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On p-adic string amplitudes in the limit p approaches to one

M. Bocardo-Gaspar,[a] H. García-Compeán[b] and W.A. Zúñiga-Galindo[a]

[a] *Departamento de Matemáticas, Unidad Querétaro, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Libramiento Norponiente #2000, Fracc. Real de Juriquilla, Santiago de Querétaro, Qro. 76230, México*

[b] *Departamento de Física, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, P.O. Box 14-740, CP. 07000, México D.F., México*

E-mail: mbocardo@math.cinvestav.mx, compean@fis.cinvestav.mx, wazuniga@math.cinvestav.edu.mx

Abstract: In this article we discuss the limit p approaches to one of tree-level p-adic open string amplitudes and its connections with the topological zeta functions. There is empirical evidence that p-adic strings are related to the ordinary strings in the $p \to 1$ limit. Previously, we established that p-adic Koba-Nielsen string amplitudes are finite sums of multivariate Igusa's local zeta functions, consequently, they are convergent integrals that admit meromorphic continuations as rational functions. The meromorphic continuation of local zeta functions has been used for several authors to regularize parametric Feynman amplitudes in field and string theories. Denef and Loeser established that the limit $p \to 1$ of a Igusa's local zeta function gives rise to an object called topological zeta function. By using Denef-Loeser's theory of topological zeta functions, we show that limit $p \to 1$ of tree-level p-adic string amplitudes give rise to certain amplitudes, that we have named Denef-Loeser string amplitudes. Gerasimov and Shatashvili showed that in limit $p \to 1$ the well-known non-local effective Lagrangian (reproducing the tree-level p-adic string amplitudes) gives rise to a simple Lagrangian with a logarithmic potential. We show that the Feynman amplitudes of this last Lagrangian are precisely the amplitudes introduced here. Finally, the amplitudes for four and five points are computed explicitly.

Keywords: Bosonic Strings, Differential and Algebraic Geometry, Effective Field Theories

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Contents

1 Introduction

The p-adic field and string theories have been studied over the time with some periodic fluctuations in their interest (for some reviews, see [1–4]). Recently a considerable amount of work has been performed on this topic in the context of the AdS/CFT correspondence [5–8].

On the other hand, Sen's conjecture asserts that the tachyonic potential has a local minimum which exactly cancels the total energy of a D-brane in string theory [9, 10]. This conjecture has been proved by using bosonic and superstring field theories [11, 12]. In the p-adic setting, Sen's conjecture is easier to verify than the classical version, see [13].

In string theory, N-point string amplitudes are an important observable, which is computed through integration over the moduli space of Riemann surfaces [14]. It is known that even at the tree-level amplitudes the convergence of these integrals have not been well understood for general N [15]. For particular values of N, for instance $N = 4$ or $N = 5$ for open and closed strings at the tree-level there are some criteria for an appropriate choice of the external momenta in such a way that the corresponding integrals converge and the corresponding amplitudes are well defined.

In [16] (see also [17]), an effective Lagrangian was proposed from which there can be derived the Feynman rules necessary to compute the N-point p-adic string amplitudes at tree-level. Later, some time-dependent solutions to the effective action have been found representing a rolling tachyon for potentials for both p even and odd [18]. Moreover this effective action has been used also with cosmological purposes, for instance inflation was studied in [19].

The p-adic strings seem to be related in some interesting ways with ordinary strings. For instance, connections through the adelic relations [20] and through the limit when $p \to 1$ [21, 22], have been discussed in the literature. In [21], the limit $p \to 1$ of the effective action was studied, it was showed that this limit gives rise to a boundary string field theory (BSFT), which was previously proposed by Witten in the context of background independent string theory [23, 24]. The limit $p \to 1$ in the effective theory can be performed without any problem. Though originally p was a prime number for the world-sheet theory, in the effective theory one can consider p just as integer or real parameter and take formally the limit $p \to 1$. The resulting theory is related to a field theory describing an open string tachyon [25]. In the limit $p \to 1$ also there are exact noncommutative solitons, some of these solutions were found in [26]. Moreover, this limit has found a very interesting physical interpretation in [27], in terms of a lattice discretization of ordinary string worldsheet. In the worldsheet theory we cannot forget the nature of p as a prime number, thus the analysis of the limit is more subtle. The correct way of taking the limit $p \to 1$ involves the introduction of finite extensions of the p-adic field $\mathbb{Q}_p$. The totally ramified extensions gives rise to a finer discretization of the worldsheet following the rules of the renormalization group [27]. In this article we will also require the use of finite extensions of the p-adic field at the level of the string amplitudes.

In [28], we showed that the p-adic open string $N-$point tree amplitudes are bonafide integrals that admit meromorphic continuations as rational functions, by relating them with multivariate local zeta functions (also called multivariate Igusa local zeta functions [29, 30]). Moreover Denef and Loeser [31] established that the limit p approaches to one of a local zeta function gives rise a new object called *topological zeta function*, which is associated with a complex polynomial. By using the theory of topological zeta functions, we show that limit $p \to 1$ of p-adic string amplitudes gives rise to new string amplitudes, that we called *Denef-Loeser open string amplitudes* which are rational functions. Taking the limit at the level of Koba-Nielsen amplitudes involves the introduction of finite extension of the p-adic field $\mathbb{Q}_p$. This task is carried out here using the results of [28].

Finally, we want to point out that the results presented in this article are essentially independent of the results given in [28]. More precisely, in order to use Denef-Loeser's theory of topological zeta functions, we do not need the convergence of the p-adic Koba-Nielsen string amplitudes which is one of the main results in [28]. Instead of this, we can regularize 'formally' the p-adic Koba-Nielsen amplitudes by expressing them as a sum of local zeta functions, without using the fact that all these functions are holomorphic in a common domain, fact that was established in [28].

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The article is organized in the following form. In section 2, we provide a brief review of the limit $p \to 1$ in the effective action following the results from [21]. In particular, we emphasize that in this limit, the theory with a logarithmic potential, given in [21], gives rise to Feynman rules that by definition generate Feynman tree-level amplitudes of the p-adic open string in the limit $p \to 1$. Section 3 will be devoted to present the extension of our results [28] to unramified finite field extensions of the p-adic field. Section 4 gives the description of the $p \to 1$ limit of the p-adic string amplitudes. For this we use the formulation of topological zeta functions [31, 32]. We also present the computation of $N = 4$ and $N = 5$ points Denef-Loeser amplitudes. In section 5, we give some final comments. In appendices A and B at the end of the article, we review some mathematical results employed along sections 3 and 4.

2 The limit $p \to 1$ in the effective action

In this section we will briefly overview some of the results from [21]. As we mentioned before in [16], it was argued than the effective action on the D-dimensional target spacetime M and from which one can obtain the p-adic scattering amplitudes at tree-level is given by

$$S(\phi) = \frac{1}{g^2}\frac{p^2}{p-1}\int d^D x\bigg(-\frac{1}{2}\phi p^{-\frac{1}{2}\Delta}\phi + \frac{1}{p+1}\phi^{p+1}\bigg), \tag{2.1}$$

where g is the coupling constant, Δ is the Laplacian on the underlying spacetime M and D is the dimension of M, which is, in principle, arbitrary. The equation of motion is

$$p^{-\frac{1}{2}\Delta}\phi = \phi^p. \tag{2.2}$$

This equation has different solitonic solutions depending of the value of p [10, 18, 33].

Remember that p is a prime number, which is a parameter in the equation of motion (2.2), since this equation is formulated in the target space $\mathbb{R}^D$, we can extend p to be a real parameter.

By considering formally that p is a real variable and comparing the Taylor expansion of $\exp(-\frac{1}{2}\Delta \log p)$ and $\exp(p \log \phi)$ at $(p-1)$, we get that the equation of motion (2.2) becomes

$$\Delta\phi = -2\phi\log\phi, \tag{2.3}$$

which can be interpreted as a 'linearization' of (2.2) in the variable p. This is a linear theory with potential

$$V(\phi) = \phi^2 \log\frac{\phi^2}{e}. \tag{2.4}$$

Thus 'the $p \to 1$ limit of effective action' yields

$$S(\phi) = \int d^D x\Big((\partial\phi)^2 - V(\phi)\Big), \tag{2.5}$$

where $(\partial\phi)^2 = \eta^{ij}\partial_i\phi \cdot \partial_j\phi$ and η^{ij} is the inverse of Minkowski metric

$$\eta_{ij} = \mathrm{diag}(-1, 1, \ldots, 1), \tag{2.6}$$

in the sense that action (2.5) leads to equations of motion (2.3). Notice that the factor $\frac{p^2}{g^2(p-1)}$ in action (2.1) does not play any role in the linearization of the equation of motion (2.2) around $p = 1$. The computation of the correlation functions of the interacting theory can be done leaving out the mentioned factor. At the end of the computation the coupling constant g can be introduced again without any problem.

