94. On Osima's Blocks of Group Characters

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Let $\mathfrak{G}$ be a group of finite order g and p be a fixed rational prime. M. Osima, in his earlier paper [4], introduced a concept of blocks of characters with regard to a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ (" $\mathfrak{H}$-blocks"). Let $\mathfrak{H}_0$ be the maximal normal subgroup of $\mathfrak{G}$ contained in $\mathfrak{H}$. It is well known that the irreducible characters[1] $\phi_1, \phi_2, \cdots, \phi_k$ of $\mathfrak{H}_0$ are distributed into the classes $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_s$ of associated characters in $\mathfrak{G}$. If $\mathfrak{B}'_1, \mathfrak{B}'_2, \cdots, \mathfrak{B}'_{s'}$ are the classes of associated irreducible characters of $\mathfrak{H}_0$ in $\mathfrak{H}$, then each class $\mathfrak{B}_\sigma$ is a collection of classes $\mathfrak{B}'_\rho$. Let $\chi_1, \chi_2, \cdots, \chi_n$ be the irreducible characters of $\mathfrak{G}$ and $\theta_1, \theta_2, \cdots, \theta_h$ be those of $\mathfrak{H}$. As is well known, there corresponds to each character χ_i exactly one class $\mathfrak{B}_\sigma$ such that

$$\chi_i(H_0) = s_{i\sigma} \sum_{\phi_\mu \in \mathfrak{B}_\sigma} \phi_\mu(H_0) \qquad (H_0 \in \mathfrak{H}_0)$$

where $s_{i\sigma}$ is a positive rational integer. If a class $\mathfrak{B}_\sigma$ corresponds to a character χ_i in this sense, we say that χ_i belongs to $\mathfrak{B}_\sigma$ by counting χ_i in $\mathfrak{B}_\sigma$. We also say that θ_λ belongs to $\mathfrak{B}_\sigma$ if θ_λ belongs to $\mathfrak{B}'_\rho$ contained in $\mathfrak{B}_\sigma$. Then the classes $\mathfrak{B}_\sigma$ are the $\mathfrak{H}$-blocks of $\mathfrak{G}$ in Osima's sense. From the definition, we see that χ_i and χ_j belong to the same $\mathfrak{H}$-block of $\mathfrak{G}$ if and only if $\chi_i(H_0)/\chi_i(1) = \chi_j(H_0)/\chi_j(1)$ for all elements H_0 of $\mathfrak{H}_0$ [4], where 1 denotes the identity of $\mathfrak{G}$.

In the following, "block" of a group will always mean block with regard to a p-Sylow subgroup of the group. While Brauer's blocks for a rational prime q will be referred always as q-blocks. The purpose of this paper is to consider a connection between blocks of $\mathfrak{G}$ and the blocks of the normalizer $\mathfrak{N}(R)$ of a p-regular element R in $\mathfrak{G}$.

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1. Let $\mathfrak{P}$ be a p-Sylow subgroup of $\mathfrak{G}$ and $\mathfrak{P}_0$ be the maximal normal p-subgroup of $\mathfrak{G}$. We shall denote by $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_s$ the blocks of $\mathfrak{G}$ with regard to $\mathfrak{P}$. For each $\mathfrak{B}_\sigma$ we set

(1.1) $$\Delta_\sigma = \sum_{\chi_i \in \mathfrak{B}_\sigma} e_i,$$

where e_i is the primitive idempotent of the center Z of the group ring of $\mathfrak{G}$ over the field Ω of g-th roots of unity which belongs to χ_i. Let $K_1, K_2, \cdots, K_n$ be the classes of conjugate elements in $\mathfrak{G}$ and $G_1,$

1) The term "irreducible character" will always mean absolutely irreducible ordinary character.

$G_2, \cdots, G_n$ be a complete system of representatives for the classes. If we interprete each class K_ν as the sum of all its elements, then we may write

$$\Delta_\sigma = \sum_\nu a_\nu^\sigma K_\nu, \tag{1.2}$$

where $a_\nu^\sigma = \frac{1}{g} \sum_{\chi_i \in \mathfrak{B}_\sigma} \chi_i(1)\bar{\chi}_i(G_\nu)$.[2] By Frobenius' theorem on induced characters, we have the following:

Lemma 1. 1) *$a_\nu^\sigma = 0$ for all classes K_ν which are not contained in $\mathfrak{P}_0$.* 2) *All $p^{a_0} a_\nu^\sigma$ are algebraic integers, where p^{a_0} is the order of $\mathfrak{P}_0$.*

The converse of this lemma also holds in the following form: *If, for a set $\mathfrak{B}$ of characters χ_i, the idempotent $\Delta = \sum_{\chi_i \in \mathfrak{B}} e_i$ of Z is expressed as a linear combination of classes K_ν contained in $\mathfrak{P}_0$, then $\mathfrak{B}$ is a collection of blocks $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.*

