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***Abstract*—In orthogonal time frequency space (OTFS) modulation, information-carrying symbols reside in the delay-Doppler (DD) domain. By operating in the DD domain, an appealing property for communication arises: time-frequency (TF) dispersive channels encountered in high mobility environments become time-invariant. The time-invariance of the channel in the DD domain enables efficient equalizers for time-frequency dispersive channels. In this paper, we propose an OTFS system based on the discrete Zak transform. We show that the presented formulation simplifies the derivation and analysis of the input-output relation of TF dispersive channel in the DD domain.**

***Index Terms*—Discrete Zak transform, discrete Fourier transform, OFDM, OTFS, time-frequency dispersive channel.**

I. Introduction

MOTIVATED by challenges encountered in wireless communication over time-variant channels such as Doppler dispersion or equalization, a new modulation technique termed orthogonal time frequency space (OTFS) was introduced in [1]. The driving idea behind OTFS is to utilize the delay-Doppler (DD) domain to represent information-carrying symbols. The interaction of the corresponding OTFS waveform with a time-frequency (TF) dispersive channel results in a two-dimensional convolution of the symbols in the DD domain [2, Sec. III-A]. OTFS thus turns a time-variant channel into a time-invariant channel in the DD domain, i.e, all symbols experience the same channel. The time-invariant waveform-channel interaction is utilized by OTFS and allows to outperform orthogonal frequency division multiplexing (OFDM) in many scenarios, as shown in [1]–[6].

In most literature, OTFS is considered as a pre- and post-processing technique for OFDM systems. However, a more fundamental relation between the DD domain and the time domain is provided by the Zak transform, as pointed out in [2]. Recently, a treatment of OTFS by means of the *continuous* Zak transform was provided in [7]. OTFS defines orthogonal signals in the DD (Zak) domain, similar to OFDM that defines orthogonal signals in the frequency domain.

The fundamental concept of OFDM of mapping of symbols onto a set of orthogonal signals in the frequency domain dates back to 1966 [8]. However, the real breakthrough of OFDM is based on its efficient *digital* implementation using fast Fourier algorithms [9]. Efficient digital implementations lead to its adoption in many communication standards such as 802.11a or 5G. Equivalently, OTFS can be efficiently implemented using the *discrete* Zak transform (DZT). In fact, the DZT itself is based on the discrete Fourier transform (DFT), and thus, it allows for an efficient implementation as well. Efficient implementations of OTFS have been studied for example in [10], which effectively resembles the DZT. However, a fundamental treatment of OTFS based on the DZT is missing in the literature. The aim of this work is to close this gap in the literature, i.e., to provide a complete treatment of OTFS solely based on the DZT. Therefore, we do not only discuss the DZT and some of its properties, but also derive the input-output relation for time-frequency dispersive channels in the DD using the DZT and its properties. Our DZT-based approach provides an intuitive understanding of OTFS and drastically simplifies its analysis. An extended version of this paper including all the proofs and several examples can be found online [11].

We organize the remainder of the paper as follows. In Sec. II, we provide a introduction to the DZT covering all properties need for OTFS. An OTFS implementation based on DZT is presented in Sec. III. We further establish the input-output relation of OTFS based on the DZT in Sec. IV. Conclusions are drawn in Sec. V.

II. Discrete Zak Transform

A. *Definition and Relations*

In the following discussion, we will treat finite-length sequences of length N as one period of a periodic sequence with period N. Following the notation in [12], we use $Z_x^{(L,K)} \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ to denote the DZT of a sequence $x \in \mathbb{C}^{\mathbb{Z}}$ with period N. The DZT of x is defined as [12, eq. (30)]

$$Z_x^{(L,K)}[n,k] \triangleq \frac{1}{\sqrt{K}} \sum_{l=0}^{K-1} x[n+lL] e^{-j2\pi \frac{k}{K} l}, \quad n,k \in \mathbb{Z}, \tag{1}$$

where K and L are chosen such that there product matches the period of the sequence, i.e., $KL = N$. The choice of the parameters K and L will be addressed later. The corresponding inverse, the inverse DZT (IDZT), is given as

$$x[n] = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} Z_x^{(L,K)}[n,k]. \tag{2}$$

It follows from (1) that the DZT for a given n, is the unitary discrete Fourier transform (DFT) of a, by a factor of L, subsampled sequence. The variable n determines the starting phase of the downsampled sequence, whereas the variable k is the discrete frequency of its DFT. Thus, the variables n and k represent time and frequency, respectively.

