent kernel for $S\chi$ appears in the final result.

Now for the details. We denote by R the resolvent operator for $S\chi$, so that

$$(I - S\chi)^{-1} = I + R,$$

and by $R(x, y)$ its kernel. Observe that this is smooth in x but discontinuous in y . Factoring out $I - S\chi$ and using the fact that determinants multiply, we obtain

$$E_1(0; J)^2 = E_2(0; J) \det(I - (S + RS)(1 - \chi)\varepsilon\chi D). \tag{14}$$

To write the last operator more explicitly we denote the intervals comprising J by (a_{2k-1}, a_{2k}) , $(k = 1, \dots, m)$, and set

$$\varepsilon_k(x) := \varepsilon(x - a_k), \quad \delta_k(y) := \delta(y - a_k), \quad R_k(x) := R(x, a_k),$$

where the last really means

$$\lim_{\substack{y \rightarrow a_k \\ y \in J}} R(x, y).$$

We use the notation $\alpha \otimes \beta$ for the operator with kernel $\alpha(x)\beta(y)$, the most general finite rank operator being a sum of these. For any operator A we have

$$A(\alpha \otimes \beta) = (A\alpha \otimes \beta), \quad (\alpha \otimes \beta)A = (\alpha \otimes A^t\beta), \tag{15}$$

where A^t is the transpose of A .

It is easy to see that the commutator $[\chi, D]$ has the representation

$$[\chi, D] = \sum_{k=1}^{2m} (-1)^k \delta_k \otimes \delta_k,$$

and so

$$\varepsilon[\chi, D] = \sum_{k=1}^{2m} (-1)^k \varepsilon_k \otimes \delta_k. \tag{16}$$

Since $\varepsilon D = I$ we have

$$(1 - \chi)\varepsilon\chi D = (1 - \chi)\varepsilon[\chi, D],$$

and we deduce the representation

$$(S + RS)(1 - \chi)\varepsilon\chi D = \sum_{k=1}^{2m} (-1)^k (S + RS)(1 - \chi)\varepsilon_k \otimes \delta_k.$$

The determinant for this is evaluated using the general formula

$$\det \left(I - \sum_{k=1}^n \alpha_k \otimes \beta_k \right) = \det(\delta_{j,k} - (\alpha_j, \beta_k))_{j,k=1,\dots,n}, \tag{17}$$

where (α_j, β_k) denotes the inner product. (This is an exercise in linear algebra. Its generalization to an arbitrary trace class operator is in [7].) Those that arise in our case are

$$((S + RS)(1 - \chi)\varepsilon_j, \delta_k) = ((1 - \chi)\varepsilon_j, (S + SR^t)\delta_k) = ((1 - \chi)\varepsilon_j, R_k), \tag{18}$$

where we have used the fact that $(S + SR^t)\chi = R$. (This is perhaps most quickly seen by writing $R = \sum_{k=1}^{\infty} (S\chi)^k$ and using $S = S^t$.) Thus we have established the formula

$$E_1(0; J)^2 = E_2(0; J) \det(\delta_{j,k} - (-1)^k ((1 - \chi)\varepsilon_j, R_k))_{j,k=1,\dots,2m}. \tag{19}$$

III. The Scaled Gaussian Symplectic Ensemble

Because of the factor 2 in the arguments of the functions in the matrix (8) we make the variable changes $x \rightarrow x/2, y \rightarrow y/2$, and find that $E_4(0; J/2)$ is the square root of the determinant for operator with kernel

$$\frac{1}{2} \chi_J(x) \begin{pmatrix} S(x-y) & DS(x-y) \\ IS(x-y) & S(x-y) \end{pmatrix} \chi_J(y).$$

Thus the operator matrix (9) of the last section is replaced by

$$\frac{1}{2} \begin{pmatrix} \chi S\chi & \chi DS\chi \\ \chi \varepsilon S\chi & \chi S\chi \end{pmatrix}.$$

Proceeding exactly as before we find that (12) is replaced by

$$\frac{1}{2} \begin{pmatrix} \varepsilon S\chi D + S\chi & S\chi \\ 0 & 0 \end{pmatrix}$$

and (13) by

$$\det \left(I - \frac{1}{2} S\varepsilon\chi D - \frac{1}{2} S\chi \right).$$

But notice that the operator here equals

$$I - S\chi - \frac{1}{2} S\varepsilon[\chi, D].$$

Using this we factor out $I - S\chi$ as before and conclude that the analogue of (14) is

$$E_4(0; J/2)^2 = E_2(0; J) \det \left(I - \frac{1}{2}(I + R)S\varepsilon[\chi, D] \right). \tag{20}$$

Using (16) once again we see that

$$(S + RS)\varepsilon[\chi, D] = \sum_{k=1}^{2m} (-1)^k (S + RS)\varepsilon_k \otimes \delta_k,$$

and we obtain the analogue of (19),

$$E_4(0; J/2)^2 = E_2(0; J) \det \left(\delta_{j,k} - \frac{1}{2}(-1)^k (\varepsilon_j, R_k) \right)_{j,k=1,\dots,2m}, \tag{21}$$

where we used the fact that in the inner product we may use the identity $\delta_k = \chi\delta_k$ since all evaluations are done by taking the limit from within the interval.

IV. The Case $J = (-t, t)$

In this case we have $m = 1, a_1 = -t, a_2 = t$, and

$$\varepsilon_1(x) = \varepsilon(x + t), \quad \varepsilon_2(x) = \varepsilon(x - t), \quad R_1(x) = R(x, -t), \quad R_2(x) = R(x, t).$$

If we define

$$\mathcal{I}_\pm := \int_{-\infty}^{\infty} R(x, t) \varepsilon(x \mp t) dx,$$

then

$$-(\varepsilon_1, R_1) = (\varepsilon_2, R_2) = \mathcal{I}_+, \quad -(\varepsilon_2, R_1) = (\varepsilon_1, R_2) = \mathcal{I}_-. \tag{22}$$

The first equalities above used the evenness of R (i.e., the fact $R(-x, y) = R(x, -y)$) and the oddness of ε .

To evaluate these integrals we find expressions for their derivatives with respect to t . Observe that $R(x, y)$ depends on the interval J as well as x and y . Formulas (2.9) and (2.22) of [14] give, for $y \in J$, (see also Lemma 2 in [11])

$$\frac{\partial}{\partial t} R(x, y) = R(x, -t)R(-t, y) + R(x, t)R(t, y),$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) R(x, y) = R(x, -t)R(-t, y) - R(x, t)R(t, y),$$

from which it follows that

$$\frac{\partial}{\partial t} R(x, t) = 2R(x, -t)R(-t, t) - \frac{\partial}{\partial x} R(x, t).$$

Hence, applying the product formula to the integrand and using $\partial\varepsilon(x \mp t)/\partial t = \mp\delta(x \mp t)$, we find that

$$\frac{d}{dt} \mathcal{I}_\pm = \int_{-\infty}^{\infty} \left[2R(x, -t)R(-t, t) - \frac{\partial}{\partial x} R(x, t) \right] \varepsilon(x \mp t) dx \mp R(\pm t, t).$$

Integrating by parts the term involving $\partial R(x, t)/\partial x$ and using parity give

$$\frac{d}{dt} \mathcal{I}_{\pm} = -2R(-t, t) \mathcal{I}_{\mp} + R(\pm t, t) \mp R(\pm t, t) .$$

Adding these two identities gives for $\mathcal{I}_+ + \mathcal{I}_-$ the differential equation

$$\frac{d}{dt} (\mathcal{I}_+ + \mathcal{I}_-) = -2R(-t, t) (\mathcal{I}_+ + \mathcal{I}_-) + 2R(-t, t) ,$$

whose general solution is

$$1 + c e^{-2\int_0^t R(-\tau, \tau) d\tau} .$$

(The function R in the integral is the resolvent kernel for the interval $(-\tau, \tau)$.) Since $\mathcal{I}_{\pm}(0) = 0$ we have $c = -1$ and so

$$\mathcal{I}_+ + \mathcal{I}_- = 1 - e^{-2\int_0^t R(-\tau, \tau) d\tau} . \tag{23}$$

Similarly if we subtract the two identities for $d\mathcal{I}_{\pm}/dt$ we obtain

$$\frac{d}{dt} (\mathcal{I}_+ - \mathcal{I}_-) = 2R(-t, t) (\mathcal{I}_+ - \mathcal{I}_-) - 2R(-t, t) ,$$

whose solution is

$$\mathcal{I}_+ - \mathcal{I}_- = 1 - e^{2\int_0^t R(-\tau, \tau) d\tau} . \tag{24}$$

Because of (22) these relations determine the inner products (ε_j, R_k) which arise in (21). For the determinant in (19) we need the inner products $((1 - \chi)\varepsilon_j, R_k)$. But observe that

$$\begin{aligned} ((1 - \chi)\varepsilon_1, R_2) &= ((1 - \chi)\varepsilon_2, R_2) = \frac{1}{2} \int_c R(x, t) \operatorname{sgn} x dx \\ &= \frac{\mathcal{I}_+ + \mathcal{I}_-}{2} = \frac{1 - e^{-2\int_0^t R(-\tau, \tau) d\tau}}{2} , \end{aligned}$$

$$((1 - \chi)\varepsilon_1, R_1) = ((1 - \chi)\varepsilon_2, R_1) = -\frac{\mathcal{I}_+ + \mathcal{I}_-}{2} = \frac{e^{-2\int_0^t R(-\tau, \tau) d\tau} - 1}{2} .$$

So we have all the inner products we need.