2. Let q be an arbitrarily fixed rational prime, different from p, and Q be an arbitrarily given element of $\mathfrak{G}$ whose order is a power of q. It follows from Lemma 1 that each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ is a collection of q-blocks B_τ of $\mathfrak{G}$. Let $B^{(\tau)}(Q)$ be the collection of q-blocks of the normalizer $\mathfrak{N}(Q)$ of Q in $\mathfrak{G}$ which determine a q-block B_τ of $\mathfrak{G}$. We set $\mathfrak{B}^{(\sigma)}(Q) = \bigcup_{B_\tau \subseteq \mathfrak{B}_\sigma} B^{(\tau)}(Q)$. By (4.16) in [2][3] and Lemma 1, we have the following:

Lemma 2. *Each $\mathfrak{B}^{(\sigma)}(Q)$ is a collection of blocks $\hat{\mathfrak{B}}_\rho$ of $\mathfrak{N}(Q)$.*

Let now R be a p-regular element of $\mathfrak{G}$ whose order is a product of powers of distinct rational primes $q_1, q_2, \cdots, q_r$. As is well known, R is decomposed uniquely into

$$R = Q_1 Q_2 \cdots Q_r \qquad (Q_i Q_j = Q_j Q_i), \tag{2.1}$$

where Q_i is the q_i-factor of R. Let a block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ be given arbitrarily. First, applying Lemma 2 for $Q = Q_1$, $\mathfrak{B}_\sigma$ and $\mathfrak{G}$, we have a collection $\mathfrak{B}^{(\sigma)}(Q_1)$ of blocks of $\mathfrak{N}(Q_1)$. Secondly, working similarly for $Q = Q_2$, $\mathfrak{B}_\sigma = \mathfrak{B}^{(\sigma)}(Q_1)$ and $\mathfrak{G} = \mathfrak{N}(Q_1)$, we have a collection $\mathfrak{B}^{(\sigma)}(Q_1, Q_2)$ of blocks of $\mathfrak{N}(Q_1 Q_2)$. Continuing this process, we have finally a collection $\tilde{\mathfrak{B}}^{(\sigma)} = \mathfrak{B}^{(\sigma)}(Q_1, Q_2, \cdots, Q_r)$ of blocks $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}} = \mathfrak{N}(R)$. If a block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$ belongs to the collection $\tilde{\mathfrak{B}}^{(\sigma)}$, we say that *the block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ is determined by the block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$.* It follows from Theorem 1 in §3 that $\tilde{\mathfrak{B}}^{(\sigma)}$ is independent of the order of $Q_1, Q_2, \cdots, Q_r$.

3. Let R be a p-regular element of $\mathfrak{G}$ and $S(R)$ be the p-regular section ("Oberklasse")[4] of R in $\mathfrak{G}$, i.e. the set of all elements of $\mathfrak{G}$

2) If α is a complex number, we denote by $\bar{\alpha}$ the conjugate complex number of α.

3) Cf. also [7, p. 181].

4) Cf. [9].

whose p-regular factors are conjugate to R in $\mathfrak{G}$. Let $\tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_v$ be the classes of conjugate elements in $\tilde{\mathfrak{G}}=\mathfrak{N}(R)$ whose orders are powers of p. We may assume that the maximal normal p-subgroup $\tilde{\mathfrak{P}}_0$ of $\tilde{\mathfrak{G}}$ is the union of the first u classes $\tilde{K}_\alpha$. We may also assume that $K_\alpha \supseteqq R\tilde{K}_\alpha$, $\alpha=1, 2, \cdots, v$; $S(R)$ is the union of $K_1, K_2, \cdots, K_v$. We denote by $S_0(R)$ the union of $K_1, K_2, \cdots, K_u$ and denote by $S_1(R)$ the union of $K_{u+1}, K_{u+2}, \cdots, K_v$; $S(R)=S_0(R)\cup S_1(R)$.

Let $\tilde{\mathfrak{B}}_1, \tilde{\mathfrak{B}}_2, \cdots, \tilde{\mathfrak{B}}_{\tilde{s}}$ be the blocks of $\tilde{\mathfrak{G}}$ with regard to a p-Sylow subgroup $\tilde{\mathfrak{P}}$ of $\tilde{\mathfrak{G}}$. Let, for each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$, Δ_σ be given by (1.1). Similarly, for each block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$, we define an idempotent $\tilde{\Delta}_\rho$ of the center $\tilde{Z}$ of the group ring of $\tilde{\mathfrak{G}}$ over Ω. We set $\tilde{\Delta}^{(\sigma)} = \sum_{\tilde{\mathfrak{B}}_\rho \subseteq \tilde{\mathfrak{B}}^{(\sigma)}} \tilde{\Delta}_\rho$, where $\tilde{\mathfrak{B}}^{(\sigma)}$ is the collection of blocks $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$ which determine $\mathfrak{B}_\sigma$, and set

$$K_\mu \Delta_\sigma = \sum_{\nu=1}^{n} a^\sigma_{\mu\nu} K_\nu \qquad (\mu=1, 2, \cdots, n). \tag{3.1}$$