Then Feynman rules which can be derived from the above Lagrangian are simple to obtain (see for instance [34]). The free theory with a source term is given by

$$S_0(\phi) = \int d^D x \big[(\partial\phi)^2 + \phi^2(x) + J(x)\phi(x)\big]. \tag{2.7}$$

The equation of motion is given by

$$(\Delta - 1)\phi(x) = \frac{1}{2}J(x). \tag{2.8}$$

We use the following notation and conventions:

$$\phi(x) = -\int d^D y G(x,y)\frac{J(y)}{2}, \qquad G(x,y) = \int \frac{d^D k}{(2\pi)^D} e^{i\boldsymbol{k}\cdot(x-y)} G(\boldsymbol{k})$$

and

$$\delta^D(x-y) = \int \frac{d^D k}{(2\pi)^D} e^{i\boldsymbol{k}\cdot(x-y)},$$

where $G(x,y)$ is the Green function of operator $\Delta - 1$ and $G(\boldsymbol{k})$ is its Fourier transform. After a standard analysis in quantum field theory one finds that the propagator x_{ij}, represent the Green function $G(\boldsymbol{k})$, and can be expressed as

$$x_{ij} = \frac{1}{\boldsymbol{k}_i \cdot \boldsymbol{k}_j + 1}, \tag{2.9}$$

where we are using the notation for the propagator from [16]. Here $\boldsymbol{k}_i$ with $i = 1, \ldots, N$ are the external momenta of the scattered particles. The products $\boldsymbol{k}_i \cdot \boldsymbol{k}_j$ for all the possible values of pairs i, j represent the different tachyons propagating in channels s, t and u. Moreover, the interactions are represented by vertices with four external lines attached to each vertex.

In [21] it was argued that action (2.5) equivalently describes the tree-level of the tachyon field (without quantum corrections) and neglecting all other fields of the BSFT action given in [23, 24]. The relation is performed through a simple field redefinition $T = -\log\phi^2$, where T is the tachyon field.

2.1 Amplitudes from the Gerasimov-Shatashvili Lagrangian

In this section we show how to extract the four and five-point amplitudes of the Gerasimov-Shatashvili Lagrangian (2.5) found in [21]. In order to do that, we first require to study the interacting theory. The generating functional of the correlation function for the free theory is given by

$$\mathcal{Z}_0[J] = \mathcal{N}[\det(\Delta - 1)]^{-1/2} \exp\left\{ -\frac{i}{4\hbar}\int d^D x \int d^D x' J(x) G_F(x-x') J(x')\right\}, \tag{2.10}$$

where $G_F(x-x')$ is the Green-Feynman function of time-ordered product of two fields of the theory, $\mathcal{N}$ is a normalization constant, $[\det(\Delta-1)]^{-1/2}$ is a suitable regularization of the divergent determinant bosonic operator, see e.g. [34].

Action (2.5) can be conveniently rewritten as

$$S(\phi) = \int d^D x \big[(\partial\phi)^2 + m^2 - U(\phi)\big], \tag{2.11}$$

where $U(\phi) = 2\phi^2 \log\phi$. We expand $U(\phi)$ in Taylor series around the origin as follows:

$$U(\phi) = A\phi^2 + B\phi^3 + C\phi^4 + D\phi^5 + \cdots, \tag{2.12}$$

where A, B, C and D are certain real constants.

In the standard formalism of QFT [34], the N-point correlation functions are proportional to

$$\langle T(\widehat{\phi}(x_1)\widehat{\phi}(x_2)\cdots\widehat{\phi}(x_N))\rangle = \frac{(-i\hbar)^N}{\mathcal{Z}[J]} \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_N)}\bigg|_{J=0}, \tag{2.13}$$

where the $\widehat{\phi}$'s are N local operators (observables) in N different points $x_1, x_2, \ldots, x_N$ of the Minkowski spacetime, $\mathcal{Z}[J]$ is the generating functional constructed using interacting Lagrangian (2.11). The functional can be computed as

$$\begin{aligned} \mathcal{Z}[J] = \exp\bigg\{ &-\frac{iB}{\hbar}\int d^D x \Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)^3 \\ &-\frac{iC}{\hbar}\int d^D x \Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)^4 - \frac{iD}{\hbar}\int d^D x \Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)^5 + \cdots \bigg\} \mathcal{Z}_0[J]. \end{aligned} \tag{2.14}$$

We assert that connected tree-level scattering amplitudes of this theory match exactly with the corresponding amplitudes of the effective action (2.1) in the limit when p tends to one.

2.2 Four-point amplitudes

The 4-point amplitudes can be computed as follows: the 4-point vertex can be obtained purely from the quartic interaction at the first order in perturbation theory. The generating functional, with the vertex labeled by x and 4 external legs attached to it, is given by

$$\mathcal{Z}[J] = \cdots - iC\hbar^3 \int d^D x \Big(\frac{\delta}{\delta J(x)}\Big)^4 \mathcal{Z}_0[J] + \cdots. \tag{2.15}$$

The corresponding 4-point amplitude is proportional to

$$\begin{aligned} &\frac{\delta^4 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} \\ &= -4! iC\hbar^3 \int d^D x \Big[-\frac{i}{2\hbar}G_F(x-x_1)\Big]\Big[-\frac{i}{2\hbar}G_F(x-x_2)\Big]\Big[-\frac{i}{2\hbar}G_F(x-x_3)\Big]\Big[-\frac{i}{2\hbar}G_F(x-x_4)\Big] \\ &= -\frac{3iC}{2\hbar}\int d^D x\; G_F(x-x_1)\; G_F(x-x_2)\; G_F(x-x_3)\; G_F(x-x_4), \end{aligned} \tag{2.16}$$

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where $G_F(x-y)$ is the Green-Feynman propagator. In the Fourier space the above amplitude corresponds to the Feynman diagram with only one vertex and four external legs. In analogy to the notation from [16], we will represent it by the letter $\overline{K}_4$.

The interaction term $B\phi^3$ in the Lagrangian has also a non-vanishing contribution to the 4-points tree amplitudes at the second order in perturbation theory. They are described by Feynman diagrams with two vertices located at points x and y connected by a propagator $G_F(x-y)$ and with two external legs attached to each vertex. In this case the amplitude is computed from the relevant part of the generating functional

$$\mathcal{Z}[J] = \cdots + \frac{B^2\hbar^4}{2}\int d^Dx \int d^Dy \left(\frac{\delta}{\delta J(x)}\right)^3\left(\frac{\delta}{\delta J(y)}\right)^3 \mathcal{Z}_0[J] + \cdots . \tag{2.17}$$

The connected 4-point amplitudes at the second order of the cubic interaction $C\phi^3$ yields

$$\begin{aligned}
\frac{\delta^4\mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} &= 18B^2\hbar^4\int d^Dx\int d^Dy\left[-\frac{i}{2\hbar}G_F(x-y)\right]\\
&\times\left\{\left[-\frac{i}{2\hbar}G_F(x-x_4)\right]\left[-\frac{i}{2\hbar}G_F(x-x_3)\right]\left[-\frac{i}{2\hbar}G_F(y-x_2)\right]\left[-\frac{i}{2\hbar}G_F(y-x_1)\right]\right.\\
&+\left[-\frac{i}{2\hbar}G_F(x-x_4)\right]\left[-\frac{i}{2\hbar}G_F(y-x_3)\right]\left[-\frac{i}{2\hbar}G_F(x-x_2)\right]\left[-\frac{i}{2\hbar}G_F(y-x_1)\right]\\
&+\left[-\frac{i}{2\hbar}G_F(x-x_4)\right]\left[-\frac{i}{2\hbar}G_F(y-x_3)\right]\left[-\frac{i}{2\hbar}G_F(y-x_2)\right]\left[-\frac{i}{2\hbar}G_F(x-x_1)\right]\\
&+\left[-\frac{i}{2\hbar}G_F(y-x_4)\right]\left[-\frac{i}{2\hbar}G_F(y-x_3)\right]\left[-\frac{i}{2\hbar}G_F(x-x_2)\right]\left[-\frac{i}{2\hbar}G_F(x-x_1)\right]\\
&+\left[-\frac{i}{2\hbar}G_F(y-x_4)\right]\left[-\frac{i}{2\hbar}G_F(x-x_3)\right]\left[-\frac{i}{2\hbar}G_F(y-x_2)\right]\left[-\frac{i}{2\hbar}G_F(x-x_1)\right]\\
&\left.+\left[-\frac{i}{2\hbar}G_F(y-x_4)\right]\left[-\frac{i}{2\hbar}G_F(x-x_3)\right]\left[-\frac{i}{2\hbar}G_F(x-x_2)\right]\left[-\frac{i}{2\hbar}G_F(y-x_1)\right]\right\}.
\end{aligned} \tag{2.18}$$

This amplitude corresponds to the scattering of particles propagating in the sum of the s, t and u channels. They together with the 4-point vertex (2.16) constitute the tree-level amplitudes arising in the 4-point p-adic amplitudes in the limit when $p \to 1$. Thus in the Fourier space the total amplitude for 4-point amplitudes consists of the sum of the amplitude given by eq. (2.16) plus the contribution (2.18) that we schematically write (in notation from [16]) as

$$A_4 = \overline{K}_4 + \sum_{i<j} x_{ij}, \tag{2.19}$$

where x_{ij} is given by (2.9).