It follows from the last displayed formulas that the determinant on the right side of (19) equals

$$\begin{vmatrix} \frac{1 + e^{-2\int_0^t R(-\tau, \tau) d\tau}}{2} & \frac{1 - e^{-2\int_0^t R(-\tau, \tau) d\tau}}{2} \\ \frac{1 - e^{-2\int_0^t R(-\tau, \tau) d\tau}}{2} & \frac{1 + e^{-2\int_0^t R(-\tau, \tau) d\tau}}{2} \end{vmatrix} = e^{-2\int_0^t R(-\tau, \tau) d\tau} . \tag{25}$$

Now in this case $J = (-t, t)$ we have

$$\frac{d}{dt} \log E_2(0; J) = -2R(t, t) ,$$

and so

$$E_2(0; J) = e^{-2\int_0^t R(\tau, \tau) d\tau} . \tag{26}$$

Thus from (19) and this fact we deduce

$$E_1(0; J) = e^{-\int_0^J [R(\tau, \tau) + R(-\tau, \tau)] d\tau} . \tag{27}$$

For the symplectic ensemble in this special case we use (22), (23) and (24) to evaluate the inner products appearing in the determinant on the right side of (21) and find that it equals

$$\begin{aligned} \left| \begin{array}{cc} 1 - \frac{\mathcal{I}_+}{2} & \frac{\mathcal{I}_-}{2} \\ \frac{\mathcal{I}_-}{2} & 1 - \frac{\mathcal{I}_+}{2} \end{array} \right| &= \left(1 - \frac{\mathcal{I}_+}{2} \right)^2 - \left(\frac{\mathcal{I}_-}{2} \right)^2 \\ &= \left(1 - \frac{\mathcal{I}_+ + \mathcal{I}_-}{2} \right) \left(1 - \frac{\mathcal{I}_+ - \mathcal{I}_-}{2} \right) = \left(\frac{e^{-\int_0^J R(-\tau, \tau) d\tau} + e^{\int_0^J R(-\tau, \tau) d\tau}}{2} \right)^2 , \end{aligned} \tag{28}$$

by (23) and (24). Hence, from (21) and (26),

$$E_4(0; J/2) = \frac{1}{2} \left(e^{-\int_0^J [R(\tau, \tau) + R(-\tau, \tau)] d\tau} + e^{-\int_0^J [R(\tau, \tau) - R(-\tau, \tau)] d\tau} \right) . \tag{29}$$

To see that (27) and (29) are just the known formulas (6.5.19) and (10.7.5) of [9] we recall the notations

$$D(t) := \prod_{i=0}^{\infty} (1 - \lambda_i), \quad D_+(t) := \prod_{i=0}^{\infty} (1 - \lambda_{2i}), \quad D_-(t) := \prod_{i=0}^{\infty} (1 - \lambda_{2i+1}),$$

where $\lambda_0 > \lambda_1 > \dots$ are the eigenvalues of the kernel $S(x - y)$ on $(-t, t)$. Of course $D(t)$ is the determinant for $S(x - y)$ while $D_{\pm}(t)$ are the determinants for the kernels

$$S_{\pm}(x, y) := \frac{1}{2} (S(x - y) \pm S(x + y)) .$$

The resolvent kernels for these are

$$R_{\pm}(x, y) = \frac{1}{2} (R(x, y) \pm R(x, -y)) ,$$

and from these we obtain the integral representations

$$D_{\pm}(t) = e^{-\int_0^J [R(\tau, \tau) \pm R(-\tau, \tau)] d\tau} \tag{30}$$

analogous to (26). Hence (27) and (29) are (6.5.19) and (10.7.5), respectively, of [9].

We mention here that the left side of (24) equals

$$-\int_J R(x, t) dx ,$$

and so in view of (30) the formula can be rewritten

$$\frac{D_-(t)}{D_+(t)} = 1 + \int_J R(x, t) dx .$$

This is equivalent to identity (A.16.6) of [9] and, although this might not be obvious on first comparing the two, our derivation of it has elements in common with the derivation, attributed to M. Gaudin, given in [9].

Finally, we mention that (26) and (30) may be expressed in terms of Painlevé V following [8]. (See [11] for a simplified treatment.)

V. The Finite N Gaussian Ensembles

The analogue of the matrix kernel (7) for the finite N GOE is given by (6.3.8) of [9] and so we can write down the analogue of (9). We shall denote here by S the operator with kernel $K_N(x, y)$ given by (1). Recall that the determinant for this operator on J equals $E_2(0; J)$ for the finite N GUE. We also write

$$\varphi(x) = \left(\frac{N}{2}\right)^{1/4} \varphi_N(x), \quad \psi(x) = \left(\frac{N}{2}\right)^{1/4} \varphi_{N-1}(x).$$

Then for N even (this case is slightly simpler) the analogue of (9) is

$$\chi \begin{pmatrix} S + \psi \otimes \varepsilon \varphi & SD - \psi \otimes \varphi \\ \varepsilon S - \varepsilon + \varepsilon \psi \otimes \varepsilon \varphi & S + \varepsilon \varphi \otimes \psi \end{pmatrix} \chi. \tag{31}$$

(See [9], Sect. 6.3.) Now we have to be careful because S does not commute with ε and D . But we have the simple relations

$$[S, D] = \varphi \otimes \psi + \psi \otimes \varphi, \quad [\varepsilon, S] = -\varepsilon \varphi \otimes \varepsilon \psi - \varepsilon \psi \otimes \varepsilon \varphi. \tag{32}$$

The first follows from (2.20) of [14] (where finite N GUE is called the ‘‘Hermite ensemble’’), and the second follows from the first upon left- and right-multiplying by ε . (We used (15) here and the antisymmetry of ε . We shall make similar use of this below.) We write the first row of the matrix in (31) as

$$D(\varepsilon S + \varepsilon \psi \otimes \varepsilon \varphi, \varepsilon SD - \varepsilon \psi \otimes \varphi) = D(S\varepsilon - \varepsilon \varphi \otimes \varepsilon \psi, S + \varepsilon \varphi \otimes \psi),$$

where we applied (32). Using this, and applying (32) now to the lower left corner, we find that the analogue of the second matrix in (10) is

$$\begin{pmatrix} (S\varepsilon - \varepsilon \varphi \otimes \varepsilon \psi)\chi & (S + \varepsilon \varphi \otimes \psi)\chi \\ (S\varepsilon - \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi)\chi & (S + \varepsilon \varphi \otimes \psi)\chi \end{pmatrix}.$$

Now we move that matrix $\begin{pmatrix} \chi D & 0 \\ 0 & \chi \end{pmatrix}$ around to the right and find the analogue of the right side of (11),

$$\begin{pmatrix} (S\varepsilon - \varepsilon \varphi \otimes \varepsilon \psi)\chi D & (S + \varepsilon \varphi \otimes \psi)\chi \\ (S\varepsilon - \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi)\chi D & (S + \varepsilon \varphi \otimes \psi)\chi \end{pmatrix}.$$

The same row and column operations as before reduce this to

$$\begin{pmatrix} (S\varepsilon - \varepsilon \varphi \otimes \varepsilon \psi)\chi D + (S + \varepsilon \varphi \otimes \psi)\chi & (S + \varepsilon \varphi \otimes \psi)\chi \\ -\varepsilon \chi D & 0 \end{pmatrix},$$

the analogue of (12). We complete the computation just as before and find that the determinant for this equals the determinant of the scalar operator

$$\begin{aligned}
 & I - S\varepsilon\chi D + (\varepsilon\varphi \otimes \varepsilon\psi)\chi D + (S + \varepsilon\varphi \otimes \psi)\chi(\varepsilon\chi D - I) \\
 &= I - S\chi - S(1 - \chi)\varepsilon\chi D - \varepsilon\varphi \otimes \chi\psi - (\varepsilon\varphi \otimes \psi)(1 - \chi)\varepsilon\chi D.
 \end{aligned}$$

Factoring out $I - S\chi$ shows that the determinant of the above equals $E_2(0, J)$ times the determinant of

$$I - (S + RS)(1 - \chi)\varepsilon\chi D - Q_\varepsilon \otimes \chi\psi - Q_\varepsilon \otimes \psi(1 - \chi)\varepsilon\chi D,$$

where

$$Q_\varepsilon := (I - S\chi)^{-1}\varepsilon\varphi,$$

and we have used the same notation R as before for the resolvent operator for $S\chi$. Using (16) and the general fact $(\alpha \otimes \beta)(\gamma \otimes \delta) = (\beta, \gamma)\alpha \otimes \delta$ we see that the above operator equals

$$\begin{aligned}
 & I - \sum_{k=1}^{2m} (-1)^k (S + RS)(1 - \chi)\varepsilon_k \otimes \delta_k - Q_\varepsilon \otimes \chi\psi \\
 & - \sum_{k=1}^{2m} (-1)^k (\psi, (1 - \chi)\varepsilon_k) Q_\varepsilon \otimes \delta_k. \tag{33}
 \end{aligned}$$

Thus $E_1(0; J)^2/E_2(0; J)$ equals the determinant of this operator.