We express the period N of the sequence x as a product KL with $K, L \in \mathbb{N}$. This factorization ensures that the sequence

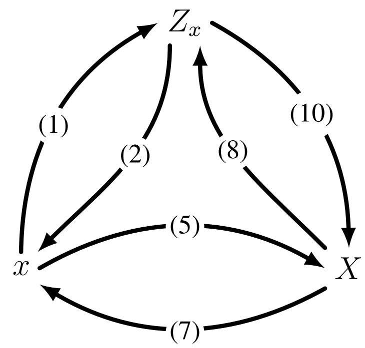

Fig. 1. Different signal representations of a sequence x and its corresponding transforms: DZT Z_x and DFT X.

can be decomposed into L subsampled sequences with period K. In general, the product KL is not unambiguously defined since different choices of K and L will result in the same product. Independent of the period, two choices are always possible and will provide some interesting insights. Firstly, the choice $K=1$ in (1) leads to

$$Z_x^{(L,1)}[n,k] = x[n], \tag{3}$$

i.e., the elements of DZT for a specific n and any k are the elements of the sequence x. Secondly, the case $L=1$ results in

$$Z_x^{(1,K)}[n,k] = \frac{1}{\sqrt{K}} \sum_{l=0}^{K-1} x[n+l] e^{-j2\pi \frac{k}{K} l}. \tag{4}$$

For $n=0$, we obtain the definition of the unitary DFT given as

$$X[k] \triangleq \frac{1}{\sqrt{K}} \sum_{l=0}^{K-1} x[l] e^{-j2\pi \frac{k}{K} l} = Z_x^{(1,K)}[0,k]. \tag{5}$$

From the circular shift property of the DFT [13, eq. (3.168)] we further have

$$Z_x^{(1,K)}[n,k] = e^{j2\pi \frac{n}{K} k} X[k]. \tag{6}$$

Following the same approach that provided the DFT (5), we can obtain the inverse DFT (IDFT). Therefore, we consider (2) for the case $L=1$ and use (6), which is

$$x[n] = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} X[k] e^{j2\pi \frac{k}{K} n}, \tag{7}$$

and which follows from the circular shift property of the DFT.

The DZT Z_x of a sequence x can also from its DFT X in (5) through (a proof is given in [11, App. A])

$$Z_x^{(L,K)}[n,k] = \frac{1}{\sqrt{L}} \sum_{l=0}^{L-1} X[k+lK] e^{j2\pi \frac{k+lK}{KL} n} \tag{8}$$

$$= e^{j2\pi \frac{n}{KL} k} Z_X^{(K,L)}[k,-n], \tag{9}$$

where $Z_X^{(K,L)}$ is the DZT of the DFT sequence X.

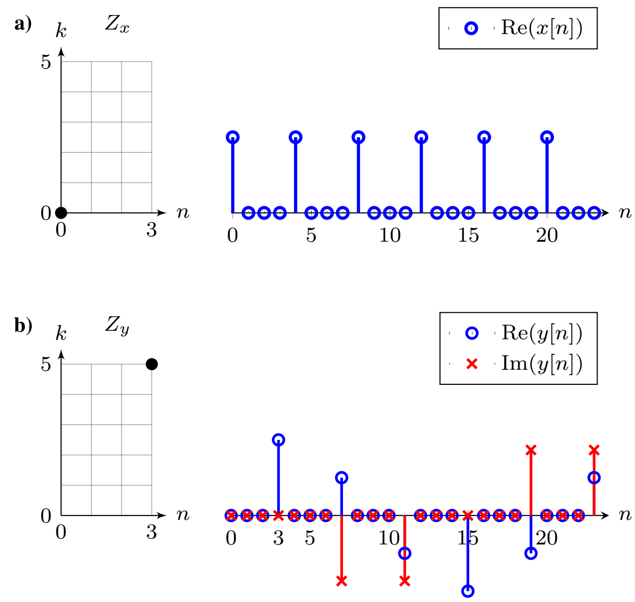

Fig. 2. Two examples of DZTs defined by a single nonzero coefficient (indicated by a dot) at $n_0 = 0$ and $k_0 = 0$ (top left) and at $n_0 = 3$ and $k_0 = 5$ (bottom left) and the corresponding sequences (right).