2.3 Five-point amplitudes

For the 5-point amplitudes there is a contribution coming from the quintic interaction term $D\phi^5$ in the Lagrangian. Thus we have at the first order in the perturbative expansion that the relevant contribution of the generating functional is given by

$$\mathcal{Z}[J] = \cdots - D\hbar^4\int d^Dx\left(\frac{\delta}{\delta J(x)}\right)^5 \mathcal{Z}_0[J] + \cdots . \tag{2.20}$$

The vertex function for the 5-point amplitude reads

$$\frac{\delta^5 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)\delta J(x_5)}\bigg|_{J=0} = -5!D\hbar^4 \int d^D x \left[-\frac{i}{2\hbar}G_F(x-x_1)\right] \\ \times \left[-\frac{i}{2\hbar}G_F(x-x_2)\right]\left[-\frac{i}{2\hbar}G_F(x-x_3)\right]\left[-\frac{i}{2\hbar}G_F(x-x_4)\right]\left[-\frac{i}{2\hbar}G_F(x-x_5)\right]. \tag{2.21}$$

Similarly to the case of 4-point amplitudes, the above amplitude is represented by a diagram with only one vertex and five external legs and the amplitude denoted by $\overline{K}_5$ in the Fourier space.

Now we study the possible terms to the 5-point tree-amplitude coming from the interaction term $B\phi^3 \times C\phi^4$. This term consist of p-adic amplitudes in the fourier space constructed from amplitudes with 2-vertices, 5 external legs and one internal leg as described in section 3 from [16].

The relevant part of the generating functional is given by

$$\mathcal{Z}[J] = \cdots - iBC\hbar^5 \int d^D x \int d^D y \left(\frac{\delta}{\delta J(x)}\right)^3 \left(\frac{\delta}{\delta J(y)}\right)^4 \mathcal{Z}_0[J] + \cdots. \tag{2.22}$$

The computation of a 5-point amplitude from this generating functional is given by

$$\frac{\delta^5 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)\delta J(x_5)}\bigg|_{J=0} \\ = -iBC(12)^2\hbar^5 \int d^D x \int d^D y \left[-\frac{i}{2\hbar}G_F(x-y)\right]\left\{\left[-\frac{i}{2\hbar}G_F(x-x_5)\right]\left[-\frac{i}{2\hbar}G_F(x-x_4)\right] \right. \\ \times \left[-\frac{i}{2\hbar}G_F(y-x_3)\right]\left[-\frac{i}{2\hbar}G_F(y-x_2)\right]\left[-\frac{i}{2\hbar}G_F(y-x_1)\right] + \cdots \\ + \left[-\frac{i}{2\hbar}G_F(y-x_5)\right]\left[-\frac{i}{2\hbar}G_F(x-x_4)\right]\left[-\frac{i}{2\hbar}G_F(x-x_3)\right] \\ \left. \times \left[-\frac{i}{2\hbar}G_F(y-x_2)\right]\left[-\frac{i}{2\hbar}G_F(y-x_1)\right] + \cdots \right\}. \tag{2.23}$$

There will be in total ten terms in equation (2.23) including the possible permutations of labels $(x_1, \ldots, x_5)$ and of the two vertices at x and y.

Finally the lacking contribution to the 5-point amplitudes comes from the third order of the cubic interaction term in the Lagrangian. We have three vertices labeled by x, y and z. Two of these vertices are connected to two external legs and to one internal line. The other is attached to two internal lines and one external. Thus the generating function in this case is given by

$$\mathcal{Z}[J] = \cdots + \frac{B^3\hbar^6}{3!} \int d^D x \int d^D y \int d^D z \left(\frac{\delta}{\delta J(x)}\right)^3 \left(\frac{\delta}{\delta J(y)}\right)^3 \left(\frac{\delta}{\delta J(z)}\right)^3 \mathcal{Z}_0[J] + \cdots. \tag{2.24}$$

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The contribution of these terms to the 5-point function results

$$\frac{\delta^5 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)\delta J(x_5)}\bigg|_{J=0} = \frac{B^3\hbar^6 a}{3!}\int d^Dx\int d^Dy\int d^Dz\left[-\frac{i}{2\hbar}G_F(x-y)\right]$$
$$\times\left[-\frac{i}{2\hbar}G_F(y-z)\right]\left\{\left[-\frac{i}{2\hbar}G_F(y-x_5)\right]\left[-\frac{i}{2\hbar}G_F(z-x_4)\right]\left[-\frac{i}{2\hbar}G_F(z-x_3)\right]\right.$$
$$\left.\times\left[-\frac{i}{2\hbar}G_F(x-x_2)\right]\left[-\frac{i}{2\hbar}G_F(x-x_1)\right]+\cdots\right\}, \tag{2.25}$$

where a is a suitable constant.

Thus we summarize the contributions to the 5-point amplitudes A_5 obtained from the Gerasimov and Shatashvili Lagrangian. This is written schematically by

$$A_5 = \overline{K}_5 + \sum_{i<j}\overline{K}_4 x_{ij} + \sum_{i<j}\sum_{k<l} x_{ij}\cdot x_{kl},$$

where the three terms in the sum correspond to the amplitudes given by eqs. (2.21), (2.23) and (2.25) respectively, and where x_{ij} is given by (2.9).

Through out this procedure we can compute a N-point tree-level p-adic string amplitude in the limit $p\to 1$ for any number of external legs N. As we have argued in this section, this amplitude can be obtained from the Gerasimov-Shatashvili action with a logarithmic potential (2.5).

Notice that the calculations involving the limit $p\to 1$ in the case of effective action are performed in $\mathbb{R}^D$, meanwhile the calculations involving the limit $p\to 1$ in the case of p-adic string amplitudes are performed in $\mathbb{Q}_p^D$, and in the p-adic topology the limit $p\to 1$ does not make sense. In section 4 we will give a rigorous procedure to get the limit $p\to 1$ in the amplitudes and we will reproduce the correct Feynman rules discussed in the present section. As a byproduct one can see that these amplitudes can be computed in a more economic and efficient way by using this rigorous procedure.

3 Koba-Nielsen string amplitudes on finite extensions of non-Archimedean local fields

3.1 p-adic string amplitudes

In [16] Brekke et al. discussed the amplitudes for the N-point tree-level p-adic bosonic string amplitudes. They also computed the four and five-points amplitudes explicitly and it was investigated how these amplitudes can be obtained from an effective Lagrangian. The open string $N-$point tree amplitudes over the p-adic field $\mathbb{Q}_p$ are defined as

$$\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}) = \int_{\mathbb{Q}_p^{N-3}}\prod_{i=2}^{N-2}|x_i|_p^{\boldsymbol{k}_1\cdot\boldsymbol{k}_i}|1-x_i|_p^{\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}\prod_{2\leq i<j\leq N-2}|x_i-x_j|_p^{\boldsymbol{k}_i\cdot\boldsymbol{k}_j}\prod_{i=2}^{N-2}dx_i, \tag{3.1}$$

where $|\cdot|_p$ is the p-adic norm (see appendix A), $\prod_{i=2}^{N-2}dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$, $\underline{\boldsymbol{k}}=(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)$ and $\boldsymbol{k}_i=(k_{0,i},\ldots,k_{25,i})$, $i=1,\ldots,N$, $N\geq 4$, are the momentum

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components of the i-th tachyon (with Minkowski inner product $\boldsymbol{k}_i \cdot \boldsymbol{k}_j = -k_{0,i}k_{0,j} + k_{1,i}k_{1,j} + \cdots + k_{25,i}k_{25,j}$) obeying

$$\sum_{i=1}^{N} \boldsymbol{k}_i = \mathbf{0}, \qquad \boldsymbol{k}_i \cdot \boldsymbol{k}_i = 2, \tag{3.2}$$

for $i = 1, \ldots, N$. A central problem in string theory is to know whether integrals of type (3.1) converge for some complex values $\boldsymbol{k}_i \cdot \boldsymbol{k}_j$. Our results in [28] allow us to solve this problem.

3.2 Non-Archimedean string zeta functions

In this subsection we extend some results of our previous work [28] from $\mathbb{Q}_p$ to $\mathbb{K}_e$, the unique unramified extension of $\mathbb{Q}_p$ of degree e. In this article we use most of the notation and conventions introduced in [28].

For a discussion about non-Archimedean local fields, the reader may consult appendix A or references [3, 40–42].

We consider $\mathbb{K}$ a non-Archimedean local field of characteristic zero. Denote by $R_{\mathbb{K}}$ the *ring of integers* of $\mathbb{K}$, this ring contains a unique maximal ideal $P_{\mathbb{K}}$, which is principal. We fix a generator π (also called a *uniformizing parameter* of $\mathbb{K}$), so $P_{\mathbb{K}} = \pi R_{\mathbb{K}}$.

Any finite extension $\mathbb{K}$ of $\mathbb{Q}_p$ is a non-Archimedean local field. Then

$$pR_{\mathbb{K}} = \pi^m R_{\mathbb{K}}, \qquad m \in \mathbb{N}. \tag{3.3}$$

If $m = 1$ we say that $\mathbb{K}$ is a *unramified* extension of $\mathbb{Q}_p$, otherwise we say that $\mathbb{K}$ is a *ramified* extension. It is well known that for every positive integer e there exist a unique unramified extension $\mathbb{K}_e$ of $\mathbb{Q}_p$ of degree e, which means that $\mathbb{K}_e$ is a $\mathbb{Q}_p$-vector space of dimension e. From now on, π stands for a local uniformizing parameter of $\mathbb{K}_e$, thus $pR_{\mathbb{K}_e} = \pi R_{\mathbb{K}_e}$, $R_{\mathbb{K}_e}/P_{\mathbb{K}_e} \cong \mathbb{F}_{p^e}$ and $|\pi|_{\mathbb{K}_e} = p^{-e}$. Thus π in $\mathbb{K}_e$ plays the role of p in $\mathbb{Q}_p$.