The analogue of the matrix kernel (8) for finite N GSE is given by (7.1.5) of [9] where now N must be odd.¹ Because of the factor $\sqrt{2}$ in the arguments of the functions in the matrix (7.1.5) of [9], we make the change of variables $x \rightarrow x/\sqrt{2}$, $y \rightarrow y/\sqrt{2}$ and find that $E_4(0; J/\sqrt{2})$ is the square root of the determinant for

$$\frac{1}{2}\chi \begin{pmatrix} S + \psi \otimes \varepsilon\varphi & SD - \psi \otimes \varphi \\ \varepsilon S + \varepsilon\psi \otimes \varepsilon\varphi & S + \varepsilon\varphi \otimes \psi \end{pmatrix} \chi.$$

Proceeding analogously to finite N GOE leads to the following formula for the operator whose determinant is $E_4(0; J/\sqrt{2})^2/E_2(0; J)$:

$$I - \frac{1}{2} \sum_{k=1}^{2m} (-1)^k (S + RS)\varepsilon_k \otimes \delta_k - Q_\varepsilon \otimes \chi\psi - \frac{1}{2} \sum_{k=1}^{2m} (-1)^k (\psi, \varepsilon_k) Q_\varepsilon \otimes \delta_k. \tag{34}$$

VI. The Case $J = (t, \infty)$

Now J has the end-point $a_1 = t$ and $a_2 = \infty$, and we write $\delta_t, \delta_\infty, \varepsilon_t, \varepsilon_\infty, R_t$ and R_∞ for the quantities $\delta_k, \varepsilon_k, R_k$ ($k = 1, 2$) of the last sections. Note that

$$\varepsilon_\infty = -\frac{1}{2}, \quad (1 - \chi)\varepsilon_t = (1 - \chi)\varepsilon_\infty = -\frac{1}{2}(1 - \chi), \quad R_\infty = 0.$$

¹ In finite $N = 2n + 1$ GSE the matrices are $2n \times 2n$ Hermitian matrices with each eigenvalue doubly degenerate [9].

With this notation the operators (33) and (34) become

$$I - Q_\varepsilon \otimes \chi\psi - \frac{1}{2}[(S + RS)(1 - \chi) + (\psi, (1 - \chi)) Q_\varepsilon] \otimes (\delta_t - \delta_\infty), \tag{35}$$

$$I - Q_\varepsilon \otimes \chi\psi + \frac{1}{2}[(S + RS)\varepsilon_t + (\psi, \varepsilon_t) Q_\varepsilon] \otimes \delta_t + \frac{1}{4}[(S + RS)1 + (\psi, 1) Q_\varepsilon] \otimes \delta_\infty, \tag{36}$$

respectively. Both operators are of the form

$$I - \sum_{k=1}^n \alpha_k \otimes \beta_k, \tag{37}$$

so to evaluate their determinants using (17) we have many inner products to evaluate. We shall introduce several new quantities now and express all the inner products, and therefore the determinants, in terms of them. Then we shall write down systems of linear differential equations (in the variable t) which in principle determine these quantities.

First, there are

$$\begin{aligned} Q &:= (I - S\chi)^{-1}\varphi, & P &:= (I - S\chi)^{-1}\psi, \\ Q_\varepsilon &:= (I - S\chi)^{-1}\varepsilon\varphi, & P_\varepsilon &:= (I - S\chi)^{-1}\varepsilon\psi, \end{aligned}$$

the third of which we have already met. We use small letters to denote the values of these functions at $x = t$:

$$q = Q(t), \quad p = P(t), \quad q_\varepsilon = Q_\varepsilon(t), \quad p_\varepsilon = P_\varepsilon(t). \tag{38}$$

(The functions q and p play important roles in the investigation of $E_2(0; J)$ [14] and we think of them here as known.)

Next, there are the inner products

$$\begin{aligned} u_\varepsilon &:= (Q, \chi\varepsilon\varphi) = (Q_\varepsilon, \chi\varphi), & v_\varepsilon &:= (Q, \chi\varepsilon\psi) = (P_\varepsilon, \chi\psi), \\ \tilde{v}_\varepsilon &:= (P, \chi\varepsilon\varphi) = (Q_\varepsilon, \chi\psi), & w_\varepsilon &:= (P, \chi\varepsilon\psi) = (P_\varepsilon, \chi\psi), \end{aligned}$$

the last four being analogous to (2.4)–(2.5) of [14] with x^j replaced by ε . Our first system of differential equations (in which q and p appear as coefficients) will connect these with q_ε and p_ε .

Finally there are two triples of integrals

$$\begin{aligned} \mathcal{R}_1 &:= \int_{-\infty}^t R(x, t) dx, & \mathcal{P}_1 &:= \int_{-\infty}^t P(x) dx, & \mathcal{Q}_1 &:= \int_{-\infty}^t Q(x) dx, \\ \mathcal{R}_4 &:= \int_{-\infty}^{\infty} \varepsilon_t(x) R(x, t) dx, & \mathcal{P}_4 &:= \int_{-\infty}^{\infty} \varepsilon_t(x) P(x) dx, & \mathcal{Q}_4 &:= \int_{-\infty}^{\infty} \varepsilon_t(x) Q(x) dx. \end{aligned}$$

(The subscripts 1 and 4 indicate that these arise in GOE and GSE, respectively.) We shall find systems of differential equations for each of these triples.

The determinants of (35) and (36). We consider the GOE operator (35) first. If we set

$$a_1 := (\psi, 1 - \chi),$$

then the operator is of the form (37) with $n = 2$ and

$$\alpha_1 = Q_\varepsilon, \quad \alpha_2 = \frac{1}{2}[(S + RS)(1 - \chi) + a_1 Q_\varepsilon], \quad \beta_1 = \chi\psi, \quad \beta_2 = \delta_t - \delta_\infty.$$

We also set

$$c_\varphi := \varepsilon\varphi(\infty) = \frac{1}{2} \int_{-\infty}^{\infty} \varphi(x) dx, \quad c_\psi := \varepsilon\psi(\infty) = \frac{1}{2} \int_{-\infty}^{\infty} \psi(x) dx. \quad (39)$$

For N even a computation gives

$$c_\varphi = (\pi N)^{1/4} 2^{-3/4 - N/2} \frac{(N!)^{1/2}}{(N/2)!}.$$

Since $Q_\varepsilon = \varepsilon\varphi + S\chi(I - S\chi)^{-1}\varepsilon\varphi$, we have $Q_\varepsilon(\infty) = c_\varphi$. Therefore, with the notations introduced above,

$$(\alpha_1, \beta_1) = \tilde{v}_\varepsilon, \quad (\alpha_1, \beta_2) = q_\varepsilon - c_\varphi.$$

To compute the inner products involving α_2 we use the fact $(S + SR^t)\chi = R$, as in the derivation of (18), to write

$$\begin{aligned} ((S + RS)(1 - \chi), \chi\psi) &= (1 - \chi, R\psi) = (1 - \chi, P - \psi) = \mathcal{P}_1 - a_1, \\ ((S + RS)(1 - \chi), \delta_t) &= (1 - \chi, R_t) = \mathcal{R}_1, \\ ((S + RS)(1 - \chi), \delta_\infty) &= (1 - \chi, R_\infty) = 0. \end{aligned}$$

Using these, we find that

$$(\alpha_2, \beta_1) = \frac{1}{2}(\mathcal{P}_1 - a_1 + a_1 \tilde{v}_\varepsilon), \quad (\alpha_2, \beta_2) = \frac{1}{2}(\mathcal{R}_1 + a_1 q_\varepsilon - a_1 c_\varphi).$$