The corresponding inverse relation is (a proof is given in [11, App. B])

$$X[k] = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} Z_x^{(L,K)}[n,k] e^{-2\pi \frac{k}{KL} n}. \tag{10}$$

Fig. 1 summarizes the relations between the sequence x, the DZT Z_x, and the DFT X. Note that the DFT X can be obtained in two ways. Either directly via (5) or indirectly using (1) and (10). The later approach resembles the Cooley-Tuckey algorithm, a fast Fourier transform algorithm. The Zak domain thereby represents a intermediate layer when calculating the DFT. This fact has been used in a modulation scheme termed *asymmetric OFDM* [14]. *Asymmetric* OFDM is based on the Cooley-Tuckey algorithm and defines symbols in this intermediate layer rather than in the frequency domain. Equivalently, OTFS also defines symbols in the Zak domain, as we will show in the Sec. III. The similarity between OTFS and *asymmetric* OFDM has also been pointed out in [15].

B. DZT Properties and Transform Pairs

The DFT X of a sequence x with length/period K is periodic with period K, i.e., $X[k] = X[k+mK]$ with $m \in \mathbb{Z}$. Since the DZT is the DFT of the downsampled sequence, see (1), it is also periodic in the frequency variable k, i.e.,

$$Z_x^{(L,K)}[n,k+mK] = Z_x^{(L,K)}[n,k], \quad m \in \mathbb{Z}. \tag{11}$$

By the circular shift property of the DFT [13, eq. (3.168)]), we further have

$$Z_x^{(L,K)}[n+mL,k] = e^{j2\pi \frac{k}{K} m} Z_x^{(L,K)}[n,k], \quad m \in \mathbb{Z}, \tag{12}$$

i.e., the DZT is periodic in n with period L up to a complex factor $e^{j2\pi(k/K)m}$. The DZT is therefore said to be *quasi*-periodic with *quasi*-period L. Due to the periodicity properties