We now describe the generalization of p-adic Koba-Nielsen amplitudes. These amplitudes are generalized as follows:

$$\boldsymbol{A}^{(N)}\left(\underline{\boldsymbol{k}}, \mathbb{K}_e\right) = \int_{\mathbb{K}_e^{N-3}} \prod_{i=2}^{N-2} |x_i|_{\mathbb{K}_e}^{\boldsymbol{k}_1 \cdot \boldsymbol{k}_i} |1 - x_i|_{\mathbb{K}_e}^{\boldsymbol{k}_{N-1} \cdot \boldsymbol{k}_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_{\mathbb{K}_e}^{\boldsymbol{k}_i \cdot \boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i \tag{3.4}$$

where $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{K}_e^{N-3}$.

Following [28], in order to study the amplitude $\boldsymbol{A}^{(N)}\left(\underline{\boldsymbol{k}}; \mathbb{K}_e\right)$, we introduce the *open string N-point zeta function*, which is defined by

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}, \mathbb{K}_e\right) := \int_{\mathbb{K}_e{}^{N-3} \smallsetminus \Lambda} F\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N, \mathbb{K}_e\right) \prod_{i=2}^{N-2} dx_i, \tag{3.5}$$

where

$$F\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N, \mathbb{K}_e\right) = \prod_{i=2}^{N-2} |x_i|_{\mathbb{K}_e}^{s_{1i}} |1 - x_i|_{\mathbb{K}_e}^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_{\mathbb{K}_e}^{s_{ij}}. \tag{3.6}$$

Here we assume that $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, with $D = \frac{(N-3)(N-4)}{2} + 2(N-3)$. Moreover $s_{ij} = s_{ji}$ for all i, j and $\boldsymbol{x} = (x_2, \ldots, x_{N-2}) \in \mathbb{K}_e^{N-3}$ and Λ is defined by

$$\Lambda := \left\{ (x_2, \ldots, x_{N-2}) \in \mathbb{K}_e^{N-3}; \prod_{i=2}^{N-2} x_i (1 - x_i) \prod_{2 \leq i < j \leq N-2} (x_i - x_j) = 0 \right\} \tag{3.7}$$

and $\prod_{i=2}^{N-2} dx_i$ is the Haar measure of $\mathbb{K}_e^{N-3}$ normalized so that the measure of $R_{\mathbb{K}_e}^{N-3}$ is 1. The name for $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$ comes from the fact that it is a finite sum of multivariate local zeta functions, as we explain below, see also appendix B. In the definition of eq. (3.5) we remove the set Λ from the domain of integration in order to use the formula $a^s = e^{s \ln a}$ for $a > 0$ and $s \in \mathbb{C}$.

For a subset I of $T = \{2, \ldots, N-2\}$, we define the zeta function

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, \mathbb{K}_e) = \int_{Sect(I)} F(\underline{\boldsymbol{s}}, \boldsymbol{x}; N, \mathbb{K}_e) \prod_{i=2}^{N-2} dx_i, \tag{3.8}$$

attached to the sector

$$Sect(I) = \left\{ (x_2, \ldots, x_{N-2}) \in \mathbb{K}_e^{N-3}; |x_i|_{\mathbb{K}_e} \leq 1 \Leftrightarrow i \in I \right\}. \tag{3.9}$$

Then $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$ is a sum over all the possible inequivalent sectors $Sect(I)$:

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e) = \sum_{I \subseteq T} \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, \mathbb{K}_e). \tag{3.10}$$

As in [28], we can show that

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e) = \sum_{I \subseteq T} p^{eM(\underline{\boldsymbol{s}})} \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e) \ \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, \mathbb{K}_e), \tag{3.11}$$

where

$$M(\underline{\boldsymbol{s}}) := |T \smallsetminus I| + \sum_{i \in T \smallsetminus I} (s_{1i} + s_{(N-1)i}) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \smallsetminus I,\ j \in T}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in I, j \in T \smallsetminus I}} s_{ij}. \tag{3.12}$$

The functions $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e)$ and $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, \mathbb{K}_e)$ are given by

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e) = \int_{R_{\mathbb{K}_e}^{|I|}} F_0(\boldsymbol{s}, \boldsymbol{x}; N, \mathbb{K}_e) \prod_{i \in I} dx_i, \tag{3.13}$$

where

$$F_0(\underline{\boldsymbol{s}}, \boldsymbol{x}; N, \mathbb{K}_e) := \prod_{i \in I} |x_i|_{\mathbb{K}_e}^{s_{1i}} |1 - x_i|_{\mathbb{K}_e}^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_{\mathbb{K}_e}^{s_{ij}} \tag{3.14}$$

and

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, \mathbb{K}_e) = \int_{R_{\mathbb{K}_e}^{|T \smallsetminus I|}} F_1(\underline{\boldsymbol{s}}, \boldsymbol{x}; N, \mathbb{K}_e) \prod_{i \in T \smallsetminus I} dx_i, \tag{3.15}$$

where

$$F_1(\underline{\boldsymbol{s}}, \boldsymbol{x}; N, \mathbb{K}_e) := \frac{\prod\limits_{\substack{2 \le i < j \le N-2 \\ i,j \in T \smallsetminus I}} |x_i - x_j|_{\mathbb{K}_e}^{s_{ij}}}{\prod\limits_{i \in T \smallsetminus I} |x_i|_{\mathbb{K}_e}^{2 + s_{1i} + s_{(N-1)i} + \sum_{2 \le j \le N-2, j \ne i} s_{ij}}}. \tag{3.16}$$

By convention $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; \varnothing, 0, \mathbb{K}_e) = 1$, $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; \varnothing, 1, \mathbb{K}_e) = 1$. Regarding the notation, for $J \subseteq T$, $J \ne \varnothing$, we denote by $R_{\mathbb{K}_e}^{|J|}$ the set $\{(x_i)_{i \in J}; x_i \in R_{\mathbb{K}_e}\}$, if $J = \varnothing$, then $R_{\mathbb{K}_e}^{|J|} = \varnothing$. We denote by $T \smallsetminus I = \{j \in T; j \notin I\}$.

The functions $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e)$ and $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, \mathbb{K}_e)$ are multivariate local zeta functions, see Apendix B. The local zeta functions are related with deep arithmetical and geometrical matters, and they have been studied extensively since the 50s, see [30, 44] and references therein.

In [28] we showed that $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{Q}_p)$ has an analytic continuation to the whole $\mathbb{C}^D$ as a rational function in the variables $p^{-s_{ij}}$, see Propositions 1, 2 and Theorem 1 in [28]. These results are valid for finite extensions of $\mathbb{Q}_p$. More precisely, all the zeta functions appearing in the right-hand side of formula (3.11) admit analytic continuations to the whole $\mathbb{C}^D$ as rational functions in the variables $p^{-es_{ij}}$. In addition, each of these functions is holomorphic on a certain domain in $\mathbb{C}^D$ and the intersection of all these domains contains an open and connected subset of $\mathbb{C}^D$. Therefore $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$ is a holomorphic function in a certain domain of $\mathbb{C}^D$ admitting a meromorphic continuation to the whole $\mathbb{C}^D$ as a rational function in the variables $p^{-es_{ij}}$, see Theorem 1 in [28].

We use $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$ as regularizations of Koba-Nielsen amplitudes $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}, \mathbb{K}_e)$, more precisely, we define

$$\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}, \mathbb{K}_e) = \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e) \,|_{s_{ij} = \boldsymbol{k}_i \cdot \boldsymbol{k}_j}\,. \tag{3.17}$$

Then $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}, \mathbb{K}_e)$ is a well defined rational function in the variables $p^{-e\boldsymbol{k}_i \cdot \boldsymbol{k}_j}$, which agree with the integral (3.4) when it converges.

4 The limit $p \to 1$ in p-adic string amplitudes

In the previous sections we have seen that the p-adic string amplitudes are essentially local zeta functions, explicitly $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e)$ and $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, \mathbb{K}_e)$ are both multivariate local zeta functions of type $\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K}_e)$ for suitable $\boldsymbol{f}$ (for more details see appendix B).

4.1 Topological zeta functions

To make mathematical sense of the limit of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{Q}_p)$ as $p \to 1$ we use the work of Denef and Loeser, see [31] and [32]. The first step is to pass from $\mathbb{Q}_p$ to $\mathbb{K}_e$, $e \in \mathbb{N}$, and consider $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$ instead of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{Q}_p)$, and compute the limit of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$ as $e \to 0$ instead of the limit of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{Q}_p)$ as $p \to 1$. In order to compute the limit $e \to 0$ is necessary to have an explicit formula for $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, \mathbb{K}_e)$ which is equivalent to have explicit formulas for integrals $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e)$ and $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, \mathbb{K}_e)$, see (3.11). These integrals are special types of multivariate local zeta functions $\boldsymbol{Z}(\underline{\boldsymbol{s}}, f, \mathbb{K}_e)$, see appendix B. Consequently, we need an explicit formula for the multivariate Igusa's local zeta function $\boldsymbol{Z}(\underline{\boldsymbol{s}}, f, \mathbb{K}_e)$, this formula is a simple variation of the explicit formula established by Denef [46], which requires Hironaka's desingularization Theorem [45], see also appendix B.1.