The GSE operator (36) has the form (37) with $n = 3$. But notice that

$$((S + RS)\varepsilon_t, \delta_\infty) = (\varepsilon_t, R_\infty) = 0, \quad ((S + RS)1, \delta_\infty) = (1, R_\infty) = 0,$$

and $(Q_\varepsilon, \delta_\infty) = c_\varphi = 0$ because for GSE N is odd and so φ is an odd function. Thus the contribution to the determinant of the last term in (36) is 0 and we may discard it. The resulting operator is of the form (37) with $n = 2$ and

$$\alpha_1 = Q_\varepsilon, \quad \alpha_2 = -\frac{1}{2}[(S + RS)\varepsilon_t + a_4 Q_\varepsilon], \quad \beta_1 = \chi\psi, \quad \beta_2 = \delta_t,$$

where we have set

$$a_4 := (\psi, \varepsilon_t).$$

The inner products are computed as for GOE above and we find that now

$$\begin{aligned} (\alpha_1, \beta_1) &= \tilde{v}_\varepsilon, \quad (\alpha_1, \beta_2) = q_\varepsilon, \\ (\alpha_2, \beta_1) &= -\frac{1}{2}(\mathcal{P}_4 - a_4 + a_4 \tilde{v}_\varepsilon), \quad (\alpha_2, \beta_2) = -\frac{1}{2}(\mathcal{R}_4 + a_4 q_\varepsilon). \end{aligned}$$

Thus the determinants of the operators (35) and (36) are expressible in terms of the constants a_1, a_4 and c_φ and the as yet to be determined quantities $\tilde{v}_\varepsilon, q_\varepsilon, \mathcal{P}_1, \mathcal{R}_1$,

\mathcal{P}_4 , and \mathcal{R}_4 . The determinants are given by

$$(1 - \tilde{v}_\varepsilon) \left(1 - \frac{1}{2} \mathcal{R}_1 \right) - \frac{1}{2} (q_\varepsilon - c_\varphi) \mathcal{P}_1, \tag{40}$$

$$(1 - \tilde{v}_\varepsilon) \left(1 + \frac{1}{2} \mathcal{R}_4 \right) + \frac{1}{2} q_\varepsilon \mathcal{P}_4, \tag{41}$$

respectively. (The constants a_1 and a_4 drop out, as we see.)

The first set of differential equations. The derivation will not be quite self-contained because we shall refer to [14] for some results derived there. First we have the analogues of (2.15)–(2.18) of [14],

$$u'_\varepsilon = -q q_\varepsilon, \quad v'_\varepsilon = -q p_\varepsilon, \quad \tilde{v}'_\varepsilon = -p q_\varepsilon, \quad w'_\varepsilon = -p p_\varepsilon, \tag{42}$$

which are proved in exactly the same way. (The primes denote d/dt .) To derive formulas for q'_ε and p'_ε we use the displayed formula between (2.27) and (2.28) of [14], which in this case gives

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \rho(x, y) = -Q(x) \cdot (\chi P)(y) - P(x) \cdot (\chi Q)(y), \tag{43}$$

where $\rho(x, y)$ is the kernel of $(I - S\chi)^{-1}$, in other words $\delta(x - y) + R(x, y)$. It follows from the above that

$$q'_\varepsilon = \frac{d}{dt} \int \rho(t, y) \varepsilon \varphi(y) dy = - \int \frac{\partial}{\partial y} \rho(t, y) \varepsilon \varphi(y) dy - q(\chi P, \varepsilon \varphi) - p(\chi Q, \varepsilon \varphi).$$

The first term on the right side equals

$$\int \rho(t, y) \varphi(y) dy = q.$$

We treat p_ε similarly, and we find that we have derived the equations.

$$q'_\varepsilon = q - q \tilde{v}_\varepsilon - p u_\varepsilon, \quad p'_\varepsilon = p - q w_\varepsilon - p v_\varepsilon. \tag{44}$$

For the boundary conditions at $t = \infty$, observe that the four functions $u_\varepsilon, v_\varepsilon, \tilde{v}_\varepsilon$ and w_ε all vanish there, whereas

$$q_\varepsilon(\infty) = c_\varphi, \quad p_\varepsilon(\infty) = c_\psi.$$

One of these always vanishes, the first if N is odd and the second if N is even.

It is easy to derive a first integral for our system of equations. Using the equations we find that

$$\begin{aligned} (p_\varepsilon q_\varepsilon)' &= p_\varepsilon q (1 - \tilde{v}_\varepsilon) - p_\varepsilon p u_\varepsilon + q_\varepsilon (1 - v_\varepsilon) - q_\varepsilon q w_\varepsilon \\ &= v'_\varepsilon (1 - \tilde{v}_\varepsilon) - w'_\varepsilon u_\varepsilon + \tilde{v}_\varepsilon (1 - v_\varepsilon) - u'_\varepsilon w_\varepsilon \\ &= -((1 - v_\varepsilon)(1 - \tilde{v}_\varepsilon))' - (u_\varepsilon w_\varepsilon)', \end{aligned}$$

and so, since all quantities vanish at ∞ ,

$$p_\varepsilon q_\varepsilon = 1 - (1 - v_\varepsilon)(1 - \tilde{v}_\varepsilon) - u_\varepsilon w_\varepsilon.$$

The second set of equations. It follows from (43) that for $y \in J$,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) R(x, y) = -Q(x)P(y) - P(x)Q(y),$$

and so

$$\frac{\partial}{\partial t} R(x, t) = -\frac{\partial}{\partial x} R(x, t) - pQ(x) - qP(x), \tag{45}$$

and from this we obtain

$$\mathcal{R}'_1 = R(t, t) + \int_{-\infty}^t \left[-\frac{\partial}{\partial x} R(x, t) - pQ(x) - qP(x) \right] dx.$$

This gives our first equation,

$$\mathcal{R}'_1 = -p \mathcal{Q}_1 - q \mathcal{P}_1. \tag{46}$$

By (2.10) of [14] we have

$$\frac{\partial Q}{\partial t} = -R(x, t)q,$$

and so

$$\mathcal{Q}'_1 = q - q \int_{-\infty}^t R(x, t) dx.$$

We treat \mathcal{P}_1 similarly, and so we have our other two equations,

$$\mathcal{Q}'_1 = q(1 - \mathcal{R}_1), \quad \mathcal{P}'_1 = p(1 - \mathcal{R}_1). \tag{47}$$

At $t = \infty$ our functions have the values

$$\mathcal{R}_1(\infty) = 0, \quad \mathcal{Q}_1(\infty) = 2c_\varphi, \quad \mathcal{P}_1(\infty) = 2c_\psi. \tag{48}$$

The system (46), (47) has a first integral. If we multiply the first equation by \mathcal{R}_1 and use the second we obtain

$$\mathcal{R}_1 \mathcal{R}'_1 = (\mathcal{P}'_1 - p)\mathcal{Q}_1 + (\mathcal{Q}'_1 - q)\mathcal{P}_1 = (\mathcal{Q}_1 \mathcal{P}_1)' + \mathcal{R}'_1.$$

Integrating gives

$$\frac{1}{2} \mathcal{R}_1^2 = \mathcal{Q}_1 \mathcal{P}_1 + \mathcal{R}_1. \tag{49}$$

(Again the constant of integration is 0 because either c_φ or c_ψ is 0.) Differentiating (46) and using (47) again give

$$\mathcal{R}''_1 = -p' \mathcal{Q}_1 - q' \mathcal{P}_1 + 2pq(\mathcal{R}_1 - 1). \tag{50}$$

Solving Eqs. (46) and (50) for \mathcal{Q}_1 and \mathcal{P}_1 and substituting the results into (49), we obtain a second-order differential equation for \mathcal{R}_1 :

$$\begin{aligned} -\left(\frac{1}{2} \mathcal{R}_1^2 - \mathcal{R}_1 \right) (q'p - p'q)^2 &= (p \mathcal{R}''_1 - p' \mathcal{R}'_1 + 2p^2q(\mathcal{R}_1 - 1)) \\ &\quad \times (q \mathcal{R}''_1 - q' \mathcal{R}'_1 + 2q^2p(\mathcal{R}_1 - 1)). \end{aligned}$$

To find a differential equation for \mathcal{P}_1 we use the second relation of (47) to express \mathcal{R}_1 in terms of \mathcal{P}_1 , then use (46) to express \mathcal{Q}_1 in terms of \mathcal{P}_1 , and then

substitute these results into (49). What results is the equation

$$\mathcal{P}_1 \left(\left(\frac{\mathcal{P}'_1}{p} \right)' - q\mathcal{P}_1 \right) = \frac{p}{2} \left(\left(\frac{\mathcal{P}'_1}{p} \right)^2 - 1 \right).$$

The third set of equations. To obtain the equations for \mathcal{R}_4 , \mathcal{Q}_4 and \mathcal{P}_4 we proceed almost exactly as in the last section, replacing the domain of integration $(-\infty, t)$ by $(-\infty, \infty)$, and inserting the factor $\varepsilon(x - t)$ in the integrands. The only difference is that when we differentiate the integrals we apply the product formula to the integrands. What results is the system

$$\mathcal{R}'_4 = -p\mathcal{Q}_4 - q\mathcal{P}_4, \quad \mathcal{Q}'_4 = -q(\mathcal{R}_4 + 1), \quad \mathcal{P}'_4 = -p(\mathcal{R}_4 + 1). \tag{51}$$

The solutions of this system are obtained from the solutions of the last by simply changing their signs. But notice that the values at $t = \infty$ are now given by

$$\mathcal{R}_4(\infty) = 0, \quad \mathcal{Q}_4(\infty) = -c_\varphi, \quad \mathcal{P}_4(\infty) = -c_\psi.$$