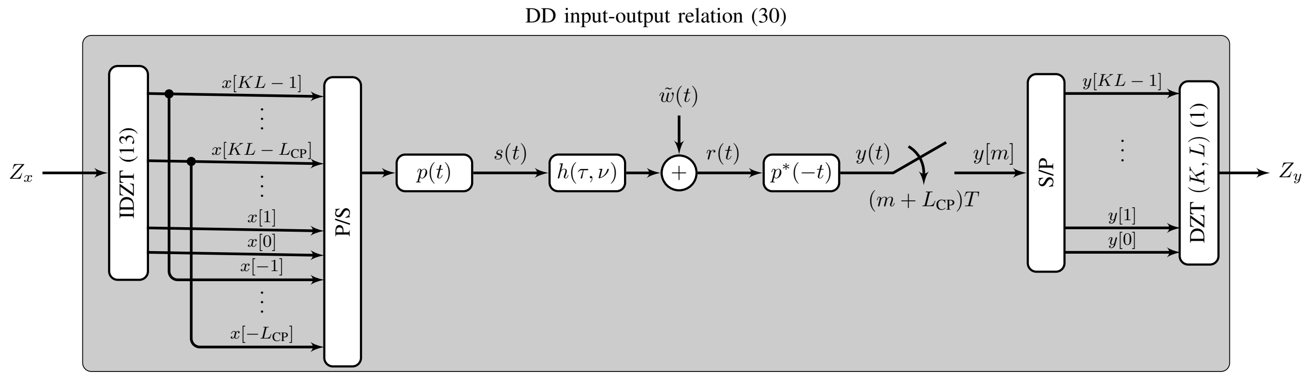

Fig. 3. OTFS system model considered in this work. The IDZT maps a sequence the symbols defined in the DD domain to a discrete sequence. A CP is added by copying the last L_{CP} samples. The resulting sequence x is converted to a serial stream by a parallel-to-serial converter (P/S) before mapped onto a pulse $p(t)$ and send over a noisy TF-dispersive channel $h(\tau,\nu)$. At the receiver, a sampled matched filter is applied before the serial stream is converted to a parallel stream by a serial-to-parallel (S/P) converter. Lastly, the sequence y is mapped to the DD domain using the DZT. The DD input-output relation is given by (30) and Theorem 1.

in (11) and (12), the DZT is fully determined by the DZT for $0 \le n \le L-1$ and $0 \le k \le K-1$, which is referred to as the *fundamental rectangle* [12]. The *quasi*-periodicity in (12) can be utilized to express the IDZT in (2) equivalently as

$$x[n+lL] = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} Z_x^{(L,K)}[n,k] e^{j2\pi \frac{k}{K} l}. \tag{13}$$

In Sec. III, we will define a signal by means of its Zak domain representation. We have already shown in (1) that the DZT is DFT of a subsampled sequence. The IDZT inverts this process, i.e., a single Zak domain component, say $Z_x^{(L,K)}[n_0,k_0]$, is a complex exponential signal whose frequency is determined by k_0 and is upsampled by a factor of L. Additionally, the starting phase is determined by n_0. Two examples of a Zak domain representations with only one nonzero element defined on the fundamental rectangle and the corresponding sequences are illustrated in Fig. 2.

In the context of OTFS, two important transform pairs are: (i) modulation, and (ii) circular convolution. The corresponding DZTs are

$$x \cdot y \Leftrightarrow \frac{1}{\sqrt{K}} \sum_{l=0}^{K-1} Z_x^{(L,k)}[n,l] Z_y^{(L,K)}[n,k-l], \tag{14}$$

and

$$x \circledast y \Leftrightarrow \sqrt{K} \sum_{m=0}^{L-1} Z_x^{(L,K)}[m,k] Z_y^{(L,K)}[n-m,k], \tag{15}$$

respectively.

III. System Model

In this section, we will use the IDZT/DZT to map the symbols in the DD domain directly to a time domain sequence and vice versa. We consider a pulse-amplitude modulation (PAM) system to map the discrete symbols onto continuous pulses, as schematically shown in Fig. 3. This approach allows the digital implementation of OTFS similar to the PAM implementation of OFDM as presented in [16, Ch. 6.4.2]. Before presenting the system model, we discuss the pre- and postprocessing approach that allows to implement OTFS in OFDM systems.

A. OTFS Preprocessing for OFDM

OTFS is typically presented as a pre- and postprocessing step for OFDM systems, see [4, Sec. II] or [5, Sec. II]. In OTFS, a frame $Z_x \in \mathbb{C}^{K\times L}$ of complex symbols is defined in the DD domain. Here we use the same notation for the symbols in the DD domain as for the Zak domain. Furthermore, we dropped the superscript (L,K) for the brevity of notation. The individual symbols are denoted by $Z_x[n,k]$, where $n = 0,1,\ldots,L-1$ and $k = 0,1,\ldots,K-1$ are the delay and Doppler indices, respectively. The frame is mapped to the TF domain via the inverse symplectic finite Fourier transform (ISFFT) defined as [5, eq. (2)]

$$X[m,l] = \frac{1}{\sqrt{KL}} \sum_{k=0}^{K-1} \sum_{n=0}^{L-1} Z_x[n,k] e^{j2\pi\left(\frac{k}{K}m - \frac{l}{L}n\right)}, \tag{16}$$

where m and l denote the symbol and subcarrier index, respectively. Practical OFDM systems are based on the IDFT/DFT and fast implementations thereof. Using the IDFT defined in (7), the discrete sequence after OFDM modulation is

$$x[n+mL] = \frac{1}{\sqrt{L}} \sum_{l=0}^{L-1} X[m,l] e^{j2\pi \frac{m}{L} n}, \tag{17}$$

for $n = 0,1,\ldots,L-1$ and $m = 0,1,\ldots,K-1$. Note that the total length of the discrete sequences representing the frame is $K \times L$.