Let $\boldsymbol{f} = (f_1, \ldots, f_r)$ be a polynomial mapping, with $f_i(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, a non-constant polynomial for $i = 1, \ldots, r$. Let (Y, h) be an embedded resolution of singularities for $D = \operatorname{Spec} \mathbb{Q}[\boldsymbol{x}] / (\prod_{i=1}^r f_i(\boldsymbol{x}))$ over $\mathbb{Q}$ with $\{E_i\}$ the irreducible components of $h^{-1}(0)$.

For any scheme V of finite type over a field $L \subset \mathbb{C}$, we denote by $\chi(V)$ the Euler characteristic of the $\mathbb{C}$-analytic space associated with V. Denef and Loeser associated to $\prod_{i=1}^r f_i(\boldsymbol{x})$ the following function (the topological zeta function):

$$\boldsymbol{Z}_{\text{top}}(\boldsymbol{s}) = \sum_{I \subseteq T} \chi\left(\overset{\circ}{E}_I\right) \prod_{i \in I} \frac{1}{v_i + \sum_{j=1}^r N_{ij} s_j}, \tag{4.1}$$

for the notation, see appendix B.

In arbitrary dimension there is no a canonical way of picking an embedded resolution of singularities for a divisor. Then, it is necessary to show that definition (4.1) is independent of the resolution of singularities chosen, this fact was established by Denef and Loeser in [31], see also [32]. By using the explicit formula (B.5)–(B.6), Denef and Loeser showed that

$$\boldsymbol{Z}_{\text{top}}(\boldsymbol{s}) = \lim_{e \to 0} \boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K}_e). \tag{4.2}$$

The limit $e \to 0$ makes sense because one can l-adically interpolate $\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K}_e)$ as a function of e. This means that there exist $\kappa \in \mathbb{N} \smallsetminus \{0\}$ and a meromorphic function in the variables $\boldsymbol{s}$ and e, $\boldsymbol{Z}_l(\boldsymbol{s}, \boldsymbol{f}, e, n)$ on $\mathbb{Z}_l^r \times (\kappa \mathbb{Z}_l)$ such that for any $\boldsymbol{s} \in \mathbb{N}^r$ and $e \in \kappa \mathbb{Z}_l$ verifies that

$$\boldsymbol{Z}_l(\boldsymbol{s}, \boldsymbol{f}, e, n) = \boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K}_e). \tag{4.3}$$

In addition, it is possible to choose κ such that $\boldsymbol{Z}_l(\boldsymbol{s}, \boldsymbol{f}, e, n)\left(v_i + \sum_{j=1}^r N_{ij} s_j\right)^n$ is a convergent series on $\mathbb{Z}_l^r \times (\kappa \mathbb{Z}_l)$.

In particular

$$\lim_{e \to 0} c_I(\mathbb{K}_e) = \chi_c\left(\overset{\circ}{E}_I \otimes \mathbb{F}_{p^e}^a, \mathcal{F}_{\chi_{\text{triv}}}\right) = \chi\left(\overset{\circ}{E}_I\right), \tag{4.4}$$

for almost all prime number p, where χ_c denotes the Euler characteristic with respect to l-adic cohomology with compact support, and $\mathcal{F}_{\chi_{\text{triv}}}$ denotes a suitable sheaf, and $\mathbb{F}_{p^e}^a$ denotes an algebraic closure of $\mathbb{F}_{p^e}$. Furthermore, they gave a description of the poles of the multivariate local zeta functions in terms of the poles of the topological zeta function: if $\boldsymbol{\rho}$ is a pole of $\boldsymbol{Z}_{\text{top}}(\boldsymbol{s})$, then for almost all prime numbers p there exist infinitely many unramified extensions $\mathbb{K}_e$ of $\mathbb{Q}_p$ for which $\boldsymbol{\rho}$ is a pole of $\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K}_e)$, see [Theorem (2.2) in [31]].

4.2 String amplitudes and topological string zeta functions

By using the fact that $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e)$ and $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, \mathbb{K}_e)$ are particular cases of $\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K}_e)$, and by applying (4.2), we define

$$\boldsymbol{Z}_{\text{top}}^{(N)}(\underline{\boldsymbol{s}}; I, 0) = \lim_{e \to 0} \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, \mathbb{K}_e) \tag{4.5}$$

and

$$\boldsymbol{Z}_{\text{top}}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1\right)=\lim_{e\rightarrow 0}\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1,\mathbb{K}_e\right), \tag{4.6}$$

which are elements of $\mathbb{Q}\left(s_{ij},i,j\in\{1,\ldots,N-1\}\right)$, the field of rational functions in the variables s_{ij}, $i,j\in\{1,\ldots,N-1\}$ with coefficients in $\mathbb{Q}$. Then, by using (3.11) we define the *open string N-point topological zeta function* as

$$\boldsymbol{Z}_{\text{top}}^{(N)}\left(\underline{\boldsymbol{s}}\right)=\sum_{I\subseteq T}\boldsymbol{Z}_{\text{top}}^{(N)}\left(\underline{\boldsymbol{s}};I,0\right)\boldsymbol{Z}_{\text{top}}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1\right). \tag{4.7}$$

Then, we have the following result: the open string N-point topological zeta function $\boldsymbol{Z}_{\text{top}}^{(N)}\left(\underline{\boldsymbol{s}}\right)$ is a rational function of $\mathbb{Q}\left(s_{ij},i,j\in\{1,\ldots,N-1\}\right)$ defined as (4.7).

We define the *Denef-Loeser open string N-point amplitudes at the tree level* as

$$\boldsymbol{A}_{\text{top}}^{(N)}\left(\underline{\boldsymbol{k}}\right)=\boldsymbol{Z}_{\text{top}}^{(N)}\left(\underline{\boldsymbol{s}}\right)|_{s_{ij}=\boldsymbol{k}_i\cdot\boldsymbol{k}_j}, \tag{4.8}$$

with $i\in\{1,\ldots,N-1\}$, $j\in T$ or $i,j\in T$, where $T=\{2,\ldots,N-2\}$. Thus the Denef-Loeser amplitudes are rational functions of the variables $\boldsymbol{k}_i\cdot\boldsymbol{k}_j$, $i,j\in\{1,\ldots,N\}$.

4.3 Denef-Loeser open string four-point amplitudes

In this subsection we calculate the open string 4-point topological zeta function. We recall that the open string 4-point zeta function is defined as

$$\boldsymbol{Z}^{(4)}(\underline{s},\mathbb{K}_e)=\int_{\mathbb{K}_e}|x_2|_{\mathbb{K}_e}^{s_{12}}|1-x_2|_{\mathbb{K}_e}^{s_{32}}dx_2. \tag{4.9}$$

From (3.11) and (3.12), we calculate the contributions of each sector attached to $I\subseteq T=\{2\}$:

$$\begin{aligned}\boldsymbol{Z}^{(4)}(\underline{s},\mathbb{K}_e)&=\boldsymbol{Z}^{(4)}(\underline{s};\{2\},0,\mathbb{K}_e)\boldsymbol{Z}^{(4)}(\underline{s};\{\varnothing\},1,\mathbb{K}_e)\\&\quad+p^{e(1+s_{12}+s_{32})}\boldsymbol{Z}^{(4)}(\underline{s};\{\varnothing\},0,\mathbb{K}_e)\boldsymbol{Z}^{(4)}(\underline{s};\{2\},1,\mathbb{K}_e),\end{aligned} \tag{4.10}$$

where we recall that $\boldsymbol{Z}^{(4)}(\underline{s};\{\varnothing\},0,\mathbb{K}_e)=1$, $\boldsymbol{Z}^{(4)}(\underline{s};\{\varnothing\},1,\mathbb{K}_e)=1$.