VII. Scaling GOE and GSE at the Edge

The goal of this section is the computation of the limiting probability distribution functions for the largest eigenvalue in the finite N Gaussian orthogonal and symplectic ensembles. The probability distribution functions for finite N GOE, GUE and GSE are precisely the functions $E_\beta(0, (t, \infty))$ of Sect. VI with $\beta = 1, 2$ and 4 , respectively. We denote them by $F_{N\beta}(t)$, and limits we are interested in are

$$F_\beta(s) := \lim_{N \rightarrow \infty} F_{N\beta} \left(\sqrt{2N} + \frac{s}{\sqrt{2N}^{1/6}} \right).$$

In [12] we showed that the limit exists when $\beta = 2$ and is given by

$$F_2(s) = \exp \left(- \int_s^\infty (x - s) q(x)^2 dx \right), \tag{52}$$

where q is the P_{II} function determined by the differential equation

$$q'' = sq + 2q^3$$

together with the condition $q(s) \sim \text{Ai}(s)$ as $s \rightarrow \infty$. (For more details on this solution see [3, 4, 12].) We shall show here that the limits exist for $\beta = 1$ and 4 also, and that

$$F_1(s)^2 = F_2(s) e^{-\int_s^\infty q(x) dx}, \tag{53}$$

$$F_4(s/\sqrt{2})^2 = F_2(s) \left(\frac{e^{\frac{1}{2} \int_s^\infty q(x) dx} + e^{-\frac{1}{2} \int_s^\infty q(x) dx}}{2} \right)^2. \tag{54}$$

(Note the similarity to formulas (25) and (28).)

The reader must see our notational difficulty: q denotes both a Painlevé function and the function defined by (38) of the last section. We resolve this difficulty by

denoting the latter function here by q_N (and denoting the function p of the last section by p_N), while retaining the notation q for the Painlevé II function.

We denote our scaling transformation by τ , so that

$$\tau(x) := \sqrt{2N} + \frac{x}{\sqrt{2N^{1/6}}}.$$

We think of s as fixed, and the functions $q_N, p_N, q_\varepsilon, u_\varepsilon, \dots, \mathcal{P}_4$ of the last section as being associated with $t = \tau(s)$. We shall show that these functions of s (perhaps after normalization) tend to limits as $N \rightarrow \infty$ and that these limits satisfy systems of differential equations which are solvable in terms of the P_{II} function q . Substituting the values of these limits into (40) and (41) will give (53) and (54). Everything will be a consequence of the following:

- (i) $\lim_{N \rightarrow \infty} N^{-1/6} q_N = \lim_{N \rightarrow \infty} N^{-1/6} p_N = q$.
- (ii) The limits $\lim_{N \rightarrow \infty} u_\varepsilon$ and $\lim_{N \rightarrow \infty} \tilde{v}_\varepsilon$ exist and are equal.
- (iii) The limits of q_ε and $\mathcal{R}_1, \dots, \mathcal{P}_4$ all exist. The limits of \mathcal{Q}_1 and \mathcal{P}_1 differ by a constant as do the limits of \mathcal{Q}_4 and \mathcal{P}_4 .
- (iv) All of the above limits hold uniformly for bounded s .

These will be established below, but suppose for the moment that they are true. Denote the common limit in (ii) by \bar{u} , the limit of q_ε by \bar{q} and the limits of $\mathcal{R}_1, \dots, \mathcal{P}_4$ by $\bar{\mathcal{R}}_1, \dots, \bar{\mathcal{P}}_4$.

Let us rewrite the first equations of (42) and (44) using our present notation and letting the prime now denote $\frac{d}{ds}$ ($= \frac{d}{dt} / \sqrt{2N^{1/6}}$):

$$u'_\varepsilon = -\frac{q_N}{\sqrt{2N^{1/6}}} q_\varepsilon, \quad q'_\varepsilon = \frac{1}{\sqrt{2N^{1/6}}} (q_N - q_N \tilde{v}_\varepsilon - p_N u_\varepsilon).$$

Taking the limits as $N \rightarrow \infty$, using (i) and (ii), gives the system

$$\bar{u}' = -\frac{q}{\sqrt{2}} \bar{q}, \quad \bar{q}' = \frac{q}{\sqrt{2}} (1 - 2\bar{u}).$$

(The interchange of the limit and the derivative is justified by the uniformity of convergence of these derivatives.) We remind the reader that when we apply our finite N results to GOE and GSE we restrict N to even or odd values, respectively, and this affects the boundary condition at $s = \infty$. In fact, we have

$$\bar{u}(\infty) = 0, \quad \bar{q}(\infty) = \begin{cases} 1/\sqrt{2} & \text{in GOE,} \\ 0 & \text{in GSE.} \end{cases}$$

The reason for the first is that u_ε vanishes at ∞ and the reason for the second is that $q_\varepsilon(\infty) = c_\varphi$, which vanishes when N is odd and can be shown to have the limit $1/\sqrt{2}$ as $N \rightarrow \infty$ through even values. Introduction of

$$\mu := \int_s^\infty q(x) dx$$

as a new independent variable reduces our system of equations for \bar{u} and \bar{q} to one with constant coefficients which is easily solved. We find that

$$\begin{aligned} \bar{u} &= \frac{1}{2}(1 - e^{-\mu}), & \bar{q} &= \frac{1}{\sqrt{2}} e^{-\mu} \quad \text{in GOE,} \\ \bar{u} &= \frac{1}{2} \left(1 - \frac{1}{2} e^\mu - \frac{1}{2} e^{-\mu} \right), & \bar{q} &= \frac{1}{2\sqrt{2}} (e^{-\mu} - e^\mu) \quad \text{in GSE.} \end{aligned}$$

Next, we consider the limiting quantities $\bar{\mathcal{R}}_1, \bar{\mathcal{Q}}_1$ and $\bar{\mathcal{P}}_1$. Recalling that these arise in GOE, when N is even, we find from (48) that

$$\bar{\mathcal{R}}_1(\infty) = 0, \quad \bar{\mathcal{Q}}_1(\infty) = \sqrt{2}, \quad \bar{\mathcal{P}}_1(\infty) = 0,$$

and so, since by (iii) the last two functions differ by a constant,

$$\bar{\mathcal{Q}}_1 = \bar{\mathcal{P}}_1 + \sqrt{2}.$$

We find now that the limiting form of the system (46) and (47) is

$$\bar{\mathcal{R}}_1' = -\frac{q}{\sqrt{2}}(2\bar{\mathcal{P}}_1 + \sqrt{2}), \quad \bar{\mathcal{P}}_1' = -\frac{q}{\sqrt{2}}(\bar{\mathcal{R}}_1 - 1).$$

The same substitution reduces this to a system with constant coefficients, and we find the solution to be

$$\bar{\mathcal{R}}_1 = 1 - e^{-\mu}, \quad \bar{\mathcal{P}}_1 = \frac{1}{\sqrt{2}}(e^{-\mu} - 1).$$

Similarly, for $\beta = 4$ when N is odd we find that

$$\bar{\mathcal{R}}_4(\infty) = 0, \quad \bar{\mathcal{Q}}_4(\infty) = 0, \quad \bar{\mathcal{P}}_4(\infty) = -\frac{\sqrt{2}}{2},$$

that $\bar{\mathcal{Q}}_4 = \bar{\mathcal{P}}_4 + \sqrt{2}/2$, that the system is

$$\bar{\mathcal{R}}_4' = -\frac{q}{\sqrt{2}} \left(2\bar{\mathcal{P}}_4 + \frac{\sqrt{2}}{2} \right), \quad \bar{\mathcal{P}}_4' = -\frac{q}{\sqrt{2}}(\bar{\mathcal{R}}_4 + 1),$$

and that the solution is

$$\bar{\mathcal{R}}_4 = \frac{1}{4}e^{\mu} + \frac{3}{4}e^{-\mu} - 1, \quad \bar{\mathcal{P}}_4 = \frac{1}{\sqrt{2}} \left(\frac{1}{4}e^{\mu} - \frac{3}{4}e^{-\mu} - \frac{1}{2} \right).$$

If we substitute the limiting values we have found into the formulas (40) and (41) which give the values of the ratios $F_{N1}(t)^2/F_{N2}(t)$ and $F_{N4}(t/\sqrt{2})^2/F_{N2}(t)$ we obtain formulas (53) and (54).