Substituting (16) in (17) and using $(1/L)\sum_{l=0}^{L-1} e^{j2\pi l/L(n-n')} = \delta[n-n']$ where $\delta[n]$ is a Kronecker delta, we obtain

$$x[n+mL] = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} Z_x[n,k] e^{j2\pi \frac{k}{K} m}, \tag{18}$$

which is exactly the IDZT defined in (13). Consequently, the composition of ISFFT and OFDM modulation based on the

DFT resembles the DZT. Therefore, using the DZT allows to skip the ISFFT and directly obtain the discrete sequence. A similar result can be obtained for the receiver where the OFDM demodulation based on the DZT combined with the symplectic finite Fourier transform results in the DZT.

B. Transmitter

We just have shown that the composition of ISSFT and IDFT resembles the IDZT. Thus, we can directly use the IDZT to map the symbols $Z_x^{L,K}[n,k]$ to the time domain using (13). To ensure a cyclic behavior of the delayed signal and avoid interference between consecutive frames, a cyclic prefix (CP) of length L_{CP} is added. The CP is added by copying the last L_{CP} samples and inserting them at the beginning of the sequence (see Fig. 3). As we will show later, the CP will turn the linear convolution of the channel into a circular convolution and allows to use the circular convolution property (31) of the DZT. The elements of the sequence x are then mapped onto time-shifted pulses $p(t)$ using PAM. The transmitted signal is given as

$$s(t) = \sum_{n=0}^{N+L_{\mathrm{CP}}-1} x[n-L_{\mathrm{CP}}]p(t-nT), \tag{19}$$

where T is the modulation interval and $p(t)$ is a square-root Nyquist pulse.

In Sec.II-A, we have discussed the implications of the choice of the parameters K and L for the DZT. Similarly, they choice of K and L influences the OTFS system under study. For the case $K=1$, the symbol of Z_x are arranged on a line along the delay axis. The IDZT does not alter the sequence and can be skipped, see (3). Thus, the system is a single carrier system. On the other hand, for $L=1$, the symbols $Z_x^{(L,K)}[n,k]$ are arranged along the Doppler axis. The IDZT is simply the IDFT (see (7)), and (19) becomes an OFDM signal as in [16, Ch. 6.4.2].

C. Channel Model

We consider TF dispersive channels and model the received signal as [17, Ch. 1.3.1]

$$r(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau,\nu)s(t-\tau)e^{j2\pi\nu t}d\tau d\nu + \tilde{w}(t) \tag{20}$$

where $h(\tau,\nu)$ is the so-called DD spreading function. The complex noise $\tilde{w}(t)$ is assumed to be white and Gaussian with power spectral density N_0. We model the channel by P discrete scattering objects. Each scattering object is associated with a path delay τ_p, a Doppler shift ν_p, and a complex attenuation factor α_p. Thus, the spreading function $h(\tau,\nu)$ becomes

$$h(\tau,\nu) = \sum_{p=0}^{P-1} \alpha_p \delta(\tau-\tau_p)\delta(\nu-\nu_p). \tag{21}$$

Substituting (21) in (20) yields:

$$r(t) = \sum_{p=0}^{P-1} \alpha_p s(t-\tau_p)e^{j2\pi\nu_p t} + \tilde{w}(t), \tag{22}$$

i.e., a superposition of scaled, delayed, and Doppler shifted replicas of the transmitted signal. The Doppler shift is given by $\nu_p = v_p f_{\mathrm{c}}/c$, where v_p, f_{c} and c are, respectively, the relative velocity of the pth scattering object, the carrier frequency, and the speed of light. The CP's length in (19) is chosen such that $L_{\mathrm{CP}}T$ is larger than or equal to the maximum delay.