By using the results given in sections 3 and 4, we obtain

$$\begin{aligned}\boldsymbol{Z}^{(4)}(\underline{s},\mathbb{K}_e)&=\boldsymbol{Z}^{(4)}(\underline{s};\{2\},0,\mathbb{K}_e)+p^{e(1+s_{12}+s_{32})}\boldsymbol{Z}^{(4)}(\underline{s};\{2\},1,\mathbb{K}_e)\\&=\int_{R_{\mathbb{K}_e}}|x_2|_{\mathbb{K}_e}^{s_{12}}|1-x_2|_{\mathbb{K}_e}^{s_{32}}dx_2+p^{e(1+s_{12}+s_{32})}\int_{R_{\mathbb{K}_e}}|x_2|_{\mathbb{K}_e}^{-2-s_{12}-s_{32}}dx_2,\end{aligned} \tag{4.11}$$

where $R_{\mathbb{K}_e}$ is the ring of integers of $\mathbb{K}_e$ and

$$\boldsymbol{Z}^{(4)}(\underline{s};\{2\},0,\mathbb{K}_e)=1-2p^{-e}+\frac{\left(1-p^{-e}\right)p^{e(-1-s_{12})}}{1-p^{e(-1-s_{12})}}+\frac{\left(1-p^{-e}\right)p^{e(-1-s_{32})}}{1-p^{e(-1-s_{32})}} \tag{4.12}$$

and

$$\boldsymbol{Z}^{(4)}(\underline{s};\{2\},1,\mathbb{K}_e)=\frac{\left(1-p^{-e}\right)p^{e(1+s_{12}+s_{32})}}{1-p^{e(1+s_{12}+s_{32})}}. \tag{4.13}$$

Taking the limit e approaches to zero, we obtain

$$\boldsymbol{Z}_{\text{top}}^{(4)}(\underline{s};\{2\},0)=-1+\frac{1}{s_{12}+1}+\frac{1}{s_{32}+1} \tag{4.14}$$

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I	I^c	$Sect(I)$
$\{2\}$	$\{3\}$	$R_{\mathbb{K}_e} \times \mathbb{K}_e \backslash R_{\mathbb{K}_e}$
$\{3\}$	$\{2\}$	$\mathbb{K}_e \backslash R_{\mathbb{K}_e} \times R_{\mathbb{K}_e}$
$\{2,3\}$	$\varnothing$	$R_{\mathbb{K}_e} \times R_{\mathbb{K}_e}$
$\varnothing$	$\{2,3\}$	$\mathbb{K}_e \backslash R_{\mathbb{K}_e} \times \mathbb{K}_e \backslash R_{\mathbb{K}_e}$

Table 1. In the table we enumerate the different subsets I, their complements $T \smallsetminus I$ and their associated region $Sect(I)$.

and

$$\boldsymbol{Z}_{\text{top}}^{(4)}(\underline{s}; \{2\}, 1) = -\frac{1}{s_{12} + s_{32} + 1}. \tag{4.15}$$

Consequently

$$\boldsymbol{Z}_{\text{top}}^{(4)}(\underline{\boldsymbol{s}}) = -1 + \frac{1}{s_{12}+1} + \frac{1}{s_{32}+1} - \frac{1}{s_{12}+s_{32}+1}. \tag{4.16}$$

By using the kinematic relations $\boldsymbol{k}_1 + \ldots + \boldsymbol{k}_4 = 0$ and $\boldsymbol{k}_i^2 = 2$ we get $\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + \boldsymbol{k}_3 \cdot \boldsymbol{k}_2 + 1 = -1 - \boldsymbol{k}_2 \cdot \boldsymbol{k}_4$, thus the Denef-Loeser string 4-point amplitude is given by

$$\boldsymbol{A}_{\text{top}}^{(4)}(\underline{\boldsymbol{k}}) = Z_{\text{top}}^{(4)}(\underline{\boldsymbol{k}}) = -1 + \frac{1}{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + 1} + \frac{1}{\boldsymbol{k}_3 \cdot \boldsymbol{k}_2 + 1} + \frac{1}{\boldsymbol{k}_2 \cdot \boldsymbol{k}_4 + 1}. \tag{4.17}$$

This result is precisely the one that is obtained by finding the scattering amplitudes from the resulting theory in the limit $p \to 1$ as we described in section 2.

4.4 Denef-Loeser open string five-point amplitudes

The open string 5 -point zeta function is given by

$$\boldsymbol{Z}^{(5)}(\underline{\boldsymbol{s}}, \mathbb{K}_e) = \int_{\mathbb{K}_e^2} |x_2|_{\mathbb{K}_e}^{s_{12}} |x_3|_{\mathbb{K}_e}^{s_{13}} |1-x_2|_{\mathbb{K}_e}^{s_{42}} |1-x_3|_{\mathbb{K}_e}^{s_{43}} |x_2-x_3|_{\mathbb{K}_e}^{s_{23}} dx_2 dx_3. \tag{4.18}$$

Formulae (3.11)–(3.12) require an explicit description of the sectors attached to all the subsets I of $T = \{2,3\}$, i.e. for any $I \in \{\{2\},\{3\},\{2,3\},\varnothing\}$. For instance, the sector corresponding to $T = \{2,3\}$ is $Sect(T) = \left\{(x_2,x_3) \in \mathbb{K}_e^2;\ |x_2|_{\mathbb{K}_e} \leq 1 \text{ and } |\mathrm{x}_3|_{\mathbb{K}_e} \leq 1\right\}$. An explicit description of all the sectors is given in table 1.

The open string 5-point topological zeta function is defined as

$$\boldsymbol{Z}_{\text{top}}^{(5)}(\underline{\boldsymbol{s}}) = \sum_{I \subseteq T} \boldsymbol{Z}_{\text{top}}^{(5)}(\underline{\boldsymbol{s}}; I, 0)\, \boldsymbol{Z}_{\text{top}}^{(5)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1). \tag{4.19}$$

Table 2 contains explicit formulae for all the integrals $\boldsymbol{Z}_{\text{top}}^{(5)}(\underline{\boldsymbol{s}}; I, 0)$ and $\boldsymbol{Z}_{\text{top}}^{(5)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1)$.

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I	$\boldsymbol{Z}_{\text{top}}^{(5)}(\underline{\boldsymbol{s}}; I, 0)$	$\boldsymbol{Z}_{\text{top}}^{(5)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1)$
$\{2\}$	$-1 + \frac{1}{1+s_{12}} + \frac{1}{1+s_{42}}$	$-\frac{1}{1+s_{13}+s_{43}+s_{23}}$
$\{3\}$	$-1 + \frac{1}{1+s_{13}} + \frac{1}{1+s_{43}}$	$-\frac{1}{1+s_{12}+s_{42}+s_{23}}$
$\{2,3\}$	$\left[\frac{1}{1+s_{12}} + \frac{1}{1+s_{13}} + \frac{1}{1+s_{23}} - 1\right] \frac{1}{2+s_{12}+s_{13}+s_{23}}$ $+\frac{1}{1+s_{12}}\left[\frac{1}{1+s_{43}} - 1\right] + \frac{1}{1+s_{13}}\left[\frac{1}{1+s_{42}} - 1\right] +$ $2 - \frac{1}{1+s_{23}} - \frac{1}{1+s_{42}} - \frac{1}{1+s_{43}} +$ $\frac{1}{2+s_{42}+s_{43}+s_{23}}\left[\frac{1}{1+s_{42}} + \frac{1}{1+s_{43}} + \frac{1}{1+s_{23}} - 1\right]$	1
$\{\varnothing\}$	1	$-\frac{1}{2+s_{52}+s_{53}+s_{23}} \times$ $\left[\frac{1}{1+s_{12}+s_{42}+s_{23}} + \frac{1}{1+s_{13}+s_{43}+s_{23}} + \frac{1}{1+s_{23}} - 1\right]$

Table 2. The topological zeta functions $\boldsymbol{Z}_{\text{top}}^{(5)}(\underline{\boldsymbol{s}}; I, 0)$ and $\boldsymbol{Z}_{\text{top}}^{(5)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1)$ is written for each subset I and its complement $T \smallsetminus I$.

Thus, the Denef-Loeser open string 5-point amplitude is given by

$$
\begin{aligned}
\boldsymbol{A}_{\text{top}}^{(5)}(\underline{\boldsymbol{k}}) = & \left[\frac{1}{1+\boldsymbol{k}_1\cdot\boldsymbol{k}_2}+\frac{1}{1+\boldsymbol{k}_4\cdot\boldsymbol{k}_2}-1\right]\left[-\frac{1}{1+\boldsymbol{k}_3\cdot\boldsymbol{k}_5}\right]+\left[-\frac{1}{1+\boldsymbol{k}_2\cdot\boldsymbol{k}_5}\right]\left[\frac{1}{1+\boldsymbol{k}_1\cdot\boldsymbol{k}_3}+\frac{1}{1+\boldsymbol{k}_4\cdot\boldsymbol{k}_3}-1\right] \\
& +\left[\frac{1}{1+\boldsymbol{k}_1\cdot\boldsymbol{k}_2}+\frac{1}{1+\boldsymbol{k}_1\cdot\boldsymbol{k}_3}+\frac{1}{1+\boldsymbol{k}_2\cdot\boldsymbol{k}_3}-1\right]\frac{1}{1+\boldsymbol{k}_4\cdot\boldsymbol{k}_5}+\frac{1}{1+\boldsymbol{k}_1\cdot\boldsymbol{k}_2}\left[\frac{1}{1+\boldsymbol{k}_4\cdot\boldsymbol{k}_3}-1\right] \\
& +\frac{1}{1+\boldsymbol{k}_1\cdot\boldsymbol{k}_3}\left[\frac{1}{1+\boldsymbol{k}_4\cdot\boldsymbol{k}_2}-1\right]+2-\frac{1}{1+\boldsymbol{k}_2\cdot\boldsymbol{k}_3}-\frac{1}{1+\boldsymbol{k}_4\cdot\boldsymbol{k}_2}-\frac{1}{1+\boldsymbol{k}_4\cdot\boldsymbol{k}_3} \\
& +\frac{1}{1+\boldsymbol{k}_1\cdot\boldsymbol{k}_5}\left[\frac{1}{1+\boldsymbol{k}_4\cdot\boldsymbol{k}_2}+\frac{1}{1+\boldsymbol{k}_4\cdot\boldsymbol{k}_3}+\frac{1}{1+\boldsymbol{k}_2\cdot\boldsymbol{k}_3}-1\right] \\
& -\frac{1}{1+\boldsymbol{k}_1\cdot\boldsymbol{k}_4}\left[\frac{1}{1+\boldsymbol{k}_5\cdot\boldsymbol{k}_2}+\frac{1}{1+\boldsymbol{k}_3\cdot\boldsymbol{k}_5}+\frac{1}{1+\boldsymbol{k}_2\cdot\boldsymbol{k}_3}-1\right].
\end{aligned} \tag{4.20}
$$