It remains to establish our claims (i)–(iv) above. We indicate by a subscript τ the result of scaling either a function or an operator. Thus,

$$S_{\tau} := \tau \circ S \circ \tau, \quad \varphi_{\tau} := \varphi \circ \tau, \quad \text{etc.}$$

It follows from results on the asymptotics of Hermite polynomials that

$$\lim_{N \rightarrow \infty} N^{-1/6} \varphi_{\tau}(x) = \lim_{N \rightarrow \infty} N^{-1/6} \psi_{\tau}(x) = A(x) \tag{55}$$

uniformly for bounded s , where $A(x)$ denotes the Airy function $\text{Ai}(x)$, and that there are estimates

$$N^{-1/6} \varphi_{\tau}(x) = O(e^{-x}), \quad N^{-1/6} \psi_{\tau}(x) = O(e^{-x}) \tag{56}$$

which hold uniformly in N and for x bounded below. (There is a better bound, in which $x^{3/2}$ appears in the exponent rather than x , but this one is more than good enough for our purposes. See [10], p. 403.)

To obtain the scaling limits of S and other quantities we shall make use of the identity

$$S(x, y) = \int_0^\infty [\varphi(x+z)\psi(y+z) + \psi(x+z)\varphi(y+z)] dz \tag{57}$$

analogous to formula (4.5) of [12],

$$\frac{A(x)A'(y) - A'(x)A(y)}{x - y} = \int_0^\infty A(x+z)A(y+z) dz$$

and proved in the same way: The formula on the top of p. 43 of [14] gives

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) S(x, y) = -\varphi(x)\psi(y) - \psi(x)\varphi(y),$$

and the same operator applied to the right side of (57) equals

$$\begin{aligned} & \int_0^\infty \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) [\varphi(x+z)\psi(y+z) + \psi(x+z)\varphi(y+z)] dz \\ &= \int_0^\infty \frac{\partial}{\partial z} [\varphi(x+z)\psi(y+z) + \psi(x+z)\varphi(y+z)] dz \\ &= -\varphi(x)\psi(y) - \psi(x)\varphi(y). \end{aligned}$$

Hence the two sides of (57) differ by a function of $x - y$ and this function must vanish since both sides tend to 0 as x and y tend to ∞ independently.

If an operator L has kernel $L(x, y)$ then L_τ has kernel

$$\frac{1}{\sqrt{2N^{1/6}}} L(\tau(x), \tau(y)),$$

and so from (57) we see that the kernel of S_τ has the representation

$$S_\tau(x, y) = \frac{1}{\sqrt{2N^{1/6}}} \int_0^\infty [\varphi(\tau(x)+z)\psi(\tau(y)+z) + \psi(\tau(x)+z)\varphi(\tau(y)+z)] dz,$$

and the substitution $z \rightarrow z/\sqrt{2N^{1/6}}$ gives

$$S_\tau(x, y) = \frac{1}{2N^{1/3}} \int_0^\infty [\varphi_\tau(x+z)\psi_\tau(y+z) + \psi_\tau(x+z)\varphi_\tau(y+z)] dz. \tag{58}$$

The asymptotic formulas (55), and the estimates (56), show that

$$\lim_{N \rightarrow \infty} S_\tau(x, y) = \int_0^\infty A(x+z)A(y+z) dz$$

pointwise, and in various function space norms as well. This will be very useful. The right side, we know, is the Airy kernel. We have been using a bar as notation for a limit, so we write the above as

$$S_\tau(x, y) \rightarrow \bar{S}(x, y) = \int_0^\infty A(x+z)A(y+z) dz. \tag{59}$$

The fact that this converges in the space $L_2(s, \infty) \otimes L_2(s, \infty)$ implies that the operator $\chi_\tau S_\tau \chi_\tau$ converges in the norm of operators on $L_2(\mathbf{R})$ to the operator $\bar{\chi} \bar{S} \bar{\chi}$, where $\bar{\chi} = \chi_{(s, \infty)}$. Since $I - \bar{\chi} \bar{S} \bar{\chi}$ is invertible it follows that also

$$(I - \chi_\tau S_\tau \chi_\tau)^{-1} \rightarrow (I - \bar{\chi} \bar{S} \bar{\chi})^{-1} \tag{60}$$

in operator norm. The same is true if the space $L_2(\mathbf{R})$ is replaced by $L_1(\mathbf{R})$, as is seen if we use the fact that the norm of an operator $\bar{\chi} L \bar{\chi}$ on this space is at most

$$\int_s^\infty \sup_{y \geq s} |L(x, y)| dx,$$

and that (59) holds with this function space norm as well.

Let us compute the scaling limit of q_N . We have

$$\begin{aligned} q_N &= (I - S\chi)^{-1} \varphi(t) = \varphi(t) + S\chi(I - \chi S\chi)^{-1} \varphi(t) \\ &= \varphi_\tau(s) + S_\tau \chi_\tau (I - \chi_\tau S_\tau \chi_\tau)^{-1} \varphi_\tau(s). \end{aligned}$$

It follows from (55) and (56) that $N^{-1/6} \varphi_\tau \rightarrow A$ pointwise and in $L_2(s, \infty)$. The second of these facts, together with (60), implies that

$$(I - \chi_\tau S_\tau \chi_\tau)^{-1} N^{-1/6} \varphi_\tau \rightarrow (I - \bar{\chi} \bar{S} \bar{\chi})^{-1} A$$

in L_2 , and then using the fact that (59) holds in $L_2(s, \infty)$ for fixed x , we deduce that

$$S_\tau \chi_\tau (I - \chi_\tau S_\tau \chi_\tau)^{-1} N^{-1/6} \varphi_\tau \rightarrow \bar{S} \bar{\chi} (I - \bar{\chi} \bar{S} \bar{\chi})^{-1} A$$

pointwise. Putting these together shows that

$$\lim_{N \rightarrow \infty} N^{-1/6} q_N = A(s) + \bar{S} \bar{\chi} (I - \bar{\chi} \bar{S} \bar{\chi})^{-1} A(s) = (I - \bar{\chi} \bar{S} \bar{\chi})^{-1} A(s).$$

The right side was precisely the *definition* of q (it transpired that it was a Painlevé function), so we have shown that

$$\lim_{N \rightarrow \infty} N^{-1/6} q_N = q.$$

Note that since by (55) both φ and ψ have the same scaling limit, the above argument applied to p_N leads to the same result,

$$\lim_{N \rightarrow \infty} N^{-1/6} p_N = q.$$

This gives assertion (i). The uniformity assertion in (iv), for these and the limits established below, we leave to the reader.

Next we consider u_ε and \tilde{v}_ε . Recalling the definition (39) we write

$$\varepsilon \varphi(x) = c_\varphi - \int_x^\infty \varphi(y) dy,$$

and so

$$(\varepsilon \varphi)_\tau(x) = c_\varphi - \int_{\tau(x)}^\infty \varphi(y) dy = c_\varphi - \frac{1}{\sqrt{2N}^{1/6}} \int_x^\infty \varphi_\tau(y) dy.$$

As $N \rightarrow \infty$ the constant c_φ converges (in fact to $1/\sqrt{2}$) and the second term converges in $L_2(s, \infty)$ (in fact to $-\int_x^\infty A(y) dy/\sqrt{2}$). Thus the function $(\varepsilon\varphi)_\tau$ converges in the space $\mathbf{R} + L_2(s, \infty)$. Also,

$$N^{-1/6}Q_\tau = N^{-1/6}(I - \chi_\tau S_\tau \chi_\tau)^{-1} \varphi_\tau \rightarrow (I - \bar{\chi} \bar{S} \bar{\chi})^{-1} A \tag{61}$$

in $L_1(s, \infty) \cap L_2(s, \infty)$. It follows from these limit relations that

$$u_\varepsilon = (Q, \chi \varepsilon \varphi) = \frac{1}{\sqrt{2}N^{1/6}} (Q_\tau, (\chi \varepsilon \varphi)_\tau)$$

converges as $N \rightarrow \infty$. Notice also that since φ and ψ have the same scaling limit, (61) holds with Q replaced by P on the left side, from which it follows that \tilde{v}_ε has the same limiting value as u_ε . This establishes (ii).