Using (19) in (22) gives

$$r(t) = \sum_{p=0}^{P-1} \alpha_p \sum_{n=0}^{N+L_{\mathrm{CP}}-1} x[n-L_{\mathrm{CP}}]p(t-nT-\tau_p)e^{j2\pi\nu_p t} + \tilde{w}(t). \tag{23}$$

D. Receiver

At the receiver, a matched filter with impulse response $p^*(-t)$ is applied. The output of the matched filter $y(t)$ is

$$y(t) = \sum_{p=0}^{P-1} \alpha_p \sum_{n=0}^{N+L_{\mathrm{CP}}-1} x[n-L_{\mathrm{CP}}] \int_{-\infty}^{\infty} p(\tau-nT-\tau_p)e^{j2\pi\nu_p\tau}p^*(\tau-t)d\tau + w(t), \tag{24}$$

where $w(t)$ is the filtered noise. Assuming that the pulse bandwidth is much larger than the maximum Doppler shift, we can approximate the integral in (24) as $e^{j2\pi\nu_p(nT+\tau_p)}h(t-nT-\tau_p)$ where $h(t)$ is the corresponding Nyquist pulse. The output of the matched filter is then

$$y(t) \approx \sum_{p=0}^{P-1} \alpha_p \sum_{n=0}^{N+L_{\mathrm{CP}}-1} x[n-L_{\mathrm{CP}}]e^{j2\pi\nu_p(nT+\tau_p)} h(t-nT-\tau_p) + w(t). \tag{25}$$

The matched filter output is sampled every T seconds and with an offset of $L_{\mathrm{CP}}T$ to discard the CP. The sampled signal $y[m] = y((m+L_{\mathrm{CP}})T)$ is

$$y[m] = \sum_{p=0}^{P-1} \alpha_p \sum_{n=-L_{\mathrm{CP}}}^{N-1} x[n]e^{j2\pi\frac{k_p}{KL}n}h_{\tau_p}[m-n] + w[m], \tag{26}$$

where $h_{\tau_p}[n] = h(nT-\tau_p)$ is the sampled Nyquist pulse and $w[m]$ are independent and identically distributed (i.i.d.) complex zero-mean Gaussian random variables with variance N_0. To shorten the notation, we combined the constant phase terms $e^{j2\pi\nu_p\tau_p}$ with the channel gain α_p in (26). Furthermore, we express ν_p as a multiple of the Doppler resolution which we define as $\Delta\nu \triangleq 1/(KLT)$, i.e., $\nu_p = \Delta\nu k_p$.

We can bound the interval for which $h(t)$ is significantly different from zero (for sufficient large L) to $\pm LT/2$. Thus, we can express $h_{\tau_p}[n]$ as

$$h_{\tau_p}[n] = \begin{cases} h(nT-\tau_p), & \text{for } -\frac{LT}{2} \le nT-\tau_p < \frac{LT}{2}, \\ 0, & \text{else.} \end{cases} \tag{27}$$

The CP allows to approximate the linear convolution in (26) by a circular convolution, equivalently to the *digital* implementation of OFDM. The sample $y[m]$ is then

$$y[m] = \sum_{p=0}^{P-1} \alpha_p y_p[m] + w[m], \tag{28}$$

where

$$y_p[m] \approx \sum_{n=0}^{KL-1} x[n] e^{j2\pi \frac{k_p}{N} n} h_{\tau_p}[m-n]. \tag{29}$$

Here, h_{τ_p} is periodized with period KL, i.e., $h_{\tau_p}[n] = h_{\tau_p}[n + KL]$. In a last step, the receiver computes the DZT of the sequence $y[m]$ before subsequent processing takes place.

IV. Delay Doppler Input-Output relation

To express the input-output relation in the DD domain for the system presented in Fig. 3, we note first that the DZT is a linear transform, and thus, we can write the DZT of (28) as

$$Z_y[n,k] = \sum_{p=0}^{P-1} \alpha_p Z_{y_p}[n,k] + Z_w[n,k], \tag{30}$$

where Z_{y_p} is the DZT of the signal y_p described in (29) and $Z_w[n,k]$ is the DZT of the noise. The elements of $Z_w[n,k]$ are i.i.d. zero-mean Gaussian random variables with variance N_0. This follows from the fact that the DZT is an unitary transform [12, Sec. VI].