5 Final remarks

In this article we have considered the limit $p \to 1$ of p-adic Koba-Nielsen amplitudes. In order to make mathematical sense of this limit, we used the theory of topological zeta functions introduced by Denef and Loeser. This requires to extend the p-adic Koba-Nielsen amplitudes to unramified extensions of $\mathbb{Q}_p$, more precisely to $\mathbb{K}_e$ the unique unramified extension of degree e of $\mathbb{Q}_p$, and then to express these amplitudes as a finite sum of multivariate Igusa's zeta functions, see formulae (3.11)–(3.12). This step is carried out using the results of [28], however, we do not need the convergence of the p-adic Koba-Nielsen amplitudes.

In this setting, using results due to Denef and Loeser, the limit $p \to 1$ becomes the limit $e \to 0$. The computation of this last limit requires explicit formulae for certain multivariate Igusa's zeta functions, the required formulae were obtained using results due to Denef. By taking the limit $e \to 0$ in the Koba-Nielsen amplitudes over $\mathbb{K}_e$, we obtain the corresponding

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string amplitudes. We computed explicitly the 4 and 5-point Denef-Loeser string amplitudes, see formulae (4.17) and (4.20). These amplitudes coincide exactly with the quantum field derivation from the Gerasimov-Shatashvili Lagrangian as presented in section 2.

The topological zeta functions are particular cases of the motivic Igusa's zeta functions constructed by Denef and Loeser in [35] using the theory of motivic integration, see [36]. Consequently, we can assert that there exist motivic Koba-Nielsen amplitudes which specializes to the topological Denef-Loeser amplitudes introduced here and to the classical p-adic Koba-Nielsen amplitudes, however, we do not know if these motivic amplitudes have any physical meaning.

Such as it was mentioned in the introduction of [28], see also [37], there are deep connections between local zeta functions with string amplitudes and quantum field theory amplitudes that still are not fully understood.

Another relevant research direction is to explore the relation between the topological amplitudes introduced here with the amplitudes coming from the BSFT Lagrangian proposed by Witten in [23, 24]. We expect a relation due to the work of Gerasimov and Shatashvili [21]. It would be also interesting to study the incorporation of a B-field to the string amplitudes such as was worked out in [38]. In this article the amplitudes are modified by a noncommutative parameter satisfying the Moyal bracket. Finally it would be also interesting to study the interplay between p-adic amplitudes in field theory [39], AdS/CFT correspondence [5] and the renormalization group in discrete world-sheet in the limit $p \to 1$, see [27], using the methods introduced here.

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A Non-Archimedean local fields

In these appendices, we review some basic ideas and results on non-Archimedean and multivariate local zeta functions that we use along this article.

We recall that the field of rational numbers $\mathbb{Q}$ admits two types of norms: the Archimedean norm (the usual absolute value), and the non-Archimedean norms (the p-adic norms) which are parameterized by the prime numbers. The field of real numbers $\mathbb{R}$ arises as the completion of $\mathbb{Q}$ with respect to the Archimedean norm. Fix a prime number p, the p-adic norm is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ \\ p^{-\gamma} & \text{if } x = p^{\gamma}\frac{a}{b}, \end{cases} \tag{A.1}$$

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where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := \infty$, is called the *p-adic order of x*. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$.

A non-Archimedean local field $\mathbb{K}$ is a locally compact topological field with respect to a non-discrete topology, which comes from a norm $|\cdot|_{\mathbb{K}}$ satisfying

$$|x+y|_{\mathbb{K}} \leq \max\left\{|x|_{\mathbb{K}}, |y|_{\mathbb{K}}\right\}, \tag{A.2}$$

for $x, y \in \mathbb{K}$. A such norm is called an *ultranorm or non-Archimedean*. Any non-Archimedean local field $\mathbb{K}$ of characteristic zero is isomorphic (as a topological field) to a finite extension of $\mathbb{Q}_p$, and it is called a p-adic field. The field $\mathbb{Q}_p$ is the basic example of non-Archimedean local field of characteristic zero. In the case of positive characteristic, $\mathbb{K}$ is isomorphic to a finite extension of the field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field $\mathbb{F}_q$, where q is a power of a prime number p.

In this article we work only with non-Archimedean fields $\mathbb{K}$ of characteristic zero. Thus from now on $\mathbb{K}$ denotes one of these fields. The *ring of integers* of $\mathbb{K}$ is defined as

$$R_{\mathbb{K}} = \{x \in \mathbb{K}; |x|_{\mathbb{K}} \leq 1\}. \tag{A.3}$$

Geometrically $R_{\mathbb{K}}$ is the unit ball of the normed space $(\mathbb{K}, |\cdot|_{\mathbb{K}})$. This ring is a domain of principal ideals having a unique maximal ideal, which is given by

$$P_{\mathbb{K}} = \{x \in \mathbb{K}; |x|_{\mathbb{K}} < 1\}. \tag{A.4}$$

We fix a generator π of $P_{\mathbb{K}}$ i.e. $P_{\mathbb{K}} = \pi R_{\mathbb{K}}$. A such generator is also called a *local uniformizing parameter of* $\mathbb{K}$, and it plays the same role as p in $\mathbb{Q}_p$.

The *group of units* of $R_{\mathbb{K}}$ is defined as

$$R_{\mathbb{K}}^{\times} = \{x \in R_{\mathbb{K}}; |x|_{\mathbb{K}} = 1\}. \tag{A.5}$$

The natural map $R_{\mathbb{K}} \to R_{\mathbb{K}}/P_{\mathbb{K}} \cong \mathbb{F}_q$ is called the *reduction mod* $P_{\mathbb{K}}$. The quotient $R_{\mathbb{K}}/P_{\mathbb{K}} \cong \mathbb{F}_q$, is a finite field with $q = p^f$ elements, and it is called the *residue field* of $\mathbb{K}$. Every non-zero element x of $\mathbb{K}$ can be written uniquely as $x = \pi^{ord(x)}u$, $u \in R_{\mathbb{K}}^{\times}$. We set $ord(0) = \infty$. The normalized valuation of $\mathbb{K}$ is the mapping

$$\begin{aligned} \mathbb{K} &\to \mathbb{Z} \cup \{\infty\} \\ x &\to ord(x). \end{aligned}$$

Then $|x|_{\mathbb{K}} = q^{-ord(x)}$ and $|\pi|_{\mathbb{K}} = q^{-1}$.

We fix $\mathfrak{S} \subset R_{\mathbb{K}}$ a set of representatives of $\mathbb{F}_q$ in $R_{\mathbb{K}}$, i.e. $\mathfrak{S}$ is a set which is mapped bijectively onto $\mathbb{F}_q$ by the reduction mod $P_{\mathbb{K}}$. We assume that $0 \in \mathfrak{S}$. Any non-zero element x of $\mathbb{K}$ can be written as

$$x = \pi^{ord(x)} \sum_{i=0}^{\infty} x_i \pi^i, \tag{A.6}$$

where $x_i \in \mathfrak{S}$ and $x_0 \neq 0$. This series converges in the norm $|\cdot|_{\mathbb{K}}$.

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We extend the norm $|\cdot|_{\mathbb{K}}$ to $\mathbb{K}^n$ by taking

$$||\boldsymbol{x}||_{\mathbb{K}} := \max_{1\leq i\leq n} |x_i|_{\mathbb{K}}, \tag{A.7}$$

for $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$.

We define $ord(\boldsymbol{x}) = \min_{1\leq i\leq n}\{ord(x_i)\}$, then $||\boldsymbol{x}||_{\mathbb{K}} = q^{-ord(\boldsymbol{x})}$. The metric space $(\mathbb{K}^n, ||\cdot||_{\mathbb{K}})$ is a complete ultrametric space.