Finally we come to the quantities $\mathcal{R}_1, \dots, \mathcal{P}_4$. These are trickier since, although our functions scale nicely on (s, ∞) for fixed s , they do not scale uniformly on $(-\infty, \infty)$. In a sense we have to separate out the parts that get integrated over $(-\infty, \infty)$. Beginning with \mathcal{R}_1 , we use

$$R(x, t) = (I - S\chi)^{-1}S(x, t) = S\chi(I - \chi S\chi)^{-1}S(x, t) + S(x, t),$$

and write our integrals over $(-\infty, t)$ as integrals over $(-\infty, \infty)$ minus integrals over (t, ∞) . Thus

$$\begin{aligned} \mathcal{R}_1 &= \int_{-\infty}^\infty S\chi(I - \chi S\chi)^{-1}S(x, t) dx - \int_t^\infty S\chi(I - \chi S\chi)^{-1}S(x, t) dx \\ &\quad + \int_{-\infty}^\infty S(x, t) dx - \int_t^\infty S(x, t) dx \\ &= \int_{-\infty}^\infty S_\tau \chi_\tau (I - \chi_\tau S_\tau \chi_\tau)^{-1} S_\tau(x, s) dx - \int_s^\infty S_\tau \chi_\tau (I - \chi_\tau S_\tau \chi_\tau)^{-1} S_\tau(x, s) dx \\ &\quad + \int_{-\infty}^\infty S_\tau(x, s) dx - \int_s^\infty S_\tau(x, s) dx. \end{aligned} \tag{62}$$

(The factors $1/\sqrt{2}N^{1/6}$ arising from the variable change are incorporated in the expression for $S_\tau(x, s)$.) We think of the first integral on the right side of (62) as the inner product of the functions

$$\int_{-\infty}^\infty S_\tau(x, \cdot) dx \quad \text{and} \quad (I - \chi S_\tau \chi)^{-1} S_\tau(\cdot, s) \tag{63}$$

on (s, ∞) . Now it follows from (58) that

$$\int_{-\infty}^\infty S_\tau(x, y) dx = \frac{1}{\sqrt{2}N^{1/6}} \left(c_\varphi \int_y^\infty \psi_\tau(z) dz + c_\psi \int_y^\infty \varphi_\tau(z) dz \right), \tag{64}$$

which converges in $L_2(s, \infty)$ as $N \rightarrow \infty$. Also $S_\tau(\cdot, s)$ converges in $L_2(s, \infty)$, so the same is true of the second function in (63). Thus the inner product itself converges, and this shows that the first term on the right side of (62) converges. The second term (62) is treated in much the same way – the only difference is that the first integral in (63) is taken over (s, ∞) . The next term on the right side of (62) is

simple – to show that it converges requires only the representation (64) with $y = s$ – and the last term is analogous to this one.

Turning to \mathcal{Q}_1 , we write it similarly as

$$\begin{aligned} & \int_{-\infty}^{\infty} S\chi(I - \chi S\chi)^{-1}\varphi(x) dx - \int_t^{\infty} S\chi(I - \chi S\chi)^{-1}\varphi(x) dx + \int_{-\infty}^{\infty} \varphi(x) dx - \int_t^{\infty} \varphi(x) dx \\ &= \int_{-\infty}^{\infty} S_\tau\chi_\tau(I - \chi_\tau S_\tau\chi_\tau)^{-1} \frac{\varphi_\tau(x)}{\sqrt{2N^{1/6}}} dx - \int_s^{\infty} S_\tau\chi_\tau(I - \chi_\tau S_\tau\chi_\tau)^{-1} \frac{\varphi_\tau(x)}{\sqrt{2N^{1/6}}} dx \\ &+ 2c_\varphi - \int_s^{\infty} \frac{\varphi_\tau(x)}{\sqrt{2N^{1/6}}} dx. \end{aligned}$$

We treat the integrals just as we did those for \mathcal{R}_1 , with $\varphi_\tau/\sqrt{2N^{1/6}}$ replacing $S_\tau(\cdot, s)$, to show tht they have limits as $N \rightarrow \infty$ and, of course, c_φ also has a limit. The quantity \mathcal{P}_1 is handled similarly, with ψ replacing φ everywhere. That the limits of \mathcal{P}_1 and \mathcal{Q}_1 differ by a constant follows immediately from the fact that ψ and φ have the same scaling limit on (s, ∞) . The discussion of the quantities $\mathcal{R}_4, \mathcal{Q}_4$ and \mathcal{P}_4 is entirely analogous, and (iii) is established.

VIII. Justification for the Determinant Manipulations

In this final section we give the rigorous justification for the determinant manipulations in Sects. II and III. We begin with GSE, which is slightly easier. The quantity of interest is the determinant for the operator on $L_2(J)$ with kernel

$$\frac{1}{2} \begin{pmatrix} S(x-y) & DS(x-y) \\ IS(x-y) & S(x-y) \end{pmatrix},$$

where

$$S(x) = \frac{\sin x}{\pi x}, \quad DS(x) = S'(x), \quad IS(x) = \int_0^\infty S(y) dy.$$

This kernel is smooth and so the operator is trace class. Our determinant is the same as that for the operator on $L_2(\mathbf{R})$

$$\frac{1}{2} \begin{pmatrix} \chi S\chi & \chi DS\chi \\ \chi IS\chi & \chi S\chi \end{pmatrix}, \tag{65}$$

where S, DS, IS are the operators with corresponding difference kernels and χ is multiplication by χ_J . We shall show that we can remove all the operators χ which appear on the left, if we interpret S, DS , and IS as acting between appropriate spaces.

Recall that the Sobolev space H_1 is given by

$$\{f \in L_2 : f \text{ is absolutely continuous and } f' \in L_2\},$$

with

$$\|f\|_{H_1} = \sqrt{\|f\|_{L_2}^2 + \|f'\|_{L_2}^2}.$$

The mapping

$$f \rightarrow f + Df \quad \left(D = \frac{d}{dx} \right) \tag{66}$$

is an isometry from H_1 onto L_2 .

Lemma 1. S_χ and DS_χ are trace class operators from L_2 to H_1 .

Proof. If the kernel $K(x, y)$ of an operator K satisfies

$$\int_{\mathbf{R}} \int_J |K(x, y)|^2 dy dx < \infty, \quad \int_{\mathbf{R}} \int_J \left| \frac{\partial}{\partial y} K(x, y) \right|^2 dy dx < \infty,$$

then $K_{\chi, J}$ is trace class for any set J of finite measure. (See, for example, pp. 118–119 of [5].) It is immediate that $D^n S_\chi : L_2 \rightarrow L_2$ is trace class for each $n \geq 0$. If we recall the isometry (66) we see that to show that $D^n S_\chi : L_2 \rightarrow H_1$ is trace class it is enough to show that $(I + D)D^n S_\chi : L_2 \rightarrow L_2$ is. But we know this.

Next we enlarge our spaces L_2 and H_1 by adjoining a single function to each:

$$\tilde{L}_2 := \{f + c\varepsilon : f \in L_2, c \in \mathbf{C}\}, \quad \tilde{H}_1 := \{f + cIS : f \in H_1, c \in \mathbf{C}\},$$

which are Hilbert spaces when endowed with the norms

$$\sqrt{\|f\|_{L_2}^2 + |c|^2}, \quad \sqrt{\|f\|_{H_1}^2 + |c|^2},$$

respectively.

Lemma 2. IS_χ is a trace class operator from L_2 to \tilde{H}_1 .

Proof. Integration by parts on the constituent intervals of J shows that for all $f \in L_2$,

$$(IS)(f)(x) = \sum (-1)^k IS(x - a_k)(If)(a_k) + \int S(x - y)(If)(y)\chi_J(y) dy, \quad (67)$$

where $(If)(x) := \int_0^x f(y) dy$. The map $f \rightarrow (If) \cdot \chi_J$ is bounded from L_2 to L_2 and so by Lemma 1 the integral on the right represents a trace class operator from L_2 to H_1 . The maps $f \rightarrow (If)(a_i)$ are continuous linear functionals on L_2 and each function $IS(x - a_i)$ belongs to \tilde{H}_1 since

$$IS(x - a_i) - IS(x) \in H_1.$$

So the sum on the right side represents a finite rank, and hence trace class, operator from L_2 to \tilde{H}_1 . This completes the proof.

Let us consider the operator represented by the matrix

$$\frac{1}{2} \begin{pmatrix} S_\chi & DS_\chi \\ IS_\chi & S_\chi \end{pmatrix}, \quad (68)$$

which is the same as (65) except that all factors χ that appeared on the left were removed. Since $\chi : \tilde{L}_2 \rightarrow L_2$ is bounded, it follows from Lemmas 1 and 2 that this is a trace class operator from $L_2 \oplus \tilde{L}_2$ to itself. (Actually, of course, the operator is trace class from $L_2 \oplus \tilde{L}_2$ to $H_1 \oplus \tilde{H}_1$, but we don't use this.)

We use now, and several times below, the fact that the determinant for an operator product AB is the same as for BA as long as one of the two factors is trace class and the other is bounded. They do *not* have to act on the same Hilbert space; one operator can map a space H to a space H' as long as the other maps H' to H . The two products then act on the different spaces H and H' .

Lemma 3. The determinants for the operator (65) on $L_2 \oplus L_2$ and the operator (68) on $L_2 \oplus \tilde{L}_2$ are equal.

Proof. Because χ is idempotent we can insert a factor $\begin{pmatrix} \chi & 0 \\ 0 & \chi \end{pmatrix}$ on the right side of (68) without changing it. We can bring this factor around to the other side and deduce that the determinant for (68) as an operator on $L_2 \oplus \tilde{L}_2$ is the same as that for (65) as an operator on $L_2 \oplus \tilde{L}_2$. But the range of this operator is contained in $L_2 \oplus L_2$, so the determinant for it is the same when considered an operator on this space.