For the signal model of a single reflector in (29), we provide the following result for the input-output relation in the DD domain for the OTFS system described in Sec. III.

Theorem 1. Given the fundamental rectangle $Z_x \in \mathbb{C}^{L\times K}$ of complex symbols in the DD domain, the input-output for the OTFS transmission over an time-frequency selective channel for a single reflector is

$$Z_{y_p}[n,k] = \sum_{m=0}^{L-1} \left(\sum_{l=0}^{K-1} Z_x[m,l] Z_{\nu_p}[m,k-l] \right) Z_{\tau_p}[n-m,k], \tag{31}$$

where Z_{τ_p} and Z_{ν_p} are the delay and Doppler spreading functions, respectively. The delay spreading function Z_{τ_p} is the DZT of the shifted and sampled impulse $h_{\tau_p}[n]$ in (27) and the Doppler spreading functions is given as

$$Z_{\nu_p}[n,k] = \frac{1}{\sqrt{K}} e^{j2\pi \frac{k_p}{KL} n} e^{-j\pi \frac{K-1}{K}(k-k_p)} \frac{\sin\left(\pi(k-k_p)\right)}{\sin\left(\frac{\pi}{K}(k-k_p)\right)}. \tag{32}$$

Proof. See appendix. ■

To visualize the spreading of a single symbol in the DD domain, we consider the following example. Let $L = K = 30$ and

$$Z_x[n,k] = \begin{cases} 1 & \text{for } n = k = L/2, \\ 0 & \text{else.} \end{cases} \tag{33}$$

The fundamental rectangle with the only nonzero element is presented in Fig. 4a). Furthermore, assume that $\tau = 0.5T$ and $\nu = 0.5\Delta\nu$. Note that this example causes the maximum spread of a single symbol in the DD domain. We can visualize the spreading of the symbol defined in (33) in two steps. Therefore, we define $Z_{\hat{y}}$ as the DZT resulting from the inner convolution in (31), which is presented in Fig. 4b), with respect to the Doppler index k. The resulting spread of the nonzero symbol is visualized in Fig. 4c). Finally, the symbol that has been spread in the Doppler domain is spread in the delay domain by the delay spreading function Z_τ which is illustrated in Fig. 4d). Note that due to the limited support of h_τ, see (27), the magnitude of Z_τ is independent of the index k. The resulting spread of the nonzero symbol in the DD domain is show in Fig. 4e).

For the particular case of $\tau_p = n_p T$ with $n_p = 0, 1, \dots, L_{\text{CP}} - 1$ and $\nu_p = k_p/(KLT)$ with $k_p \in \mathbb{Z}$, Z_{y_p} simplifies to

$$Z_{y_p}[n,k] = e^{j2\pi \frac{k_p}{KL}(n-n_p)} Z_x[n-n_p, k-k_p], \tag{34}$$

i.e., the received symbols are the in the DD domain translated transmitted symbols.

V. Conclusion

In this work, we presented an OTFS implementation based on the discrete Zak transform that allows for an efficient digital implementation of OTFS. Furthermore, we derived the input-output relation for the symbols in the delay-Doppler domain solely based on discrete Zak transform properties. The discrete Zak transform approach allows for a concise description of OTFS compared to the pre- and post-processing approach.

Our presented discrete Zak transform approach can be used to study and evaluate OTFS from a different perspective, potentially leading to OTFS performance improvements. For example, considering Nyquist pulse with large roll-off factors allows controlling the interference in the delay domain. Additionally, applying windows to the subsampled sequences of the DZT reduces the interference in the Doppler domain.