Then, by Theorem 2 in [3] and Lemmas 1 and 2, we obtain the following:

Theorem 1. *For $\alpha=1, 2, \cdots, u$, we have*

$$K_\alpha \Delta_\sigma = \sum_{\beta=1}^{u} a^\sigma_{\alpha\beta} K_\beta$$

and

$$\tilde{K}_\alpha \tilde{\Delta}^{(\sigma)} = \sum_{\beta=1}^{u} a^\sigma_{\alpha\beta} \tilde{K}_\beta.$$

For $\alpha=u+1, u+2, \cdots, v$, we have

$$K_\alpha \Delta_\sigma = \sum_{\beta=u+1}^{v} a^\sigma_{\alpha\beta} K_\beta$$

and

$$\tilde{K}_\alpha \tilde{\Delta}^{(\sigma)} = \sum_{\beta=u+1}^{v} a^\sigma_{\alpha\beta} \tilde{K}_\beta.$$

Let $\tilde{\chi}_1, \tilde{\chi}_2, \cdots, \tilde{\chi}_{\tilde{n}}$ be the irreducible characters of $\tilde{\mathfrak{G}}$ and $\tilde{\theta}_1, \tilde{\theta}_2, \cdots$ $\tilde{\theta}_{\tilde{h}}$ be those of $\tilde{\mathfrak{P}}$. We set

$$\tilde{\chi}_j(P) = \sum_{\lambda=1}^{\tilde{h}} \tilde{r}_{j\lambda} \tilde{\theta}_\lambda(P) \qquad (P \in \tilde{\mathfrak{P}}) \tag{3.2}$$

and

$$\chi_i(RP) = \sum_{\lambda=1}^{\tilde{h}} r^R_{i\lambda} \tilde{\theta}_\lambda(P) \qquad (P \in \tilde{\mathfrak{P}}) \tag{3.3}$$

[5, 6]. Setting

$$\tilde{w}_{\lambda\mu} = \sum_{j=1}^{\tilde{n}} \tilde{r}_{j\lambda} \bar{\tilde{r}}_{j\mu} \qquad (\lambda, \mu=1, 2, \cdots, \tilde{h}), \tag{3.4}$$

by Theorem 1 we have

$$\sum_{\chi_i \in \mathfrak{B}_\sigma} r^R_{i\lambda} \bar{r}^R_{i\mu} = \begin{cases} \tilde{w}_{\lambda\mu} & (\tilde{\theta}_\lambda, \tilde{\theta}_\mu \in \tilde{\mathfrak{B}}_\rho \subseteq \tilde{\mathfrak{B}}^{(\sigma)}), \\ 0 & \text{(elsewhere)}. \end{cases} \tag{3.5}$$

In particular,

$$\sum_{\chi_i \in \mathfrak{B}_\sigma} r_{i\lambda}^R \bar{r}_{i\lambda}^R = 0 \qquad (\tilde{\theta}_\lambda \notin \tilde{\mathfrak{B}}^{(\sigma)}),$$

hence

$$r_{i\lambda}^R = 0 \qquad (\chi_i \in \mathfrak{B}_\sigma,\ \tilde{\theta}_\lambda \notin \tilde{\mathfrak{B}}^{(\sigma)}).$$

Thus we obtain the following theorem.[5]

Theorem 2. *If an irreducible character $\tilde{\theta}_\lambda$ of $\tilde{\mathfrak{P}}$ belongs to a block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$, then $r_{i\lambda}^R$ can be different from zero only for irreducible characters χ_i of $\mathfrak{G}$ which belong to the block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ determined by the block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$.*

By Theorem 1, we also have the following refinements of some of the orthogonality relations for group characters.

Theorem 3. 1) *If two elements L and M of $\mathfrak{G}$ belong to different p-regular sections of $\mathfrak{G}$, then*

$$\sum_{\chi_i \in \mathfrak{B}_\sigma} \chi_i(L)\bar{\chi}_i(M) = 0 \tag{3.6}$$

for each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ [8].[6]

2) *If L and M belong to the same p-regular section $S(R)$ of $\mathfrak{G}$ and if exactly one of the p-factors of them belongs to the maximal normal p-subgroup of $\mathfrak{N}(R)$, then* (3.6) *also holds for each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.*

Theorem 4. *If χ_i and χ_j are two irreducible characters of $\mathfrak{G}$ which belong to different blocks of $\mathfrak{G}$, then*

$$\sum_{G \in S_0(R)} \chi_i(G)\bar{\chi}_j(G) = 0$$

and

$$\sum_{G \in S_1(R)} \chi_i(G)\bar{\chi}_j(G) = 0$$

for each p-regular section $S(R)$ of $\mathfrak{G}$.[7]

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5) Prof. M. Osima has pointed out the fact that the theorem follows also from Theorem 1 in [1].

6) We have a refinement of this result, which is a dual theorem of Theorem 2 in [1].

7) This theorem is an inprovement of a result in [8].

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