It follows from the lemma that we can replace (65) by (68). Since $D(IS) = S$ we can write it as the product

$$\frac{1}{2} \begin{pmatrix} D & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} IS\chi & S\chi \\ IS\chi & S\chi \end{pmatrix}.$$

It follows from Lemmas 1 and 2 that the factor on the right is trace class from $L_2 \oplus \tilde{L}_2$ to $\tilde{H}_1 \oplus \tilde{H}_1$ while the factor on the left is bounded from $\tilde{H}_1 \oplus \tilde{H}_1$ to $L_2 \oplus \tilde{L}_2$ since $D : \tilde{H}_1 \rightarrow L_2$ is bounded. It follows that the determinant for (68) on $L_2 \oplus \tilde{L}_2$ is the same as the determinant for

$$\frac{1}{2} \begin{pmatrix} IS\chi & S\chi \\ IS\chi & S\chi \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & I \end{pmatrix} = \frac{1}{2} \begin{pmatrix} IS\chi D & S\chi \\ IS\chi D & S\chi \end{pmatrix}$$

on $\tilde{H}_1 \oplus \tilde{H}_1$. The matrix entries are operators on the same space, \tilde{H}_1 , so that we can perform what amounts to row and column operations on them. (This was not an accident!) Multiplying on the left by the matrix $\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$ and on the right by its inverse, $\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$, we see that the last determinant is the same as that for

$$\frac{1}{2} \begin{pmatrix} IS\chi D + S\chi & S\chi \\ 0 & 0 \end{pmatrix},$$

and so it is the determinant of

$$\begin{pmatrix} I - \frac{1}{2}IS\chi D - \frac{1}{2}S\chi & -\frac{1}{2}S\chi \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & -\frac{1}{2}S\chi \\ 0 & I \end{pmatrix} \begin{pmatrix} I - \frac{1}{2}IS\chi D - \frac{1}{2}S\chi & 0 \\ 0 & I \end{pmatrix}.$$

Both factors on the right are operators of the form $I +$ trace class operator on $\tilde{H}_1 \oplus \tilde{H}_1$, and the factor on the left is of the form $I +$ nilpotent operator and so has determinant 1. Hence the determinant of the product equals the determinant of the second factor, which equals

$$I - \frac{1}{2}IS\chi D - \frac{1}{2}S\chi.$$

We rewrite the operator as

$$I - S\chi - \frac{1}{2}(IS\chi D - S\chi). \tag{69}$$

A variant of (67) is

$$(IS\chi)(f')(x) = \sum (-1)^k IS(x - a_k)f(a_k) + \int S(x - y)f(y)\chi_j(y) dy,$$

which gives

$$IS\chi D - S\chi = \sum (IS)_k \otimes \delta_k,$$

where $(IS)_k(x) := IS(x - a_k)$ and $\delta_k(x) := \delta(x - a_k)$. The tensor product denotes the operator which takes a function in \tilde{H}_1 , evaluates it at a_k and multiplies this by $(IS)_k$. A similar interpretation holds for any tensor product $u \otimes v$, where u belongs to a Hilbert space and v to its dual space.

We extend the domain of S to all of \tilde{L}_2 by defining

$$(Sf)(x) = \int S(x - y)f(y) dy,$$

the integral being conditionally convergent. It is easy to see that $S : \tilde{L}_2 \rightarrow \tilde{H}_1$ and $(IS)_k = S\varepsilon_k$, where $\varepsilon_k(x) := \varepsilon(x - a_k)$. Thus we can write (69) as

$$I - S\chi - \frac{1}{2} \sum (-1)^k S\varepsilon_k \otimes \delta_k. \tag{70}$$

Recall that our operators act on \tilde{H}_1 . Now $I - S\chi$ is invertible as an operator on this space as well as on L_2 , since the eigenfunctions of $S\chi$ belonging to nonzero eigenvalues are the same for the two spaces. For the same reason both interpretations of the operator give the same value for the determinant. We denote the inverse of the operator, as before, by $I + R$. Factoring out $I - S\chi$ shows that the determinant of (70) equals $E_2(0, J)$ times the determinant of

$$I - \frac{1}{2} \sum (-1)^k (S + RS)\varepsilon_k \otimes \delta_k. \tag{71}$$

Recall the definition $R_k(x) := R(x, a_k)$.

Proposition. *The determinant of (71) on \tilde{H}_1 equals the determinant of*

$$I - \frac{1}{2} \sum (-1)^k \varepsilon_k \otimes R_k \tag{72}$$

on \tilde{L}_2 .

Proof. We use inner product notation (u, v) to denote the action of a dual vector v on a vector u . The determinants of (71) and (72) are scalar determinants whose entries contain the inner products

$$((S + RS)\varepsilon_j, \delta_k), \quad (\varepsilon_j, R_k),$$

respectively, in position (j, k) . We shall show that these are equal.

We begin with the observation that $R^t\chi = \chi R$ when these are thought of as acting on L_2 . This is so because R is the resolvent operator for $S\chi$ and S is symmetric. It follows from this that if f and g belong to L_2 , with g supported in J , then

$$((S + RS)f, g) = (f, (S\chi + S\chi R)g) = (f, Rg),$$

the last by the resolvent identity. Suppose h has integral 1 and is compactly supported (in \mathbf{R}^+ if k is odd, in \mathbf{R}^- if k is even), set $g(x) = nh(n(x - a_k))$ and let $n \rightarrow \infty$. We obtain

$$((S + RS)f, \delta_k) = (f, R_k).$$

Replace f by $f_n := \varepsilon_j\chi_{[-n, n]}$ and let $n \rightarrow \infty$. Since

$$(S + RS)f_n \rightarrow (S + RS)\varepsilon_j$$

uniformly on compact sets we deduce the desired identity

$$((S + RS)\varepsilon_j, \delta_k) = (\varepsilon_j, R_k).$$

Thus (21) is completely proved.

We now turn to GOE. The reason this is slightly awkward is that the operator K on $L_2(J)$ with kernel

$$\begin{pmatrix} S(x - y) & DS(x - y) \\ IS(x - y) - \varepsilon(x - y) & S(x - y) \end{pmatrix} \tag{73}$$

is not trace class. So its classical Fredholm determinant, which is what we want, is not given by $\det(I - K)$, which is not defined. It is instead given in terms of its regularized determinant \det_2 by the formula

$$\det_2(I - K) e^{-\text{tr} K},$$

where $\text{tr} K$ denotes the sum of the integrals of the diagonal entries of the kernel of K . Rather than deal with regularized determinants we approximate the kernel by a smooth kernel, evaluate the resulting Fredholm determinant, and pass to the limit at the end. Thus we replace the term $\varepsilon(x - y)$ in (73) by $\eta_n(x) := \eta(n(x - y))$, where η is a smooth function which equals ε outside a finite interval. Then the resulting operator K_n is trace class, and $\det_2(I - K) e^{-\text{tr} K}$ is equal to the limit of $\det(I - K_n)$ as $n \rightarrow \infty$.

Proceeding as before, we find that $\det(I - K_n)$ is equal to the determinant of

$$\begin{pmatrix} I - IS\chi D - S\chi & -S\chi \\ \eta_n\chi D & I \end{pmatrix}$$

on $\tilde{H}_1 \oplus \tilde{H}_1$, where η_n denotes convolution by $\eta_n(x)$. The operator can be factored as

$$\begin{pmatrix} I & -S\chi \\ 0 & I \end{pmatrix} \begin{pmatrix} I - IS\chi D - S\chi + S\chi\eta_n\chi D & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ \eta_n\chi D & I \end{pmatrix},$$

all factors being of the form $I +$ trace class operator on $\tilde{H}_1 \oplus \tilde{H}_1$, and so its determinant equals the determinant of the operator

$$I - S\chi - IS\chi D + S\chi\eta_n\chi D$$

on \tilde{H}_1 . But $\chi\eta_n\chi \rightarrow \chi\varepsilon\chi$ in the norm for operators on L_2 (where ε denotes convolution by $\varepsilon(x)$) and it follows from this and the second part of Lemma 1 that the above operator converges to

$$I - S\chi - IS\chi D + S\chi\varepsilon\chi D \tag{74}$$

in the trace norm for operators on \tilde{H}_1 . Consequently it is the determinant of this which will be our final answer.

We have already seen (cf. (69) and (70)) that

$$IS\chi D = S\chi + \sum (-1)^k S\varepsilon_k \otimes \delta_k.$$

But

$$(\varepsilon\chi D f)(x) = \int \varepsilon(x - y) f'(y) dy = (\chi f)(x) + \sum (-1)^k \varepsilon_k(x) f(a_k),$$

and so

$$S\chi\varepsilon\chi D = S\chi + \sum (-1)^k S\chi\varepsilon_k \otimes \delta_k .$$

Thus

$$IS\chi D - S\chi\varepsilon\chi D = \sum (-1)^k S(1 - \chi)\varepsilon_k \otimes \delta_k .$$

And now, just as at the end of GSE, we conclude that the determinant of (74) equals $E_2(0, J)$ times the determinant of

$$I - \sum (-1)^k (S + RS)(1 - \chi)\varepsilon_k \otimes \delta_k ,$$

which in turn equals the determinant of

$$I - \sum (-1)^k (1 - \chi)\varepsilon_k \otimes R_k .$$

This completes the justification of (19).

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