Appendix

To prove Theorem 1, we start by expressing the sequence y in (29) as

$$y = \left(x \cdot u_{\nu_p}\right) \circledast h_{\tau_p}, \tag{35}$$

where $u_{\nu_p}[n] = e^{j2\pi(k_p/N)n}$. Using the modulation property (14) and the convolution property (15), we can express the DZT of y as

$$Z_y[n,k] = \sum_{m=0}^{L-1} \left(\sum_{l=0}^{K-1} Z_x[m,l] Z_\nu[m,k-l] \right) Z_\tau[n-m,k]. \tag{36}$$

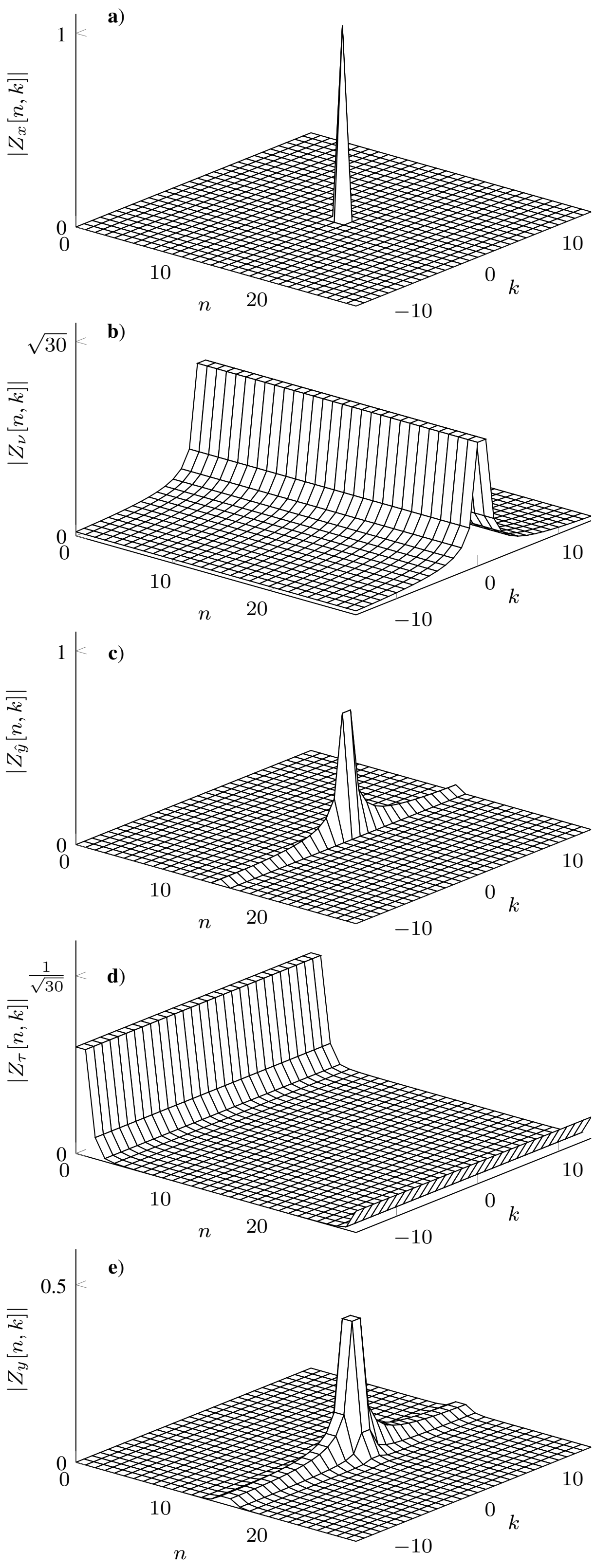

Fig. 4. Example of a spread of a symbol a) in the DD domain due to fractional delay and Doppler shift. The spread can be first evaluated in the Doppler domain c) using the Doppler spreading function in b). The spread symbol in the Doppler domain is further spread in the delay by the the delay spread function in d). The overall spread in the DD domain is shown in e).

Here, Z_ν is the DZT of the sequence u_ν, which is

$$Z_\nu[n,k] = \frac{1}{\sqrt{K}} \sum_{l=0}^{K-1} e^{j2\pi \frac{k_p}{KL}(n+lL)} e^{-j2\pi \frac{k}{K} l} \tag{37}$$

$$= \frac{1}{\sqrt{K}} e^{j2\pi \frac{k_p}{KL} n} \sum_{l=0}^{K-1} e^{-j2\pi \frac{k-k_p}{K} l}. \tag{38}$$

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