For $r \in \mathbb{Z}$, denote by $B_r^n(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{K}^n; ||\boldsymbol{x}-\boldsymbol{a}||_{\mathbb{K}} \leq q^r\}$ *the ball of radius* q^r *with center at* $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{K}^n$, and take $B_r^n(\boldsymbol{0}) := B_r^n$. Note that $B_r^n(\boldsymbol{a}) = B_r(a_1)\times\cdots\times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{K}; |x_i - a_i|_{\mathbb{K}} \leq q^r\}$ is the one-dimensional ball of radius q^r with center at $a_i \in \mathbb{K}$. The ball B_0^n equals the product of n copies of $B_0 = R_{\mathbb{K}}$. In addition, $B_r^n(\boldsymbol{a}) = \boldsymbol{a} + (\pi^{-r}R_{\mathbb{K}})^n$. We also denote by $S_r^n(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{K}^n; ||\boldsymbol{x} - \boldsymbol{a}||_{\mathbb{K}} = q^r\}$ *the sphere of radius* q^r *with center at* $\boldsymbol{a} \in \mathbb{K}^n$, and take $S_r^n(\boldsymbol{0}) := S_r^n$. We notice that $S_0^1 = R_{\mathbb{K}}^{\times}$ (the group of units of $R_{\mathbb{K}}$), but $\left(R_{\mathbb{K}}^{\times}\right)^n \subsetneq S_0^n$, for $n \geq 2$. The balls and spheres are both open and closed subsets in $\mathbb{K}^n$. In addition, two balls in $\mathbb{K}^n$ are either disjoint or one is contained in the other.

The topological space $(\mathbb{K}^n, ||\cdot||_{\mathbb{K}})$ is totally disconnected, i.e. the only connected subsets of $\mathbb{K}^n$ are the empty set and the points. A subset of $\mathbb{K}^n$ is compact if and only if it is closed and bounded in $\mathbb{K}^n$. The balls and spheres are compact subsets. Thus $(\mathbb{K}^n, ||\cdot||_{\mathbb{K}})$ is a locally compact topological space.

As we mentioned before, any finite extension $\mathbb{K}$ of $\mathbb{Q}_p$ is a non-Archimedean local field. Then

$$pR_{\mathbb{K}} = \pi^m R_{\mathbb{K}}, \qquad m \in \mathbb{N}. \tag{A.8}$$

If $m = 1$ we say that $\mathbb{K}$ is a *unramified* extension of $\mathbb{Q}_p$. In other case, we say that $\mathbb{K}$ is a *ramified* extension. It is well known that for every positive integer e there exists a unique unramified extension $\mathbb{K}_e$ of $\mathbb{Q}_p$ of degree e, which means that $\mathbb{K}_e$ is a $\mathbb{Q}_p$-vector space of dimension e. From now on, π denotes a local uniformizing parameter of $\mathbb{K}_e$, thus $pR_{\mathbb{K}_e} = \pi R_{\mathbb{K}_e}$, $R_{\mathbb{K}_e}/P_{\mathbb{K}_e} \cong \mathbb{F}_{p^e}$ and $|\pi|_{\mathbb{K}_e} = p^{-e}$. For an in-depth exposition of non-Archimedean local fields, the reader may consult [40, 41], see also [3, 42].

B Multivariate Igusa zeta functions

Let $\mathbb{K}$ be a p-adic field as before. Let $f_i(\boldsymbol{x}) \in \mathbb{K}[x_1, \ldots, x_n]$ be a non-constant polynomial for $i = 1, \ldots, r$, and let Φ be a Bruhat-Schwartz function, i.e. a locally constant function with compact support. We set $\boldsymbol{f} = (f_1, \ldots, f_r)$ and $\boldsymbol{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$. The multivariate local zeta function attached to $(f_1, \ldots, f_r, \Phi)$ (also called multivariate Igusa local zeta function) is defined as

$$\boldsymbol{Z}_\Phi(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K}) = \int_{\mathbb{K}^n\smallsetminus\cup_{i=1}^r f_i^{-1}(\boldsymbol{0})} \Phi(\boldsymbol{x}) \prod_{i=1}^r |f_i(\boldsymbol{x})|_{\mathbb{K}}^{s_i}\, d^n\boldsymbol{x} \tag{B.1}$$

for $\mathrm{Re}(s_i) > 0$, $i = 1, \ldots, r$, here $d^n\boldsymbol{x}$ denotes the normalized Haar measure of $\mathbb{K}^n$. This integral defines a holomorphic function of $(s_1, \ldots, s_r)$ in the half-space $\mathrm{Re}(s_i) > 0$, $i = 1, \ldots, r$. In the case $r = 1$, the local zeta functions were introduced by Weil, for general

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$\boldsymbol{f}$ were first studied by Igusa [29]. In the multivariate case, i.e. for $r \geq 1$, the local zeta functions were studied by Loeser [43]. The Igusa local zeta functions are related with the number of solutions of polynomial congruences mod p^m and with exponential sums mod p^m. There are many intriguing conjectures relating the poles of local zeta functions with the topology of complex singularities, see e.g. [29, 44].

If Φ is the characteristic function of $R_{\mathbb{K}}^n$ we use the simplified notation $\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K})$.

B.1 Embedded resolution of singularities

In this section $\mathbb{L}$ is an arbitrary field of characteristic zero and $f_i(\boldsymbol{x}) \in \mathbb{L}[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, is a non-constant polynomial for $i = 1, \ldots, r$. The main tool in the study of local zeta functions is Hironaka's resolution of singularities theorem [45]. Put $X =$`Spec` $\mathbb{L}[\boldsymbol{x}]$ (the n-dimensional affine space over $\mathbb{L}$), $D =$`Spec` $\mathbb{L}[\boldsymbol{x}] / (\prod_{i=1}^r f_i(\boldsymbol{x}))$ (the divisor attached to polynomials $f_1,\ldots,f_r$). An *embedded resolution of singularities* for D over $\mathbb{L}$ consists of a pair (Y, h), where Y is a smooth algebraic variety (an integral smooth closed subscheme of the projective space over X), $h : Y \to X$ is the natural map, which satisfies that the restriction $h : Y \smallsetminus h^{-1}(D) \to X \smallsetminus D$ is an isomorphism, and the reduced scheme $h^{-1}(D)_{\text{red}}$ associated to $h^{-1}(D)$ has normal crossings, i.e. its irreducible components are smooth and intersect transversally. Let E_i, $i \in T$, be the irreducible components of $h^{-1}(D)_{\text{red}}$. For each $i \in T$, let N_{ij} be the multiplicity of E_i in the divisor $f_j \circ h$ on Y, and $v_i - 1$ the multiplicity of E_i in the divisor $h^*(dx_1 \wedge \ldots \wedge dx_n)$. The $(N_{i1}, \ldots, N_{ir}, v_i)$, $i \in T$, are called *the numerical data* of the resolution (Y, h). For $i \in T$ and $I \subset T$ we define

$$\overset{\circ}{E}_i = E_i \smallsetminus \bigcup_{j \neq i} E_j, \quad E_I = \bigcap_{i \in I} E_i, \quad \overset{\circ}{E}_I = E_I \smallsetminus \bigcup_{j \in T \smallsetminus I} E_j. \tag{B.2}$$

If $I = \emptyset$, we put $E_\emptyset = Y$.

B.2 Rationality of local zeta functions

Theorem A (Loeser [43]). Let $\mathbb{K}$ be a p-adic field. The local zeta function $\boldsymbol{Z}_\Phi(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K})$ admits a meromorphic continuation to the whole $\mathbb{C}^r$ as a rational function of $q^{-s_1}, \ldots, q^{-s_r}$, more precisely

$$\boldsymbol{Z}_\Phi(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K}) = \frac{P_\Phi(q^{-s_1}, \ldots, q^{-s_r})}{\prod_{i \in T} \left(1 - q^{-v_i - \sum_{j=1}^r N_{ij} s_j}\right)}, \tag{B.3}$$

where P_Φ is a polynomial in the variables $q^{-s_1}, \ldots, q^{-s_r}$. The real parts of the poles of $\boldsymbol{Z}_\Phi(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K})$ belong to a union of hyperplanes of the form

$$v_i + \sum_{j=1}^r N_{ij} \operatorname{Re}(s_j) = 0, \qquad i \in T. \tag{B.4}$$

Theorem B (Denef [46]). Let $f_i(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, be a non-constant polynomial for $i = 1, \ldots, r$. Let (Y, h) be an embedded resolution of singularities for $D =$ *Spec* $\mathbb{Q}[\boldsymbol{x}] / (\prod_{i=1}^r f_i(\boldsymbol{x}))$ over $\mathbb{Q}$, with numerical data $\{(N_{i1}, \ldots, N_{ir}, v_i); i \in T\}$. Then, there exists a finite set of primes $S \subset \mathbb{Z}$ such that for any non-Archimedean local field $\mathbb{K} \supset \mathbb{Q}$ with $P_{\mathbb{K}} \cap \mathbb{Z} \notin S$, we have

$$\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, \mathbb{K}) = q^{-n} \sum_{I \subseteq T} c_I(\mathbb{K}) \prod_{i \in I} \frac{(q-1)\, q^{-v_i - \sum_{j=1}^{r} N_{ij} s_j}}{1 - q^{-v_i - \sum_{j=1}^{r} N_{ij} s_j}}, \tag{B.5}$$

where $q = q(\mathbb{K})$ denotes the cardinality of the residue field $\overline{\mathbb{K}}$ and

$$c_I(\mathbb{K}) = Card\left\{a \in \overline{Y}\left(\overline{\mathbb{K}}\right); a \in \overline{E}_i\left(\overline{\mathbb{K}}\right) \Leftrightarrow i \in I\ \right\}, \tag{B.6}$$

where the bar denotes the reduction mod $P_{\mathbb{K}}$ for which we refer to [46, section 2].

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